Panorama of braided fusion categories.

BIRS Workshop

Skew braces, braids and the Yang-Baxter equation

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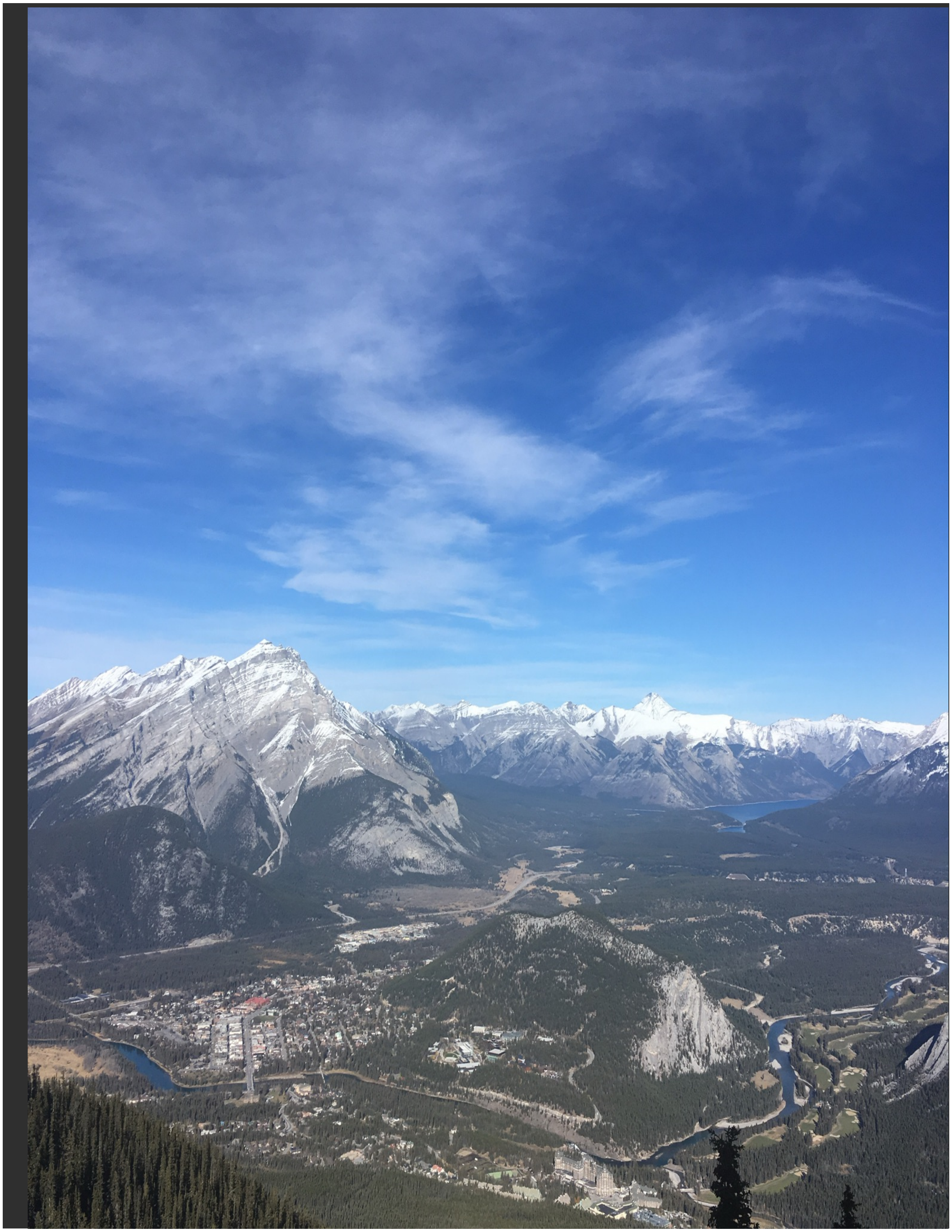

Categories in which you can take direct sums and tensor products

For example, representations of a group G or Hopf algebra H with usual $\oplus$ and $\otimes$, where

G acts on $V \otimes W$ by

$$g \cdot (v \otimes w) = g \cdot v \otimes g \cdot w, \quad g \in G$$

H acts on $V \otimes W$ by

$$h \cdot (v \otimes w) = \sum h_{(1)} \cdot v \otimes h_{(2)} \cdot w,$$

where $h \in H$ and $\Delta h = \sum h_{(1)} \otimes h_{(2)}$ is the comultiplication.

