Knot Theory Informed by Random Models and Experimental Data

Chaim Even-Zohar (Technion),
Joel Hass (University of California, Davis),
Anastasiia Tsvietkova (Rutgers University, Newark)

March 31 - April 5, 2024

1 Overview

Over its history, knot theory has yielded many questions and conjectures that drove the development not only of this field of study, but also of several other fields of mathematics. One way to generate new questions is through a probabilistic viewpoint and through the use of experimental data. Several new random models of knots and 3-manifolds have appeared recently, and probabilistic, experimental, and computer-aided studies of knots have played increasingly important roles. Such approaches allow to establish geometric and topological properties of knots beyond well-studied families, and often suggest a perspective extending known cases. The main topic of the workshop was the probabilistic and experimental study of geometric and topological properties of knots, links, surfaces, and other manifolds, and the interplay of such properties with probability and combinatorics. It also looked at associated computational questions concerning complexity and algorithms for knots and 3-manifolds.

2 Research Presentations

2.1 45-minute talks

All workshop participants who expressed interest in giving 45-minute talks were invited to do so. The speakers are listed below with titles and abstracts of their talks.

Roots of Alexander polynomials of random positive 3-braids
Nathan Dunfiled, Department of Mathematics, University of Illinois Urbana-Champaign

Motivated by an observation of Dehornoy, we study the roots of Alexander polynomials of knots and links that are closures of positive 3-strand braids. We give experimental data on random such braids and find that the roots exhibit marked patterns, which we refine into precise conjectures. We then prove several results along those lines, for example that generically at least 69 per cent of the roots are on the unit circle, which appears to be sharp. We also show there is a large root-free region near the origin. We further study the equidistribution properties of such roots by introducing a Lyapunov exponent of the Burau representation of random positive braids, and a corresponding bifurcation measure. In the spirit of Deroin and Dujardin, we conjecture that the bifurcation measure gives the limiting measure for such roots, and prove this on a region with positive limiting mass. We use tools including work of Gambaudo and Ghys on the signature function of links, for which we prove a central limit theorem. This is joint work with Giulio Tiozzo.

New Theories of how proteins knot

Erica Flapan, Pomona College and AMS

How knotted proteins fold has remained controversial since deeply knotted proteins were first identified two decades ago. The first theory of how protein knots fold suggested that a protein chain twists into a loop and one terminus threads through the loop to produce a twist knot. A more complex folding pathway involving two loops was later introduced for the 6_1 knot in the protein DehI by Sulkowska who showed using molecular dynamics simulations that a a loop flipping mechanism could describe its folding pathway. Motivated by these results, we developed a more general loop flipping theory which potentially could describe the folding of all twist knots and a large number of non-twist knots. The recently developed artificial intelligence program AlphaFold predicted the first 6_3 protein knot. This knot is not a twist knot, and molecular dynamics simulations of three representative 6_3-knotted proteins conducted by Sulkowska indicated that they did not fold according to our loop flipping theory. We have now extended our loop flipping theory to two new theories which describe the successful trajectories of the molecular dynamics of the folding of these 6_3-knotted proteins, and can be applied to many additional knots.

A knotting complexity estimator for random knots on the cubic lattice

Yuanan Diao, Department of Mathematics and Statistics, The University of North Carolina at Charlotte

Average crossing numbers and average squared writhes are estimators often used to gauge the complexity of a randomly generated knot. An obvious limitation on such measures is that they cannot detect knotting complexity directly since in theory a random knot with high ACN can be a relatively simple knot or even the unknot. In this talk, I will discuss an alternative knotting complexity estimator that is related to the braid index of the random knot that can detect knotting when it falls into certain range. In the case that the random knot is generated as a self-avoiding polygon on the cubic lattice, I shall describe an algorithm on how to compute this estimator.

Generating (and Computing with) Very Large Ensembles of Random Polygonal Knots

Clayton Shonkwiler, Department of Mathematics, Colorado State University, Fort Collins

The symplectic theory of polygon spaces, which was developed by Kapovich and Millson [46] among others, can be used to define a fast and provably ergodic Markov chain on spaces of polygonal knots in 3-space [13]. While such Markov chains have traditionally been of interest for numerical experimentation with simple ring polymer models, they can also be used to find an abundance of examples of small but complex knots.

A key insight is that this Markov chain based on symplectic geometry can easily be modified to sample polygons in rooted spherical confinement, meaning that one vertex is fixed at the center of a sphere of radius R and all other vertices are constrained to lie inside the sphere. Putting polygons in tight confinement tends to increase the probability of complicated knots, so this serves as a form of enriched sampling [33]. Of course, it remains a challenge to quickly and reliably identify the knot types of complicated polygons.

In a series of numerical experiments [11, 33, 70, 71], we have generated trillions of random polygonal knots in tight confinement and found many examples of knots realized with fewer segments or local maxima than previously seen. As a result, this work has dramatically improved our knowledge of two elementary knot invariants, the stick number and the superbridge index. For example, we have computed the majority of known superbridge indices of prime knots through 10 crossings and given the first table of stick number bounds for all 11-crossing knots.

[This work was supported by the National Science Foundation (DMS–2107700) and the Simons Foundation (#354225)]

Hyperbolicity of Staked Knots

Colin Adams, Williams College

Take a knot projection on the sphere and skewer it with sticks perpendicular to the sphere. This is a staked knot. We define hyperbolicity for staked knots, determine hyperbolicity for alternating staked knots, consider volumes and relate staked knots to knots in handlebodies, knotoids and generalized knotoids.

Crossing numbers of satellite knots

Efstratia Kalfagianni, Department of Mathematics, Michigan State University

My talk focused on recent work with Christine Lee (Texas State University) and Rob McConkey (Colorado State University) on crossing numbers of satellites for adequate (in particular alternating) knots.

