Cohomological Duality in the Local Langlands Correspondence for p-adic Groups

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The Langlands Philosophy and L-functions

- From automorphic representations to Galois representations:

$$\begin{array}{ccc} \pi \in \mathrm{Irr}(G(\mathbb{A}_{\mathbb{Q}})) & \leftrightsquigarrow & \phi \in \mathrm{Irr}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) \\ \downarrow & & \downarrow \\ L(\pi, s) & = & L(\phi, s) \end{array}$$

- The L-function breaks up according to local fields:

$$L(s, \pi) = \prod_{\langle p \rangle \operatorname{Spec} \mathbb{Z}} L(s, \pi|_{\mathbb{Q}_p})$$

with $\mathbb{Q}_p \cong \mathbb{R}$ when $p = \langle 0 \rangle$, and the p-adic numbers otherwise. Indeed,

$$\pi \cong \bigotimes_{\langle p \rangle \operatorname{Spec} \mathbb{Z}} \pi_p$$

Vogan's Conception of the Local Langlands Correspondence (LLC)

- The classical conception: a finite-to-one map

$$\left\{ \begin{array}{c} \text{Smooth, irreducible} \\ \mathbb{C}\text{-representations of} \\ G = G(F) \end{array} \right\} \twoheadrightarrow \left\{ \begin{array}{c} \text{Admissible group} \\ \text{homomorphisms } W_F' \to {}^L G \\ \text{(Langlands Parameters)} \end{array} \right\}$$

- Vogan's reinterpretation: a bijection

$$\left\{ \begin{array}{c} \text{Smooth, irreducible} \\ \mathbb{C}\text{-representations of} \\ G = G(F) \text{ with} \\ \text{central character } \chi \end{array} \right\} \leftrightsquigarrow \left\{ \begin{array}{c} \text{Simple, equivariant,} \\ \text{perverse sheaves in} \\ \mathbf{Per}_{\widehat{G}}(X_\lambda)^\circ \end{array} \right\}$$

- We call the left-hand-side the **spectral side**, and the right-hand-side the **geometric side**.

The Spectral Category

- Recall that the **Bernstein centre** of G is the ring

$$Z(\mathbf{Rep}(G)) := \mathbf{End}(\mathbb{1}_{\mathbf{Rep}(G)}),$$

  the endomorphism algebra of the identity endofunctor on $\mathbf{Rep}(G)$.
- The Bernstein centre acts on any irrep via a central character

$$\chi : Z(\mathbf{Rep}(G)) \to \mathbb{C}$$

- Only finitely many isomorphism classes of irreps share any given χ.
- The spectral category is then

$$\mathbf{Mod}(\mathrm{Ext}^{\bullet}_{G}(\Sigma, \Sigma))$$

  where Σ is the direct sum of a representative from each isomorphism class of these irreducibles.

The Geometric Category

- A restricted Langlands correspondence gives a map $\chi \mapsto \lambda$,

$$\lambda : W_F \longrightarrow {}^L G := \hat{G} \rtimes W_F$$

  is an **infinitesimal parameter**.

- We then define the **Vogan variety**, given by

$$V_\lambda := \{x \in \mathrm{Lie}(\hat{G}) \mid \lambda(w)\, x\, \lambda(w)^{-1} = |w|_F x,\ \forall w \in W_F\}$$

  equipped with an action of the algebraic group

$$H_\lambda := \{g \in \hat{G} \mid \lambda(w)\, g\, \lambda(w)^{-1} = g,\ \forall w \in W_F\}$$

- We then consider the indecomposable Abelian subcategory of H_λ-equivariant perverse sheaves on V_λ whose simple objects are in bijection with the L-packets attached to χ, up to equivalence, and we have

$$\mathbf{Per}_{H_\lambda}(V_\lambda)^\circ \hookrightarrow \mathbf{Per}_{H_\lambda}(V_\lambda) \simeq \mathbf{Per}_{\widehat{G}}(X_\lambda)$$

- We get the equivalent category $\mathbf{Per}_{\widehat{G}}(X_\lambda)$ via the base change

$$X_\lambda := \widehat{G} \times_{H_\lambda} V_\lambda$$

- For any Abelian category $\mathcal{A}$, let $\mathbf{Irr}(\mathcal{A})$ be the set of all isomorphism classes of simple objects.
- Vogan's LLC is then a canonical bijection between the finite sets

$$\mathbf{Irr}(\mathbf{Rep}_\chi(G)) \equiv \mathbf{Irr}(\mathbf{Per}_{\widehat{G}}(X_\lambda))^\circ$$

- That being said, what should we make of the categories themselves?