There are generalizations of these structures: weak-, or quasi- Hopf algebras.

Also, there are more involved constructions, e.g. tilting modules over quantum groups.

How can we axiomatize such categories?

It is natural to ask $\otimes$ to be "associative".
But asking that $(u\otimes v)\otimes w = u\otimes(v\otimes w)$ is too restrictive — this fails even for vector spaces.

Instead we ask for natural isomorphism (associativity constraint)

$$a_{u,v,w} : (u\otimes v)\otimes w \xrightarrow{\sim} u\otimes(v\otimes w)$$

satisfying a coherence condition

$$\begin{tikzcd}
& ((u\otimes v)\otimes w)\otimes x \arrow[dl, "a_{u,v,w}"'] \arrow[dr, "a_{u\otimes v,w,x}"] & \\
(u\otimes(v\otimes w))\otimes x \arrow[d, "a_{u,v\otimes w,x}"'] & & (u\otimes v)\otimes(w\otimes x) \arrow[d, "a_{u,v,w\otimes x}"] \\
u\otimes((v\otimes w)\otimes x) \arrow[rr, "a_{v,w,x}"'] & & u\otimes(v\otimes(w\otimes x))
\end{tikzcd}$$

called "pentagon".

There are also isomorphisms involving the unit object 1:

$$l_u : 1\otimes u \xrightarrow{\sim} u, \quad r_u : u\otimes 1 \xrightarrow{\sim} u$$

Can we make $\otimes$ "commutative"?

Yes, we can — but we need to be consistent with replacing $=$ with $\cong$.
In other words: $\cong \neq =$.

So we ask for natural isomorphism:

$$c_{u,v} : u\otimes v \xrightarrow{\sim} v \otimes u$$

(commutativity constraint or braiding):

$$\begin{tikzcd}
 & u\otimes(v\otimes w) \arrow[r, "c_{u,v\otimes w}"] & (v\otimes w)\otimes u \arrow[dr, "a_{v,w,u}"] & \\
(u\otimes v)\otimes w \arrow[ur, "a_{u,v,w}"] \arrow[dr, "c_{u,v}"'] & & & v\otimes(w\otimes u) \\
 & (v\otimes u)\otimes w \arrow[r, "a_{v,u,w}"] & v\otimes(u\otimes w) \arrow[ur, "c_{u,w}"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & (u\otimes v)\otimes w \arrow[r, "c_{u\otimes v,w}"] & w\otimes(u\otimes v) \arrow[dr, "a^{-1}_{w,u,v}"] & \\
u\otimes(v\otimes w) \arrow[ur, "a^{-1}_{u,v,w}"] \arrow[dr, "c_{v,w}"'] & & & (w\otimes u)\otimes v \\
 & u\otimes(w\otimes v) \arrow[r, "a^{-1}_{u,w,v}"] & (u\otimes w)\otimes v \arrow[ur, "c_{u,w}"'] &
\end{tikzcd},$$

called "hexagons".

Linearity, rigidity (existence of duals)

A category equipped only with associativity and braiding is called braided monoidal.

One can also ask for direct sum $\oplus$ making it an Abelian category such that $\otimes$ is biadditive w.r.t. $\oplus$. Also, we want $\mathrm{Hom}(U, V)$ to be a vector space over a field $\mathbb{k}$ and $\otimes$ bilinear on morphisms.

Next, we postulate existence of (left) dual object V^* for each V along with $ev_V : V^* \otimes V \to 1$, $coev_V : 1 \to V \otimes V^*$ (evaluation and coevaluation, satisfying natural relations) and right dual objects this gives a rigid braided tensor category

If, in addition, the category is semisimple with finitely many simples, we call it a braided fusion category:

$$V \otimes U \simeq \bigoplus_W N_{VU}^{W} \, W$$

$\leftarrow$ fusion of V, W produces a bunch of new objects

Relation with Yang-Baxter and braid groups

MacLane's strictness theorem: any monoidal category is equivalent to one with strict associativity:

$$(U\otimes V)\otimes W = U\otimes(V\otimes W), \quad a_{U,V,W} = 1$$

Example (Linear algebra)

Fin-dim vector spaces — $\mathbb{N}$

Linear transformations — matrices

Tensor product — Kronecker product

But, unlike associativity, braiding cannot be strictified.