Theorem 1. *[47] Suppose that K is an adequate knot with crossing number $c(K)$ and writhe $\mathrm{wr}(K)$ and let $W_-(K)$ (resp. $W_+(K)$) denote the negative (resp. positive) untwisted Whitehead double of K. Then, the crossing number $c(W_\pm(K))$ satisfies the following inequalities.*

$$4c(K) + 1 \leq c(W_\pm(K)) \leq 4c(K) + 2 + 2|\mathrm{wr}(K)|.$$

Furthermore, if $\mathrm{wr}(K) = 0$ we have $c(W_\pm(K)) = 4c(K) + 2$.

Given co-prime integers p, q let $K_{p,q}$ denote the (p, q)-cable of K. In other words, $K_{p,q}$ is the simple closed curve on $\partial N(K)$ that wraps p times around the meridian and q-times around the canonical longitude of K.

Theorem 2. *[48] For any adequate knot K with crossing number $c(K)$, and any coprime integers p, q, we have $c(K_{p,q}) > q^2 \cdot c(K)$. In particular, if $p = 2\mathrm{wr}(K) \pm 1$, then $c(K_{p,2}) = 4c(K) + 1$.*

The lower bounds in Theorems 1 and 2 come from the degree of the colored Jones polynomial of K and specifically from the *Jones diameter* [41]. It has been long known that for every knot K, $dj_K \leq 2c(K)$. In [47] we show that $dj_K = 2c(K)$ if and only if K is adequate. This result is a key ingredient in the proofs of Theorems 1 and 2. In [8] Baker, Motegi and Takata use similar methods to obtain lower bounds for crossing numbers of *Mazur doubles*. They show that if K is an adequate knot with $\mathrm{wr}(K) = 0$, then the crossing number of the Mazur double of K is either $9c(K) + 2$ or $9c(K) + 3$. [The author acknowledges partial research support through NSF Grants DMS-2004155 and DMS-2304033.]

Excluding cosmetic surgeries on knots and 3-manifolds

Jessica Purcell, Monash University, Australia

The cosmetic surgery conjecture for knots states that if two Dehn fillings of a knot complement in the 3-sphere are orientation-preserving homeomorphic, then the two Dehn filling slopes must be identical.

We use hyperbolic geometry and knot invariants to give a practical procedure for checking the cosmetic surgery conjecture on any one-cusped manifold. We have used the procedure to verify the cosmetic surgery conjecture for all knots up to 19 crossings, and all one-cusped 3-manifolds in the SnapPy census. We have also verified related conjectures. This is joint work with David Futer and Saul Schleimer.

Average genus and average signature of a 2-bridge knot

Moshe Cohen, Mathematics Department, State University of New York at New Paltz

We show that the average genus of a 2-bridge knot with crossing number c approaches $c/4 + 1/12$ as c approaches infinity [20]. We prove that the distribution of genera of all 2-bridge knots with a given crossing number approaches a normal distribution [19]. We show that the average absolute value of a 2-bridge knot with crossing number c approaches the $\sqrt{2c/\pi}$. This is joint work with Adam Lowrance and portions with his undergraduate students Abigail DiNardo, Steven Raanes, Izabella Rivera, Andrew Steindl, and Ella Wanebo.

Random polygons in spherical confinement

Claus Ernst, Western Kentucky University

In the talk a model of random equilateral polygons in spherical confinement is introduced [23, 24, 25]. A large sample of such polygons has been generated and analyzed. In the talk a summary of the analysis the large sample of such polygons is presented. The dependence of the knot spectrum on the polygonal length and the radius of confinement is illustrated [28, 30]. In addition, geometric properties such as curvature, torsion [29], average crossing number and writhe [27] - of the random polygons are discussed. How are these influenced, not only by the polygonal length and the radius of confinement, but also by the knotting complexity. While the generation process is rigorous (based on theorems), the analysis of the sample of such polygons is only based on the numerical results. There are no known theorems in this area.

Probabilistic methods in low-dimensional hyperbolic geometry

Bram Petri, Sorbonne Universié and Institut Universitaire de France

I will talk about how to use probability theory to attack extremal problems in hyperbolic geometry.

Singularity of measures for Cannon-Thurston maps

Joseph Maher, Department of Mathematics, College of Staten Island, CUNY

Cannon and Thurston showed that a hyperbolic 3-manifold that fibers over the circle gives rise to a sphere filling curve. The universal cover of the fiber surface is quasi-isometric to the hyperbolic plane, whose boundary is a circle, and the universal cover of the 3-manifold is 3-dimension hyperbolic space, whose boundary is the 2-sphere. Cannon and Thurston showed that the inclusion map between the universal covers extends to a continuous map between their boundaries, whose image is dense. In particular, any measure on the circle pushes forward to a measure on the 2-sphere using this map. We consider Lebesgue measure on the circle, and the hitting measures associated to random walks on the surface group, and show that their push forwards onto the 2-sphere are mutually singular with the Lebesgue measure on the 2-sphere, and with the hitting measures from a random walk on the 3-manifold group.

This is joint work with Vaibhav Gadre, Thomas Haettel, Catherine Pfaff and Caglar Uyanik.

On the length spectrum of random hyperbolic 3-manifolds

Anna Roig Sanchis, laboratory IMJ-PRG, Sorbonne Université

We are interested in studying the behaviour of geometric invariants of hyperbolic 3-manifolds, such as the length of their geodesics. A way to do so is by using probabilistic methods, that is, through the study of random manifolds. There are several models of random manifolds. In this talk, I will explain one of the principal probabilistic models for 3 dimensions and I will present a result concerning the length spectrum -the set of lengths of all closed geodesics- of a 3-manifold constructed under this model.

Exploring large random knots: New algorithms and questions.
Jason Cantarella, University of Georgia

A physically natural model for random knots is the random equilateral polygon. The probability of finding a polygon with n edges to have a given knot type K is well-described by [77],

$$P_k(n) = C_k n^{(v_0 + \mathrm{np}(K))} \exp(-n/n0)(1 - b_k n^{-1/2} + g_k n^{-1})$$

where $\mathrm{np}(k)$ is the number of prime components, and $v_0 \sim -0.19$ and $n_0 \sim 260$ are universal constants. So what is the connection of C_k to traditional knot invariants? We describe new methods and implementations which sample polygon space directly in $O(n^2)$ times and compute values of the Alexander polynomial in $O(n^{1.18})$ time [14], and provide reweighted samples of polygon space in linear time using ideas from conformal geometry [15].