Generalised Steinberg Representations

- From now on, let G be split semisimple.

Definition

The **generalised Steinberg representations** of G are those irreps given by

$$\sigma_P := \mathrm{Ind}_P^G(\mathbb{1}_{M_P}) / \sum_{P \subsetneq Q} \mathrm{Ind}_Q^G(\mathbb{1}_{M_Q})$$

for a parabolic subgroup $P \subset G$, and associated Levi M_P.

- In particular, they are in bijection with the parabolics of G (after fixing a Borel, up to equivalence).
- These irreps are collected by the central character

$$\chi : Z(\mathbf{Rep}(G)) \to \mathbb{C}; \quad f(x_0, \ldots, x_n) \mapsto f(q^{(n-1)/2}, \ldots, q^{(1-n)/2}).$$

Properties of σ_P

- The irrep σ_T is the usual Steinberg representation, and σ_G is the trivial representation of G.
- They give all isomorphism classes of those irreps π so that the group

$$H^\bullet(G,\pi) = \mathrm{Ext}^\bullet_G(\mathbb{1}_G,\pi)$$

is non-trivial (this is another characterisation of the generalised Steinberg representations).

The Yoneda Algebra for Steinberg Representations

- Let Σ be the direct sum of all generalised Steinberg representations.
- Following [5] and [2], we have

$$\mathrm{Ext}^i_G(\sigma_{P_I}, \sigma_{P_J}) = \begin{cases} \mathbb{C} & \text{if} \quad i = |I \cup J| - |I \cap J| \\ 0 & \textit{otherwise}. \end{cases}$$

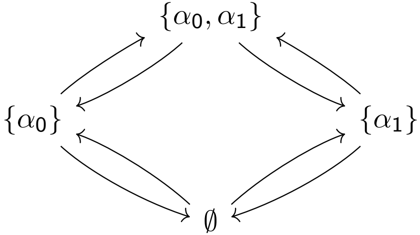
  where P_I is meant to denote the parabolic associated with $I \subset R^+$, where R^+ is the set of positive simple roots associated with G.
- It will often be easier to write $\sigma_I = \sigma_{P_I}$.
- Furthermore, there is the perfect pairing

$$\mathrm{Ext}^i_G(\sigma_I, \sigma_J) \otimes \mathrm{Ext}^j_G(\sigma_J, \sigma_K) \to \mathrm{Ext}^{i+j}_G(\sigma_I, \sigma_K)$$

- This gives the structure of the algebra $\mathrm{Ext}^\bullet_G(\Sigma, \Sigma)$.

Example: $R^+ = \{\alpha_0, \alpha_1\}$

- Now consider the case with only two simple roots $R^+ = \{\alpha_0, \alpha_1\}$.
- In this case, the category $\mathbf{Mod}(\mathrm{Ext}^\bullet_G(\Sigma, \Sigma))$ is equivalent to the representations of the quiver



- Relations: Any non-trivial cycle is equal to zero and all "diagrams commute".
- In general, the quiver will be a double quiver given by a hypercube, with the same relations.

The Vogan Variety for the Steinberg Case

- Assuming that $|R^+| = n - 1$, the associated Vogan variety is given by

$$V_\lambda = \left\{ \begin{pmatrix} 0 & x_1 & 0 & \dots & 0 \\ 0 & 0 & x_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & x_{n-1} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \mid x_i \in \mathbb{C} \right\} \subset \mathrm{Lie}(\hat{G}); \quad H_\lambda \cong \widehat{T}$$

  The action is given on each coordinate of V by $g \cdot x_i = \alpha_i(g) x_i$.

- The H_λ orbits of V_λ are in bijection with subsets of R^+ and of the form

$$C_I \cong C_I^1 \times C_I^2 \times \cdots \times C_I^n$$

  where $C_I^i \cong \{0\}$ if $\alpha_i \in I$ and $C_I^i \cong \mathrm{Spec}\,\mathbb{C}[x]_x$ otherwise.

Simple objects of $\mathbf{Per}_{H_\lambda}(V_\lambda)^\circ$

- These orbits are in bijection with the subsets $I \subset R^+$.
- Thus, the simple objects of $\mathbf{Per}_{H_\lambda}(V_\lambda)^\circ$ are all of the form

$$\mathrm{IC}(\mathbb{1}_{C_I}) := {}^I i_* {}^I j_{!*} \mathbb{1}_{C_I}[\dim C_I] \cong {}^I i_* \mathbb{1}_{\overline{C}_I}[\dim C_I],$$

where

$$C_I \xrightarrow{{}^I j} \overline{C}_I \xrightarrow{{}^I i} V_\lambda$$

and where ${}^I j_{!*} \mathbb{1}_{C_I} \cong \mathbb{1}_{\overline{C}_I}[\dim\ C_I]$ since ${}^I j$ is smooth.