In a strict category hexagon axioms for braiding give rise to the equation

$$\boxed{\begin{aligned}&(c_{V,W}\otimes id_U)(id_V\otimes c_{U,W})(c_{U,V}\otimes id_W)\\ &\quad = (id_W\otimes c_{V,U})(c_{V,W}\otimes id_U)(id_V\otimes c_{U,W})\end{aligned}}$$

which for $U=V=W$ gives a braid group relation.

But note that $c_{V,V}$ are not linear operators but isomorphisms in a category.

It follows that there are braid group homomorphisms

$$B_n \to \mathrm{Aut}(V^{\otimes n}), \quad n=1,2,\ldots$$

The simplest examples: metric groups

Suppose that a braided fusion category $\mathcal{C}$ is <u>pointed</u>, i.e., all its simple objects are invertible w.r.t. $\otimes$: $X \otimes X^* \simeq \mathbb{1}$.

Then simple objects form an Abelian group A:

$$a \otimes b = ab$$

The associativity and braiding are scalars:

$$\alpha(a,b,c)\, \mathrm{id}_{abc} : (a\otimes b)\otimes c \xrightarrow{\sim} a \otimes (b \otimes c)$$

$$c_{a,b} = \gamma(a,b)\, \mathrm{id}_{ab} : a \otimes b \xrightarrow{\sim} b \otimes a.$$

It follows from axioms that

$$q : A \to \mathbb{k}^\times, \quad q(a) = c_{a,a}$$

is a quadratic form on A, that is,

$$\frac{q(ab)}{q(a)q(b)} = B(a,b), \quad q(a) = q(a^{-1})$$

where $B(a,b) = c_{b,a} c_{a,b} \leftarrow$ the squared braiding

<u>Theorem</u> (Joyal - Street):

There is an equivalence:

$$\left\{ \begin{array}{l} \text{Pointed braided} \\ \text{fusion categories} \end{array} \right\} \rightleftarrows \left\{ \begin{array}{l} \text{Pairs } (A, q) \text{ with} \\ A \text{ an Abelian group} \\ q : A \to \mathbb{k}^\times \text{ a} \\ \text{quadratic form} \end{array} \right\}$$

(Pre-metric groups)

More examples of braided fusion categories

① Rep(G), fin-dim representations of a finite group G — with symmetric braiding

$$c_{U,V} : U \otimes V \xrightarrow{\sim} V \otimes U \; : \; u \otimes v \to v \otimes u$$

② Rep(H, R), where H is a semisimple Hopf algebra and $R \in H \otimes H$ is an R-matrix with the braiding

$$U \otimes V \to V \otimes U \; : \; u \otimes v = R_{21}(v \otimes u)$$

③ Category $\mathcal{C}(\mathfrak{g}, \ell)$ of integrable reps of the affine Lie algebra $\hat{\mathfrak{g}}$ at level ℓ (can also be realized as certain quotients of the rep. categories of $U_q(\mathfrak{g})$ for q a root of unity.

The simplest non-pointed examples of this type are

- The Ising category $\mathcal{C}(sl_2, 2)$ with 3 simple object $1, a, X$, such that $a \otimes a = 1$, $a \otimes X = X$, $X \otimes X = 1 \oplus a$
- The Fibonacci category $\mathcal{C}(sl_2, 3)_+$ with 2 simples $1, X$: $X \otimes X = 1 \oplus X$.