These are open source and available for use, and we hope that the workshop community will be interested in comparing notes and finding the right questions to ask with them! (Joint work with Clay Shonkwiler, Henrik Schumacher, Tetsuo Deguchi, Erica Uehara).

Extensions to polygon generation in spherical confinement
Uta Ziegler, Western Kentucky University

There are currently two independently developed, different, and rigorous (based on theorems) methods to generate random, equilateral, freely-jointed, rooted, volume-less polygons in spherical confinement [18, 25]. In this context, a polygon is rooted in spherical confinement, if exactly one of its vertices is at the center of the confinement sphere. Extensive data was generated and analyzed by one of the groups [28, 30, 29, 27]. Due to the rootedness of the randomly generated polygons, the radii of the confinement spheres for which data could be collected was restricted to $R \geq 1.0$. This presentation introduces two new models that try to show what happens in the following situations:

1. A model to study random polygons generated under extreme confinement conditions ($1/2 \leq R < 1$).
2. A model which biases the random polygon generation towards generating 'thicker' polygons [74].

This talk also includes topological and geometric data obtained from analyzing polygons generated using the proposed models. The results are presented with a focus on establishing that the suggested models capture many features of equilateral random polygons in tight spherical confinement [34]. Neither model is rigorous and this gives rise to a number of open questions.

Knot invariants for knots given by random braids
Marina Ville, University of Tours

(1) Estimate of the expectation of the Casson invariant $E(c_2)$ for knots defined by n-braids using models given by Lamm's *rosette braids* ([La]).

Theorem 3.

1. *For knots given by closures of s-braids of length L (Fig. 1), we have $E(c_2) \sim \frac{s}{24} L$.*
2. *For knots given by checkerboard diagrams with sizes of length $2b$ and $2nb$ (Fig. 2), $E(c_2) \sim \frac{b}{16} L$.*

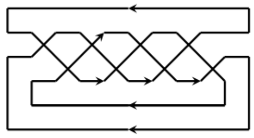

Fig. 1

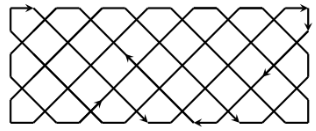

Fig. 2

(2) Random walks on the braid groups. We use the symplectic representation ρ_n of a braid group $B(n)$ in the symplectic group $Sp(2l, \mathbb{Z})$ with $l = [\frac{n-1}{2}]$. If n is odd, ρ_n coincides with $\mathcal{B}_{-1}$, that is the Burau representation for $t = -1$, and if n is even, it is a quotient of $\mathcal{B}_{-1}$. With mild assumptions on a probability μ on the braid group, we have

Theorem 4. *Let P be a polynomial in $(2l)^2$ variables with coefficients in $\mathbb{Z}$ which does not vanish identically on $Sp(2l, \mathbb{Z})$. Then the set $\{\beta \in B(n) : P(\rho_n(\beta)) = 0\}$ is transient for the right random μ-walk.*

Corollary 1. *Almost all 3-braids β verify: $\forall n \in \mathbb{N}$, $\forall i, j \in \{1, 2\}$, $sign\big((\beta\sigma_i\beta^{-1}\sigma_j)^n\big) = 0$*

Corollary 2. *Let C be a positive number and n an integer. If n is odd, the set of n-braids β which close in a link $\hat{\beta}$ with $|det(\hat{\beta})| < C$ is transient.*

Work in progress. 1) If n is even, then: a) the set $\{\beta \in B(n) : 0 < |det(\hat{\beta})| < C\}$ is transient; b) Corollary 2 is true for knots. 2) If n is odd, and p is a prime number, I study the probability for a n-braid β to close in a p-colorable link.

2.2 10-minute talks

The following participants expressed interest in giving 10-minute talks, and all of them were invited to do so. The speakers below are listed with titles of their talks, and with abstracts of some of the talks.

Computing character varieties and schemes in $\mathrm{SL}_2(\mathbb{C})$
Joan Porti, Departament de Matemàtiques, Universitat Autònoma de Barcelona, and Centre de Recerca Matemàtica (UAB-CRM), Spain

Given a finitely presented group $\Gamma = \langle \gamma_1, \ldots, \gamma_n \mid r_1, \ldots, r_m \rangle$, its variety of representations is

$$\hom(\Gamma, \mathrm{SL}_2(\mathbb{C})) \subset \mathrm{SL}_2(\mathbb{C}) \times \cdots \times \mathrm{SL}_2(\mathbb{C}) \subset \mathbb{C}^{4n}.$$

It is an algebraic subset of $\mathbb{C}^{4n}$ and it has more structure than a variety, it is an affine scheme [58] with perhaps several components and multiple points (eg $x^2 = 0$ is different from $x = 0$ as a scheme). The group $\mathrm{SL}_2(\mathbb{C})$ acts on $\hom(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ by conjugation and the quotient in the algebraic setting (e.g. the GIT quotient) $X(\Gamma) = \hom(\Gamma, \mathrm{SL}_2(\mathbb{C}))//\mathrm{SL}_2(\mathbb{C}))$. is called the *scheme of characters*. It has this name because its points can be understood as characters of representations of Γ into $\mathrm{SL}_2(\mathbb{C})$.

In [42] Fico an Montesions gave an algorithm to compute the variety of characters instead of the scheme (e.g. without distinguishing double points from simple ones). The goal of this work is to give an algorithm for computing the scheme of characters [43].