- In particular, the Langlands correspondence is given by the map

$$\sigma \mapsto \mathrm{IC}(\mathbb{1}_{C_I})$$

Calculating extensions between simple objects

- For a pair of subsets $I, J \subset R^+$, which to calculate in $D^b_H(V_\lambda)$

$$\mathrm{Ext}^k_H(\mathrm{IC}(\mathbb{1}_{C_I}), \mathrm{IC}(\mathbb{1}_{C_J})) := \mathrm{Hom}_H({}^I i_* \mathbb{1}_{\overline{C}_I}[d_I], {}^J i_* \mathbb{1}_{\overline{C}_J}[d_J + k]),$$

  where $d_I := \dim C_I$ for any $I \subset R^+$.
- Since there is a fully-faitful forgetful functor $D^b_H(V_\lambda) \to D^b_c(V_\lambda)$, we have

$$\mathrm{Ext}_{V_\lambda}(\mathrm{IC}(\mathbb{1}_{C_I}), \mathrm{IC}(\mathbb{1}_{C_J})) \cong \mathrm{Hom}_{D(V_\lambda)}({}^I i_* \mathbb{1}_{\overline{C}_I}[d_I], {}^J i_* \mathbb{1}_{\overline{C}_J}[d_J + k])$$

  i.e., we can perform the calculation in $D^b_c(V_\lambda)$.

Calculating extensions between simple objects (cont.)

- For any subvarieties $W, Y \subset V$, it is easy to see that

$$\mathbb{1}_W\big|_Y \cong \mathbb{1}_{W\cap Y}$$

  and that $\overline{C}_I \cap \overline{C}_J = \overline{C}_{I\cup J}$.

- Hence, using the adjoint $i^* \dashv i_*$ we have

$$\begin{aligned}\mathrm{Hom}_{D(V_\lambda)}({}^I i_* \mathbb{1}_{\overline{C}_I}[d_I], {}^J i_* \mathbb{1}_{\overline{C}_J}[d_J + k]) \\ &\cong \mathrm{Hom}_{D(\overline{C}_J)}({}^J i^{*I} i_* \mathbb{1}_{\overline{C}_I}[d_I], \mathbb{1}_{\overline{C}_J}[d_J + k]) \\ &\cong \mathrm{Hom}_{D(\overline{C}_J)}({}^{I\cup J} i_* \mathbb{1}_{\overline{C}_{I\cup J}}[d_I], \mathbb{1}_{\overline{C}_J}[d_J + k])\end{aligned}$$

Calculating extensions between simple objects (cont.)

- For any (shifted) local system $\mathcal{L}[k] \in \mathbf{Loc}(C)[k]$, its Verdier dual is given by
$$\mathbb{D}(\mathcal{L}[k]) \cong \mathcal{L}^*[2\dim C - k],$$
and is compatible with the six functor formalism.
- Using Verdier duality in our homset, we get

$$\begin{aligned}
\mathrm{Hom}_{D(\overline{C}_J)}&({}^{I\cup J}i_* \mathbb{1}_{\overline{C}_{I\cup J}}[d_I], \mathbb{1}_{\overline{C}_J}[d_J + k]) \\
&\cong \mathrm{Hom}_{D(\overline{C}_J)}(\mathbb{D}\mathbb{1}_{\overline{C}_J}[d_J + k], \mathbb{D}{}^{I\cup J}i_* \mathbb{1}_{\overline{C}_{I\cup J}}[d_I]) \\
&\cong \mathrm{Hom}_{D(\overline{C}_J)}(\mathbb{1}_{\overline{C}_J}[2d_j - d_j - k], {}^{I\cup J}i_* \mathbb{1}_{\overline{C}_{I\cup J}}[2d_{I\cup J} - d_I]) \\
&= \mathrm{Hom}_{D(\overline{C}_J)}(\mathbb{1}_{\overline{C}_J}[d_j - k], {}^{I\cup J}i_* \mathbb{1}_{\overline{C}_{I\cup J}}[2d_{I\cup J} - d_I])
\end{aligned}$$

Calculating extensions between simple objects (cont.)