Squared braidings, centralizers, and non-degeneracy

There is a categorical analogue of orthogonality:

We say that $U, V \in \mathcal{C}$ centralize each other if

$$c_{V,U} c_{U,V} = \mathrm{id}_{U \otimes V}.$$

(this notion is due to Müger)

For any fusion subcategory $\mathcal{D} \subset \mathcal{C}$, its centralizer

$$\mathcal{D}' = \{X \in \mathcal{C} \mid c_{V,X} c_{X,V} = \mathrm{id} \text{ for all } V \in \mathcal{D}\}$$

is a fusion subcategory of $\mathcal{D}$ — an analogue of the orthogonal complement

Definition

- $\mathcal{C}$ is symmetric if $\mathcal{C}' = \mathcal{C}$
- $\mathcal{C}$ is non-degenerate if $\mathcal{C}' = \mathrm{Vect}_k$, i.e. the only objects centralizing all others are multiples of 1.

Examples

- $\mathcal{C}(A, q)$ is non-degenerate $\Leftrightarrow$ q is non-degen. We say that (A, q) is metric group
- $\mathcal{C}(\mathfrak{g}, \ell)$ are non-degenerate.

The center construction

One can construct an interesting braided fusion category from any fusion category $\mathcal{C}$ (possibly w/out braiding).

<u>Definition</u> The <u>center</u> of $\mathcal{C}$ is

$$\boxed{Z(\mathcal{C}) = \left\{ (X,\gamma) \;\middle|\; X \in \mathcal{C},\ \gamma = \{\gamma_V : V \otimes X \xrightarrow{\sim} X \otimes V\} \text{ satisfying some coherence} \right\}}$$

It has a natural tensor product

$$(X,\gamma) \otimes (X',\gamma') = (X \otimes X', \tilde{\gamma}), \quad \text{where}$$

$$\tilde{\gamma}_V : V \otimes X \otimes X' \xrightarrow{\gamma_V} X \otimes V \otimes X' \xrightarrow{\gamma'_V} X \otimes X' \otimes V$$

and braiding

$$\gamma'_X : (X,\gamma) \otimes (X',\gamma') \xrightarrow{\sim} (X',\gamma') \otimes (X,\gamma)$$

This braiding is non-degenerate.

<u>Example</u>:

If $\mathcal{C} = \mathrm{Rep}(H)$, where H is a s/s Hopf algebra, then

$$Z(\mathrm{Rep}(H)) = \mathrm{Rep}(D(H), R)$$

the representations of the Drinfeld double of H.

Embedding a braided category into its center.

For a braided category $\mathcal{C}$ let $\mathcal{C}^{rev}$ denote $\mathcal{C}$ equipped with the reversed braiding:

$$c^{rev}_{u,v} = c^{-1}_{v,u}, \qquad u, v \in \mathcal{C}.$$

We have braided embeddings

$$\mathcal{C} \to \mathcal{Z}(\mathcal{C}) : X \mapsto (X, c_{-,X})$$

$$\mathcal{C}^{rev} \to \mathcal{Z}(\mathcal{C}) : X \mapsto (X, c^{-1}_{X,-})$$

So, contrary to a ring theory intuition, a braided category sits inside its center:

$$\boxed{\begin{array}{l}\text{Theorem (Müger)} \\ \mathcal{C} \text{ is non-degenerate} \iff \mathcal{Z}(\mathcal{C}) \simeq \mathcal{C} \boxtimes \mathcal{C}^{rev}\end{array}}$$

More generally, if $\mathcal{D} \subset \mathcal{C}$ with $\mathcal{D}, \mathcal{C}$ non-degenerate, then

$$\mathcal{C} \simeq \mathcal{D} \boxtimes \mathcal{D}'.$$

Localization

If (A, q) is a metric group and $B \subseteq A$ is an <u>isotropic</u> subgroup, i.e., $B \subset B^\perp$ and $q|_B = 1$, then $(B^\perp/B, \tilde{q})$ with $\tilde{q}(x+B) := q(x)$ is again a metric group.

There is a categorical analogue of this construction:

If $\mathcal{C}$ is non-degenerate and $\mathcal{E} \subset \mathcal{B}$ is such that $\mathcal{E} \subset \mathcal{E}'$ and $\mathcal{E} = \mathrm{Rep}(G)$ (i.e. $\mathcal{E}$ is Tannakian), then $\mathcal{C}_{\mathcal{E}}^{loc} = \mathcal{E}' \boxtimes_{\mathcal{E}} \mathrm{Vect}_{\mathbb{k}}$ is a braided non-degenerate category, called a <u>localization</u> of $\mathcal{C}$.