In the talk I also provide an example that motivates why to look at the scheme instead of the variety, namely why double points may give geometric information. This is joint work with Michael Heusener, from

Clermont-Ferrand (France). [Partially supported by FEDER-AEI (grant numbers PID2021-125625NB-100 and María de Maeztu Program CEX2020-001084-M)]

Satanic points on knots
Ryan Budney, University of Victoria

I describe an undergraduate summer project where Sean Lee implemented an algorithm to visualize the type-2 invariant of knots. The idea is given a knot in R^3, one considers the 5-tuples of points on the knot such that they both sit on a round circle in R^3, but also that the circular ordering along the circle vs. the knot is that of a "pentagram". This manifold of circular pentagrams turns out to generically be an oriented 1-dimensional submanifold of C_5K, i.e. the ordered 5-tuples of points on the knot. If one composes with a projection map $C_5K \to C_1K = K$, the degree of this map is the type-2 invariant. The algorithm uses a gradient descent approach. Provided one has a computer with an NVIDIA graphics card with 2000+ cores, and a Web-browser that supports the WEBGPU language, it computes this 1-manifold quite quickly. Several pre-computed examples are included as well. The web app is available here: https://sean564.github.io/top/.

Random meander links
Anastasiia Tsvietkova, Rutgers University, Newark

We suggest a new random model for links based on meander diagrams and graphs. We then prove that trivial links appear with vanishing probability in this model, no link L is obtained with probability 1, and there is a lower bound for the number of non-isotopic knots obtained for a fixed number of crossings. A random meander diagram is obtained through matching pairs of parentheses, a well-studied problem in combinatorics. Hence tools from combinatorics can be used to investigate properties of random links in this model, and, moreover, of the respective 3-manifolds that are link complements in 3-sphere. We use this for exploring geometric properties of a link complement. Specifically, we give expected twist number of a link diagram and use it to bound expected hyperbolic and simplicial volume of random links. The tools from combinatorics that we use include Catalan and Narayana numbers, and Zeilberger's algorithm. This is joint work with Nicholas Owad. Some related papers: [21, 36, 66, 67].

Random Manifolds from Coloring
Chaim Even-Zohar, Technion, Israel

We describe a new model for generating various manifolds and submanifolds in any dimension and codimension, by assigning colors to the vertices of a given combinatorial manifold, and considering the Voronoi regions corresponding to different sets of colors. We discuss several questions about which manifolds can and cannot occur in this model. Joint work with Joel Hass [37].

Finding slice knots by random search
Joel Hass, University of California, Davis

We discuss a method of generating random submanifolds of a triangulated manifold by assigning random colors to vertices. As an application we can search for surfaces in the 4-ball of low genus spanning a given knot in the 3-sphere. We discuss computer expeeriments using this framework. This is joint work with Chaim Even-Zohar [37].

Braids, elliptic curves, and solving the quartic equation.
Peter Huxford, University of Chicago

Let $\mathrm{Conf}_n\mathbb{C}$ denote the unordered configuration space of n distinct points in the complex plane $\mathbb{C}$. I will discuss the problem of classifying the holomorphic maps $\mathrm{Conf}_n\mathbb{C} \to \mathrm{Conf}_m\mathbb{C}$ for various values of m and n. Several famous examples of such maps arise from the solving the quartic equation, and also from torsion points on elliptic curves. In a certain sense these are the only known non-trivial examples. I will report on recent [45] and ongoing joint work with Jeroen Schillewaert on this classification problem, which builds upon results of Chen and Salter [17] and Lin [56].

Hyperbolic knots producing five essential tori after Dehn surgery

Mario Eudave-Muñoz, Universidad Nacional Autónoma de México, México

Some years ago K. Motegi asked if there exist an upper bound on the number of JSJ pieces of manifolds which are obtained by Dehn-surgery on hyperbolic knots in S^3. Another way of formulating this question is asking if there is an upper bound on the number of disjoint non-parallel incompressible tori that can appear after performing Dehn surgery on an hyperbolic knot in S^3. Any such a torus comes from a punctured torus properly embedded in the knot exterior, which fills up to a closed surface after performing Dehn surgery along the slope given by the punctured torus. L. Valdez-Sánchez [75] considered the case of once punctured tori lying in a knot exterior, that is, of Seifert surfaces, and showed that in the exterior of a hyperbolic knot there are at most five disjoint non-parallel incompressible genus one Seifert surfaces and showed an infinite family of knots which realize this bound. More recently, R. Aranda, E. Ramírez-Losada and J-Rodríguez-Viorato [5] have shown that in a hyperbolic knot exterior there are at most six disjoint, non-parallel, nested, incompressible, properly embedded twice-punctured tori. Here we show an infinite family of hyperbolic knots each having a surgery producing a graph manifold which contains five disjoint non-parallel incompressible tori, all of which come from nested twice-punctured tori properly embedded in the knot exterior. The examples are constructed via tangles and double branched covers. For details see [38]. This is a joint work with Masakazu Teragaito.

Right-angled links in thickened surfaces

Rose Kaplan-Kelly, George Mason University

In this talk, we will consider a generalization of alternating links and their complements in thickened surfaces. We will define what it means for such a link to be right-angled generalized completely realizable (RGCR). We will then show that this property is equivalent to the link having two totally geodesic checkerboard surfaces and equivalent to a set of restrictions on the link's alternating projection diagram.

Knots in Self-Avoiding Polygons

Neal Madras, Department of Mathematics and Statistics, York University, Toronto, Canada

This talk is a brief introduction to a lattice model for knots, with a summary of some key results.

An N-step self-avoiding polygon (SAP) is a simple closed curve consisting of N edges of the lattice $\mathbb{Z}^d$ ($d \geq 2$). SAPs in $\mathbb{Z}^3$ serve as a simple model of conformations of ring polymer molecules. See [60] for more about this model. Let p_N be the number of N-step SAPs modulo translation. It is not hard to prove the existence of $\lim_{N\to\infty}(p_N)^{1/N} =: \mu$. That is, $p_N = \mu^{N+o(N)}$.

Let K be a knot type (e.g. trefoil or unknot). Let $p_N[K]$ be the number of N-step SAP's in $\mathbb{Z}^3$ of knot type K. One can prove that for the unknot O, $\mu[O] := \lim_{N\to\infty} p_N[O]^{1/N}$ exists and is strictly less than μ. More generally, for any fixed knot type K, $p_N[K]$ is exponentially smaller than p_N [68, 73]. An important open problem is to prove that for any knot $K \neq O$, the limit $\lim_{N\to\infty} p_N[K]^{1/N}$ exists and equals $\mu[O]$. Simulations and theoretical arguments [65] indicate that $\frac{p_N[K]}{p_N[O]} \asymp N^{f(K)}$ as $N \to \infty$ where $f(K)$ is the number of prime knots in the knot K. This has only been proved for SAP knots in the narrowest possible tube, i.e. $\mathbb{Z} \times [0, 2] \times [0, 1]$, very recently [10].