- Again using the adjoint $i^* \dashv i_*$, we get

$$\begin{aligned}\mathrm{Hom}_{D(\overline{C}_J)}(\mathbb{1}_{\overline{C}_J}[d_j - k], {}^{I\cup J}i_*\mathbb{1}_{\overline{C}_{I\cup J}}[2d_{I\cup J} - d_I]) \\ \cong \mathrm{Hom}_{D(\overline{C}_{I\cup J})}({}^{I\cup J}i^*\mathbb{1}_{\overline{C}_J}[d_j - k], \mathbb{1}_{\overline{C}_{I\cup J}}[2d_{I\cup J} - d_I]) \\ \cong \mathrm{Hom}_{D(\overline{C}_{I\cup J})}(\mathbb{1}_{\overline{C}_{I\cup J}}[d_j - k], \mathbb{1}_{\overline{C}_{I\cup J}}[2d_{I\cup J} - d_I])\end{aligned}$$

- Thus, we get that $\mathrm{Ext}^n_H(\mathrm{IC}(\mathbb{1}_{C_I}), \mathrm{IC}(\mathbb{1}_{C_J})) = 0$ unless

$$k = d_I + d_J - 2d_{I\cup J}$$

which is easily calculated to be

$$d_I + d_J - 2d_{I\cup J} = |I \cup J| - |I \cap J|$$

Main result

- Thus, by the equivalence $D^b_{H_\lambda}(V_\lambda) \simeq D^b_{\widehat{G}}(X_\lambda)$, we get the following theorem:

Theorem (S.)

Let Σ denote the direct sum of all generalized Steinberg representations σ_I, let X_λ its corresponding Vogan variety, and let $\mathcal{IC}$ the direct sum of all representations of the form $IC(\mathbb{1}_{C_I}) \in \mathbf{Per}_{\widehat{G}}(X_\lambda)$. Then, there is an isomorphism of Yoneda algebras

$$Ext^\bullet_G(\Sigma, \Sigma) \cong Ext^\bullet_{\widehat{G}}(\mathcal{IC}, \mathcal{IC})$$

Complimentary results

- The extensions of perverse sheaves, in fact, gives a full description of the category

$$\mathbf{Mod}(\mathrm{Ext}^{\bullet}_{H_\lambda}(\mathcal{IC},\mathcal{IC})) \simeq \mathbf{Per}_{H_\lambda}(V_\lambda)^{\circ}$$

- The Aubert dual and Fourier transform of $\sigma_I \mapsto \mathrm{IC}(\mathbb{1}_{C_I})$ give

$$\mathrm{Au}(\sigma_I) \cong \sigma_{I^c} \quad \mathrm{Ft}\,(\mathrm{IC}(\mathbb{1}_{C_I})) \cong \mathrm{IC}(\mathbb{1}_{C_{I^c}})$$

  where $I^c = R^+ \setminus I$. The involutions are thus compatible and we have a Cartesian square

$$\begin{array}{ccc} \mathrm{Ext}^{\bullet}_{G}(\Sigma,\Sigma) & \xrightarrow{\sim} & \mathrm{Ext}^{\bullet}_{\widehat{G}}(\mathcal{IC},\mathcal{IC}) \\ \downarrow{\scriptstyle \mathrm{Ft}} & & \downarrow{\scriptstyle \mathrm{Au}} \\ \mathrm{Ext}^{\bullet}_{G}(\mathrm{Ft}\,\Sigma,\mathrm{Ft}\,\Sigma) & \xrightarrow{\sim} & \mathrm{Ext}^{\bullet}_{\widehat{G}}(\mathrm{Au}\,\mathcal{IC},\mathrm{Au}\,\mathcal{IC}) \end{array}$$

Thank you

Thank You!

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Identifying the principle block $\mathbf{Per}_H(V)^\circ$.

- There is a surjective homomorphism of algebraic groups given by

$$f : H \to \mathbb{G}^n; \quad t \mapsto (\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t))$$

- So that the following diagram commutes:

$$\begin{array}{ccc} H \times V & \xrightarrow{f \times 1} & \mathbb{G}_m^{n-1} \times V \\ & {\scriptstyle a} \searrow \quad \swarrow {\scriptstyle m} & \\ & V & \end{array}$$

- Then, if $\mathcal{F} \in \mathbf{Per}_{\mathbb{G}_m^n}(V)$, then there is an isomorphism $\varepsilon : m^*\mathcal{F} \xrightarrow{\sim} p^*\mathcal{F}$. Then, $(f \times 1)^*\varepsilon$ provides and isomorphism $a^*\mathcal{F} \cong p^*\mathcal{F}$, showing that $\mathcal{F} \in \mathbf{Per}_H(V)$.
- Further, this embedding is fully-faithful since both groups are connected, and is Serre since f is affine.
- Set $\mathbf{Per}_H(V)^\circ := \mathbf{Per}_{\mathbb{G}_m^{n-1}}(V)$.