This allows to separate a non-group-theoretical part of $\mathcal{C}$.

There is a construction in the opposite direction, called <u>gauging</u> (more involved — existence is not guaranteed, may be non-unique).

Some problems, recent results, and machinery.

– no particular order, omissions are not intentional.

* The Witt group of non-degenerate braided fusion categories

(Davydov - Müger - N - Ostrik)

Non-degenerate braided fusion categories form a monoid under tensor product $\boxtimes$. Its quotient by the submonoid of centers is a group

$$\boxed{\mathcal{W} := \langle \text{non-deg braided fusion categories} \rangle / \sim}$$

The categorical Witt group

where $\mathcal{C}_1 \sim \mathcal{C}_2$ if $\mathcal{C}_1 \boxtimes \mathcal{Z}(\mathcal{A}_1) \simeq \mathcal{C}_2 \boxtimes \mathcal{Z}(\mathcal{A}_2)$ for some fusion $\mathcal{A}_1, \mathcal{A}_2$.

We know the isomorphism type of $\mathcal{W}$:

$$0 \to \mathbb{Z}_{16} \to \mathcal{W} \to \mathcal{W}_{class} \oplus \mathbb{Z}_2^{\infty} \oplus \mathbb{Z}^{\infty} \to 0,$$

where $\mathcal{W}_{class}$ is the classical Witt group of quadratic forms

[Davydov-N-Ostrik, Rowell-Ng-Wang-Zhang, Johnson-Freyd - Reutter]

The open problem is to describe generators and relations of $\mathcal{W}$.

Is $\mathcal{W}$ generated by $[\mathcal{C}(\mathfrak{g}, \ell)]$?

* Minimal non-degenerate extensions

Given a degenerate braided fusion $\mathcal{C}$ one can try to embed it into a non-degenerate one of minimal possible dimension $\left(= \dim(\mathcal{C}) \dim(\mathcal{C}')\right)$

[Without the minimality assumption, one can embed $\mathcal{C} \hookrightarrow Z(\mathcal{C})$]

Minimal non-deg extensions do not always exist. When they do, there is a bijection

$$\left\{ \begin{array}{l} \text{min. extensions} \\ \text{of } \mathcal{C} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{min. extensions} \\ \text{of } \mathcal{C}' \end{array} \right\}$$

(Lan-Kong-Wen).

A recent theorem of Johnson-Freyd and Reutter says that if $\mathcal{C}' = \mathrm{sVect}_{\mathbb{K}}$ (super-vector spaces) then minimal extensions do exist, and form a torsor over $\mathbb{Z}_{16}$.

* Braid group images

Given a braided category $\mathcal{C}$ one can study the braid group images in $\mathrm{Aut}^{br}(V^{\otimes n})$.

Some questions one can ask:

- When are these finite? A conjecture of Naidu-Rowell is that this finiteness $\iff$ $\dim(\mathcal{C}) \in \mathbb{Z}$.
  This was confirmed by Etingof-Rowell-Witherspoon and Green-Nikshych for (weakly) group-theoretical categories

- What are the images? Which groups can occur as images?
  This goes back to Vaughan Jones who computed them for $\mathrm{Rep}(U_q(\mathfrak{sl}_2))$
  - For $\mathrm{Rep}(D^\omega(G))$, G a p-group [Etingof-Rowell-Witherspoon] showed that the braid group images are p-groups.
  - Symplectic groups over finite fields occur as braid group images [Green]

* Modular group representations

A non-degenerate braided fusion category $\mathcal{C}$ is modular if it has a ribbon element ("twist")

this notion is due to Turaev

$\theta_V : V \xrightarrow{\sim} V$ such that

$\theta_{V\otimes W} \simeq (\theta_V \otimes \theta_W)\, c_{WV} c_{VW}$, $\theta_{V^*} = \theta_V^*$

(One can think of this as an analogue of a quadratic form)

In this case the matrices

$$S = \{ S_{XY} = \mathrm{tr}(c_{YX} c_{XY}) \}_{X,Y \in \mathrm{Irr}(\mathcal{C})}$$

$$T = \{ \theta_X \}_{X \in \mathrm{Irr}(\mathcal{C})}$$

generate a representation of $SL_2(\mathbb{Z})$.