We also know that the commonest knot type is not exponentially rare [59]. More precisely, for each N, let K_N be the knot type K that maximizes $p_N[K]$. Then $\lim_{N\to\infty} p_N[K_N]^{1/N} = \mu$.

Unknotting Number is NP-hard

Jaeyun Bae, Rutgers University, Newark

We prove that determining the computational complexity of unknotting number of a given knot is NP-hard. 3-SAT problem is known as NP-hard problem and we use Karp reduction or polynomial transformation from 3-SAT problem to unknotting number problem to show NP-hardness of unknotting number. Also, for any $n \in \mathbb{Z}$ we can create a knot whose unknotting number is n. This is joint work with Anastasiia Tsvietkova. Some related papers: [51, 50]

Topological patterns in trivalent trees

Roy Deutch, Technion, Israel

A Trivalent Tree (a.k.a. Unrooted Binary Tree, Phylogenetic Bifurcating Tree, etc.) is an undirected tree such that every internal vertex has a degree three. For a trivalent tree T with n leaves, If one picks a subset of k out of the n leaves of T, considers the minimal subtree that spans these leaves, and suppresses any vertex of degree two, then this gives rise to a tree of order k. These subtrees are called *leaf-induced* or *topological* subtrees. They appear in several applications and algorithmic problems, such as Phylogenetics.

In a large trivalent tree, it is natural to consider the *pattern densities* of small subtrees. For trivalent trees T and S with n leaves and k leaves (respectively) let $\#S(T)$ be the number of subtrees of T that are isomorphic to S, now the *density* of S in T can be defined by $\bar{\#}S(T) = \#S(T)/\binom{n}{k}$. We raise and discuss several extremal and probabilistic questions regarding the pattern densities of topological subtrees.

Polynomial bound on number of flypes for alternating link

Touseef Haider, Rutgers University, Newark Joint work with Anastasiia Tsvietkova.

A toolkit for exploring grid diagrams

Margaret Doig, Creighton University

Preserving topology while pivoting: Sampling a fixed knot-type

Andrew Rechnitzer, University of British Columbia

Ropelength of alternating knots

Yuanan Diao, University of North Carolina at Charlotte

Braid indices of pretzel links

Claust Ernst, Western Kentucky University

Random Borsuk graphs

Francisco Martinez

Trace Fields vs. Triangulations

Kathleen Petersen, University of Minnesota, Duluth

2.3 Lectures

The following one-hour long lectures were given in the first days of the workshop. The lectures had both research and educational components.

Knot theory: Computation and experimentation
Benjamin Burton, University of Queensland, Australia

Knots and computational complexity
Yoav Rieck, University of Arkansas

3 Open Problems

Below are some of the open research problems that were suggested during the open problem session by the workshop participants. More problems can be found in BIRS video of the session, freely available online.

1. (C. Ernst) Confined random polygons

   (a) The probability of non trivial knotting of non confined random equilateral polygons goes to one (exponentially fast) with increasing length of the random polygons [22]. This has to be also true in confinement, however there is no proof of this.

   (b) It is known that with increasing length the typical knot type of an unconfined random equilateral polygon will be that of a composite knot [22]. However, numerical evidence suggests that this is not true of confined random equilateral polygons. What can we prove in this context?

   (c) We know the exact asymptotic (as length goes to infinity) expected value of curvature and torsion for unconfined random equilateral polygons. How does this value depend on the radius of confinement for confined random equilateral polygons?

   (d) The confined random equilateral polygons are generated to their exact probability distribution under the condition that they are rooted at the center of the sphere (that is one of the vertices is the center of the sphere) and the confinement radius is greater or equal to the edge length. Can a rigorous generation process be developed that removes the rooted condition, or that works for radii smaller than the edge length?

2. (U. Ziegler) Is there a more rigorous model for the situation described above? While this question has not been solved: how might we assess how a sample of polygons generated using the proposed model differs from a true sample?

3. (N. Dunfield) For knots in S^3, it is known that deciding if the Seifert genus $g(K)$ of K is at most g_0 is in **NP**$\cap$**co-NP** (see: [4], [52]). This raises the important question of whether this question is in **P**, that is, whether $g(K)$ can be determined in polynomial time in the number of crossings of K. Even if this is not the case, it seems plausible that $g(K)$ can be *generically* computed in polynomial time. This leads to the question of whether there are cheap-to-compute bounds on $g(K)$ that are generically sharp. For

example, for any knot the Alexander polynomial $\Delta_K(t)$ in $\mathbb{Z}[t]$ can be computed in polynomial time and gives:

$$2g(K) \geq \deg(\Delta_K(t))$$

Suppose we pick any reasonable model of random knot (see [35] for a survey). Is it the case that $2g(K) = \deg(\Delta_K(t))$ with probability tending to 1 as the size of the knot goes to infinity?

4. (B. Petri)

Let Γ be the fundamental group of a complete hyperbolic 3-manifold of finite volume (either compact or non-compact). For instance, the fundamental group of the figure eight knot complement: $\Gamma_8 = \pi_1(\mathbb{S}^3 - K_8) \simeq \langle x, y |\ yxy^{-1}xy = xyx^{-1}yx\rangle$. It follows from the fact that Γ is finitely generated that the number

$$s_n(\Gamma) := \#\left\{H < \Gamma;\ [\Gamma : H] = n\right\}$$

of index n subgroups of Γ is finite for all n. However, how this quantity grows as a function of n is unclear:

Question 1: How does $s_n(\Gamma)$ grow as a function of n ?

It does follow from Agol's largeness theorem [3] that the growth of the regularized function $s_{\leq n}(\Gamma) = \#\left\{H < \Gamma;\ [\Gamma : H] \leq n\right\}$ satisfies $(n!)^a \leq s_{\leq n}(\Gamma) \leq (n!)^b$ for some constants $b \geq a > 0$ depending on Γ. However, no effective bounds on these constants are known. In general, this question goes under the name of subgroup growth. For an introduction to the subject, we recommend the book by Lubotzky–Segal [57].

An answer to Question 1 would also allow one to ask finer questions, like:

Question 2: Can the volume $\mathrm{vol}\left(\Gamma\backslash\mathbb{H}^3\right)$ be extracted from the sequence $\left(s_n(\Gamma)\right)_{n\in\mathbb{N}}$?

If the answer to this question is "yes", that would imply that the volume of a hyperbolic manifold is a profinite invariant, which is a question that has been asked before (see for instance [12, Question 3.18] and Liu)

Another natural question one could hope to answer using this is:

Question 3: What is the geometry and topology of a random cover of degree n of $\Gamma\backslash\mathbb{H}^3$ like? For instance:

How do the betti numbers distribute? Does the torsion in homology grow with high probability?

How does the length spectrum behave?

Does the resulting sequence of random manifolds have a uniform spectral gap in its Laplacian with probability tending to 1 as $n \to \infty$?

The answers to the analogues to all of the questions above are known in dimension two [32, 64, 54, 63, 61, 62, 44, 69]. For related results for torus knot complements, right-angled Artin groups, right-angled Coxeter groups and 3-manifold groups, we refer to [9, 6, 7, 39].

The link with permutation representations: In the cases in which counting results are known, this usually goes through counting permutation representations.

To this end, let $\mathfrak{S}_n$ denote the symmetric group on n letters, and set

$$\mathcal{T}_n(\Gamma) = \left\{\varphi \in \mathrm{Hom}(\Gamma, \mathfrak{S}_n);\ \text{the action of } \varphi(\Gamma) \text{ on } \{1, \ldots, n\} \text{ is transitive}\right\},$$

$t_n(\Gamma) = \#\mathcal{T}_n(\Gamma)$ and

$$\mathcal{S}_n(\Gamma) = \{H < \Gamma;\ [\Gamma : H] = n\},$$

so that $s_n(\Gamma) = \#\mathcal{S}_n(\Gamma)$. The map $\mathcal{T}_n(\Gamma) \to \mathcal{S}_n(\Gamma)$ given by $\varphi \mapsto \mathrm{stab}_{\varphi(\Gamma)}\{1\}$ is surjective and $(n-1)!$-to-1. This means in particular that $s_n(\Gamma) = \frac{t_n(\Gamma)}{(n-1)!}$. Moreover, the sequence $t_n(\Gamma)$ can be computed in terms of the sequence $h_n(\Gamma) := \#\mathrm{Hom}(\Gamma, \mathfrak{S}_n), n \in \mathbb{N}$. We refer to [57, Section 1.1] for proofs of these facts.

The upshot of all of this, is that the questions above come down to counting the number of solutions to a given equation in the symmetric group. For instance, in the case of the figure eight knot complement, we can make the identification $\mathrm{Hom}(\Gamma_8, \mathfrak{S}_n) = \left\{\pi_1, \pi_2 \in \mathfrak{S}_n;\ \pi_2\pi_1\pi_2^{-1}\pi_1\pi_2 = \pi_1\pi_2\pi_1^{-1}\pi_2\pi_1\right\}$. So in this case computing $h_n(\Gamma)$ (and hence $t_n(\Gamma)$ and $s_n(\Gamma)$) is the same as computing the number of solutions in $\mathfrak{S}_n$ to a single equation in two variables.

5. (Y. Diao)

   The ropelength conjecture of alternating links

   Conjecture. [31] There exists a positive constant $\alpha > 0$ such that for any alternating link L with minimum crossing number $Cr(L)$, the ropelength of L is bounded below by $\alpha \cdot Cr(L)$. This conjecture has been proved to be true for alternating knots, but remains open for alternating links with two or more components.

Other problems mentioned in the session can be found in the video on BIRS website. These include the complexity of knot equivalence (B. Burton), the complexity of unknotting (Y. Rieck), the complexity of determining if a given link is alternating (A. Tsvietkova), the probability that the a randomly embedded twisted ladder $M_4 \to \mathbb{R}^3$ has Möbius form (E. Flapan), random knotoids and stake knots (C. Adams), and classification of random knots and links models (M. Doig, M. Cohen).

4 Outcome of the Meeting

The meeting brought together researchers from several areas of study who found a common theme in the use of experiments, probabilistic methods and algorithms in the study of links and 3-manifolds with finite volume. There were numerous conversations and discussions between participants that are likely to lead to new collaborations, including interdisciplinary ones, and new scientific results. The workshop participants represented institutions from various geographical regions (including USA, Mexico, Canada, France, Israel, Australia), and were on various career stages (graduate students, postdocs, junior, middle-career and senior tenure-line faculty members, and people working in industry). There was a number of participants from groups underrepresented in mathematics and computer science.

The format of the meeting was developed with an intention to accommodate this diversity and to help all the participants to learn about each other's research. Another intention was to give exposure to many different facets of the chosen topics, and to many different results, including the results by junior participants and participants from underrepresent groups. Hence the workshop included lectures, longer 45-minute research talks, shorter 10-minute research talks, and open problems session. Several participants submitted feedback after the workshop testifying that this was an efficient format. The focus of the workshop was on in-person communication and learning, however a number of participants joined lectures and research talks online, and contributed questions and remarks through the audio/video system.