This representation factors through the congruence subgroup $SL_2(\mathbb{Z}/N\mathbb{Z})$ for some N determined by $\mathcal{C}$.

[Sommerhausser - Zhu, Ng - Schauenburg].

...

Ng - Rowell - Wang - Wen used this result to reconstruct modular data from representations of $SL_2(\mathbb{Z}/n\mathbb{Z})$

Topological Quantum Field Theory (TQFT) and invariants of 3-manifolds.

Turaev showed that a modular category gives rise to the following construction:

If M is an oriented 3-mfld with boundary $\partial M = S_1 - S_2$ (a cobordism) then we have a linear operator

$$\tau(M) : \tau(S_1) \to \tau(S_2)$$

which is a functor

$$\left\{ \begin{matrix} \text{closed oriented} \\ \text{2-mflds,} \\ \text{cobordisms} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{Vector spaces,} \\ \text{linear operators} \end{matrix} \right\}$$

In particular, if M is closed, $\tau(\emptyset) = \mathbb{K}$, $\tau(M) \in \mathbb{K}$ – an invariant of M.

Conversely, all such functors come from modular categories.

Ocneanu rigidity and rank finiteness

Fusion categories and functors between them do not admit deformations (this is called Ocneanu rigidity). Thus,

- there are finitely many fusion categories with given fusion rules
- there are finitely many braidings on a given fusion category

[Etingof-N-Ostrik]

The rank of a fusion category C is the number of (iso classes) of simple objects in C.

Bruillard-Ng-Rowell-Wang: there are finitely many non-deg braided fusion categories of any given rank.

This was extended to arbitrary ~~braided~~ fusion categories by Jones-Morrison-N-Rowell

. . .

So it is natural to try to classify braided fusion categories either by dimension or by rank.

* Classifying module categories over braided fusion categories

A module category over C is an Abelian $\mathbb{k}$-linear category $\mathcal{M}$ along with a tensor functor

$$C \longrightarrow \mathrm{End}(\mathcal{M}).$$

complete reducibility!

Rigidity = no deformations

Problem: Classify module categories over a given fusion category C.

- For group-theoretical C this was done by Ostrik
- A very important problem is to classify module categories over $C(\mathfrak{g}, \ell)$:
  - For $\mathfrak{g} = sl_2$ – ADE classification [Capelli-Itzykson-Zuber, Kirillov-Ostrik]
  - $\mathfrak{g} = sl_3$: [Ocneanu, Gannon, Evans-Pugh]
  - $\mathfrak{g} = sl_4$: [Gannon, Copeland-Edie-Michell]
  - Recently, a lot of progress for general sl_N was made by Edie-Michell

(the list above is probably incomplete...

* Tensor product of module cats over a braided category and 2-categorical Picard groups

Module categories over a braided category $\mathcal{C}$ can be tensored — just like modules over a commutative ring:

$$\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \simeq \bigoplus_{\lambda} N^{\lambda}_{\lambda\nu}\, \lambda .$$

In particular, invertible $\mathcal{C}$-module cats form a 2-categorical group $\mathrm{Pic}(\mathcal{C})$, the Picard group of $\mathcal{C}$.

- For non-degenerate $\mathcal{C}$ there is a group isomorphism $\boxed{\mathrm{Pic}(\mathcal{C}) \simeq \mathrm{Aut}^{br}(\mathcal{C})}$.

- Gaugings of $\mathcal{C}$ by a finite group G (i.e. braided categories $\mathcal{B}$ containing $\mathcal{E} = \mathrm{Rep}(G)$ such that $\mathcal{E}' \boxtimes_{\mathcal{E}} \mathrm{Vect} \simeq \mathcal{C}$) are in bijection with monoidal 2-functors $G \to \mathrm{Pic}(\mathcal{C})$

- Computing $\mathrm{Pic}(\mathcal{C})$ even in simplest cases is an interesting largely open problem...