References

[1] N. A'Campo, Tresses, monodromie et le groupe symplectique, Comm. Math. Helvetici (54) 318-327 (1979)

[2] C. Adams, A. Calderon, N. Mayer, Generalized bipyramids and hyperbolic volumes of alternating k-uniform tiling links, Topology Appl. 271 (2020), 107045

[3] I. Agol, The virtual Haken conjecture, Documenta Mathematica 18, 2013, 1431–0635

[4] I. Agol, J. Hass, W. Thurston, The computational complexity of knot genus and spanning area, Trans. Amer. Math. Soc. 358, 2006, 3821–3850

[5] R. Aranda, E. Ramírez-Losada and J. Rodríguez-Viorato, Bound on the number of twice-punctured tori in a knot exterior, arXiv:2304.09312v1

[6] H. Baik, B. Petri, J. Raimbault, Subgroup growth of right-angled Artin and Coxeter groups, J. Lond. Math. Soc. (2), 2020, 556–588

[7] ______ , Subgroup growth of virtually cyclic right-angled Coxeter groups and their free products, Combinatorica 39, 2019, 779–811

[8] K. L. Baker, K. Motegi, T. Takata, The Strong Slope C onjecture and crossing numbers for Mazur doubles of knots, Low dimensional topology and Number Theory, Springer Proc. Math. Stat., to appear

[9] E. Baker, B. Petri, Statistics of finite degree covers of torus knot complements, Ann. H. Lebesgue 6, 2023, 1213–1257

[10] N. R. Beaton, K. Ishihara, M. Atapour, J. W. Eng, M. Vazquez, K. Shimokawa, and C. E. Soteros (2022), Entanglement statistics of polymers in a lattice tube and unknotting of 4-plats, arXiv:2204.06186

[11] R. Blair, T. D. Eddy, N. Morrison, C. Shonkwiler, Knots with exactly 10 sticks, *J. of Knot Theory and Its Ramifications*, 29(3):2050011, 2020

[12] M. Boileau, S. Friedl, The profinite completion of 3-manifold groups, fiberedness and the Thurston norm, Ann. of Math. Stud. 205, 2020, 21–44

[13] J. Cantarella, C. Shonkwiler, The symplectic geometry of closed equilateral random walks in 3-space, *The Annals of Applied Probability*, 26(1):549–596, 2016

[14] J. Cantarella, H. Schumacher, C. Shonkwiler, A faster direct sampling algorithm for equilateral closed polygons and the probability of knotting, arXiv 2309.10163, 2024

[15] J. Cantarella, H. Schumacher, CoBarS: Fast reweighted sampling for polygon spaces in any dimension, arXiv 2310.19134, 2023

[16] A. Champanerkar, I. Kofman, J. S. Purcell, Geometry of biperiodic alternating links, J. Lond. Math. Soc. (2) 99 (2019), no. 3, 807–830

[17] L. Chen, N. Salter, Holomorphic maps between configuration spaces of Riemann surfaces, arXiv 2301.08333, 2023

[18] J. Cantarella, C. Shonkwiler, The symplectic geometry of closed equilateral random walks in 3-space, Annals of Applied Probability, 26(1):549–596, 2016

[19] M. Cohen, A. DiNardo, A. M. Lowrance, S. Raanes, I. M. Rivera, A. J. Steindl, E. S. Wanebo, *The distribution of genera of 2-bridge knots*, arXiv:2307.09399, 2023

[20] M. Cohen, A. Lowrance, *The average genus of a 2-bridge knot is asymptotically linear*, arXiv:2205.06122, 2022

[21] O. Dasbach, A. Tsvietkova, Simplicial volume of links from link diagrams, Mathematical Proceedings of Cambridge Philosophical Society 166 (2019), no. 1, 75–81

[22] Y. Diao, The Knotting of Equilateral Polygons in R^3, *Journal of Knot Theory and Its Ramifications*, doi: 10.1142/S0218216595000090, 1995

[23] Y. Diao, C. Ernst, A. Montemayor, E. Rawdon, U. Ziegler, The knot spectrum of confined random equilateral polygons. *Molecular Based Mathematical Biology*, 2(1):19–33, 2014.

[24] Y. Diao, C. Ernst, A. Montemayor, U. Ziegler, Generating equilateral random polygons in confinement, *J. Phys. A: Math. Theor*, 44:405202, 2011

[25] ______, Generating equilateral random polygons in confinement II. *J. Phys. A: Math. Theor*, 45:275203, 2012

[26] ______, Generating equilateral random polygons in confinement III, *J. Phys. A: Math. Theor*, 45:465003, 2012

[27] Y. Diao, C. Ernst, E. J. Rawdon, U. Ziegler, Average crossing number and writhe of knotted random polygons in confinement, *Reactive and Functional Polymers*, 131:430–444, 2018
[28] ______, Relative frequencies of alternating and nonalternating prime knots and composite knots in random knot spaces. *Experimental Mathematics*, 27(4):454–471, 2018
[29] ______, Total curvature and total torsion of knotted random polygons in confinement, *J. of Physics A: Mathematical and Theoretical*, 51(15):154002, 2018.
[30] Y. Diao, C. Ernst, U. Ziegler, E. J. Rawdon, The knot spectrum of random knot spaces, In Philipp Reiter, Simon Blatt, and Armin Schikorra, editors, *New Directions in Geometric and Applied Knot Theory*, De Gruyter, Berlin, 2018
[31] Y. Diao, H. Morton, *The Braid Indices of the Reverse Parallel Links of Alternating Knots*, to appear in *Algebraic and Geometric Topology*
[32] J. D. Dixon, Models of random knots, Math. Z. 110, 1969, 199–205
[33] T. D. Eddy, C. Shonkwiler, New stick number bounds from random sampling of confined polygons, *Experimental Mathematics*, 31(4):1373–1395, 2022
[34] C. Ernst, E. J. Rawdon, U. Ziegler, Knotting spectrum of polygonal knots in extreme confinement, *J. Phys. A Math. Theor.*, 54(23):235202, 2021
[35] C. Even-Zohar, Models of random knots, J. Appl. Comput. Topol.1 (2017), no.2, 263–296
[36] C. Even-Zohar, J. Hass, N. Linial, T. Nowik, Universal Knot Diagrams, J. Knot Theory Ramifications 28 (2019), no. 7, 30 pp.
[37] C. Even-Zohar, J. Hass, Random colorings in manifolds, Israel J. of Mathematics 256.1 153-211 (2023)
[38] M. Eudave-Muñoz, M. Teragaito, Hyperbolic knots with a large toroidal surgery, arXiv:2310.10087
[39] S. Friedl, J. Park, B. Petri, Bram, J. Raimbault, A. Ray, Subgroup growth of virtually cyclic right-angled Coxeter groups and their free products, Combinatorica 4, 2019, 779–811
[40] H.-C. Gan, Alternating links with totally geodesic checkerboard surfaces, Trans. Amer. Math. Soc. 373 (2020), no. 4, 2349–2397
[41] S. Garoufalidis, The Jones slopes of a knot, Quantum Topol., Vol. 2, 2011, no. 1, p. 43–69
[42] F. González-Acuña, J. M. Montesinos-Amilibia, On the character variety of group representations in $\mathrm{SL}(2, \mathbf{C})$ and $\mathrm{PSL}(2, \mathbf{C})$. *Math. Z.*, 214(4):627–652, 1993
[43] M. Heusener, J. Porti, The scheme of characters in SL_2, *Trans. Amer. Math. Soc.*, 376(9):6283–6313, 2023
[44] W. Hide, M. Magee, Near optimal spectral gaps for hyperbolic surfaces, Ann. of Math. (2) 198, 2023, 791–824
[45] P. Huxford, J. Schillewaert, Braid groups, elliptic curves, and resolving the quartic, arXiv 2309.12999, 2023
[46] M. Kapovich, J. Millson. The symplectic geometry of polygons in Euclidean space. *J. of Differential Geometry*, 44(3):479–513, 1996
[47] E. Kalfagianni, C. R. S. L. Lee, Jones diameter and crossing number of knots, Adv. Math., Vol. 417, 2023
[48] E. Kalfagianni, R. Mcconkey, Crossing numbers of cable knots, arXiv 2309.03814, 2003
[49] R. Kaplan-Kelly, Right-angled links in thickened surfaces, arXiv:2203.08963, 2022
[50] D. Koenig and A. Tsvietkova, NP-hard problems naturally arising in knot theory, Trans. Amer. Math. Soc. Ser. B, Vol. 8, 2021, 420–441
[51] M. Lackenby, Links with splitting number one, Geom. Dedicata, Vol. 214, 2021, 319–351
[52] ______, The efficient certification of knottedness and Thurston norm, Adv. Math. 387, 2021, 1090–2082
[53] C. Lamm, *Fourier knots*, arXiv:1210.4543, 2012
[54] M. W. Liebeck, A. Shalev, Fuchsian groups, coverings of Riemann surfaces, subgroup growth, random quotients and random walks, J. Algebra 276, 2004, 552–601
[55] W. B. R. Lickorish, An introduction to knot theory, Grad. Texts in Mathematics, Vol. 175, Springer-Verlag, New York, 1997, x+201 pp.
[56] V. Lin, Configuration spaces of $\mathbb{C}$ and $\mathbb{C}P^1$: some analytic properties arXiv 0403120, 2004
[57] A. Lubotzky, D. Segal, Subgroup growth, Progress in Mathematics 212, 2003, 263–296, Birkhauser Verlag, Basel
[58] A. Lubotzky, A. R. Magid, Varieties of representations of finitely generated groups, *Mem. Amer. Math. Soc.*, 58(336):xi+117, 1985
[59] N. Madras (2023), Adsorption of lattice polymers with quenched topologies, *J. Statist. Phys.* **190**, 7

[60] N. Madras, G. Slade, *The Self-Avoiding Walk*, Birkhäuser, 1993
[61] M. Magee, D. Puder, The asymptotic statistics of random covering surfaces, Forum of Mathematics. Pi 11, 2023
[62] M. Magee, F. Naud, D. Puder, A random cover of a compact hyperbolic surface has relative spectral gap $\frac{3}{16} - \varepsilon$, Geom. Funct. Anal. 32, 2022, 595–661
[63] T. W. Mueller, J.-C. Puchta, Character theory of symmetric groups and subgroup growth of surface groups, J. London Math. Soc. (2) 66, 1994, 623–640
[64] A. Nica, On the number of cycles of given length of a free word in several random permutations, Random Structures Algorithms 5, 1994, 703–730
[65] E. Orlandini, M.C. Tesi, E.J. Janse van Rensburg, S.G. Whittington (1998), Asymptotics of knotted lattice polygons, *J. Phys. A: Math. Gen.* **31**, 5953–5967.
[66] N. Owad, Families of not perfectly straight knots, J. Knot Theory Ramifications 28 (2019), no. 3, 1950027, 12 pp.
[67] N. Owad, A. Tsvietkova, Random meander model for links, Discrete & Computational Geometry, published online June 2024, 1–20
[68] N. Pippenger (1989), Knots in random walks, *Discrete Appl. Math.* **25**, 273–278
[69] D. Puder, T. Zimhoni, Local Statistics of Random Permutations from Free Products, IMRN. 5, 2022, 4242–4300
[70] C. Shonkwiler, All prime knots through 10 crossings have superbridge index ≤ 5. *J. of Knot Theory and Its Ramifications*, 31(4):2250023, 2022
[71] ______, New superbridge index calculations from non-minimal realizations. *J. of Knot Theory and Its Ramifications*, 31(10):2250063, 2022
[72] M. Soret, M. Ville, Random walks and the symplectic representation of the braid group, preprint arXiv:2203.00984
[73] D.W. Sumners and S.G. Whittington (1988), Knots in self-avoiding walks, *J. Phys. A: Math. Gen.* **21**, 1689–1694
[74] S. Veeramachaneni, Generating Random Walks and Polygons with Thickness in Confinement, *Masters Theses & Specialist Projects, Paper 1482, https://digitalcommons.wku.edu/theses/1482*, 2015
[75] L. G. Valdez-Sánchez, Seifert surfaces for genus one hyperbolic knots in the 3-sphere, Algebr. Geom. Topol. 19(5) (2019), 2151–2231
[76] M. Ville, Casson invariant of knots defined by random braids, in prep.
[77] A. Xiong, A. J. Taylor, M. R. Dennis, S. G. Whittington, Knot probabilities in equilateral random polygons, J. of Physics A: Mathematical and Theoretical, vol. 54, 2021