Problems in Lefschetz and Liaison Theory 24rit01

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This was a Research in Teams project, and the team members were Mats Boij, Juan Migliore, Rosa María Miró-Roig and Uwe Nagel.

The proposed problems were too broad in scope for a one-week research stay, so the team members decided to focus on problems in Lefschetz theory and leave for a future meeting the proposed problems in Liaison theory. Indeed, they zeroed in on a very interesting problem which draws from different aspects of algebraic geometry, commutative algebra and homological algebra.

1 Overview of the Field

Let $R = k[x_1, \dots, x_n]$ be the graded polynomial ring, where k is an algebraically closed field of characteristic zero. Let $I \subseteq R$ be a homogeneous ideal and let $A = R/I$ be the quotient algebra. The ring A is said to be *artinian* if it is finite dimensional as a k-vector space. Let $\mathfrak{m}$ be the ideal generated by the variables x_i. The *socle* of A is ideal $(I : \mathfrak{m})/I$ of A. The artinian algebra A is *Gorenstein* if its socle is 1-dimensional, which necessarily occurs in the last degree in which A is nonzero. Since R is a graded ring and I is a homogeneous ideal, A is also a graded ring: $A = \bigoplus_{i \geq 0} A_i$. Each component A_i is a finite dimensional vector space, and since A is artinian, only finitely many of these components are nonzero. The degree in which the last nonzero component occurs is called the *socle degree* of A. An important invariant of A is its *Hilbert function*, defined as $h_A(t) = \dim_k A_t$.

Let ℓ be a general linear form. Then multiplication by ℓ defines a vector space homomorphism from any degree component A_i to the next: $\times\ell : A_i \to A_{i+1}$. The algebra A is said to have the *Weak Lefschetz Property (WLP)* if $\times\ell$ has maximal rank for each i. It has the *Strong Lefschetz Property (SLP)* if $\times\ell^t : A_i \to A_{i+t}$ has maximal rank for each i and each t. Notice that it follows trivially that SLP implies WLP. A useful fact is that any algebra satisfying the WLP has unimodal Hilbert function (see for instance [8] Remark 3.3).

Over the last half century (but especially the last 20 years or so) the Lefschetz properties have received a lot of attention, to the extent that there is no way to do it justice here. We refer to [7] and [9] for (somewhat outdated at this point) overviews. The general question is to determine what properties of an artinian graded algebra A force A to have the WLP or SLP. Types of algebras that have been studied are complete intersections, monomial ideals, ideals generated by powers of linear forms, ideals arising from combinatorics in various ways, etc. But one very important category of algebras that has been studied is the category of artinian Gorenstein algebras. This includes as a subcategory the artinian complete intersections.

To briefly describe the known results for artinian Gorenstein algebras concerning the Lefschetz properties, we will consider the number of variables, n. When $n = 2$, it is a standard fact that the notions of complete intersection and Gorenstein coincide. In fact *all* algebras (Gorenstein or not) have the SLP (hence WLP) [8] (remember that we assume k is algebraically closed and of characteristic zero).

When $n = 3$ it is known that all complete intersections have WLP [8], and some partial results are known for Gorenstein algebras (see section 2).

When $n \geq 4$ is has been known for many years that not all artinian Gorenstein algebras have the WLP. In fact, for $n \geq 5$ the Hilbert functions do not even need to be unimodal. (It is an open question, not directly related to the current project, whether in codimension 4 the Hilbert functions *do* need to be unimodal.) Some partial results for the WLP for artinian complete intersections when $n = 4$ were given by the participants of this team in [4].

In our project the focus was on WLP, although SLP also played a role.

2 Recent Developments and Open Problems

As mentioned above, there are many exciting open problems in the area of Lefschetz theory, and we have barely scratched the surface. However, the part of Lefschetz theory dealing with Gorenstein algebras is one of the most active and important parts.

In [5] it was shown that an artinian Gorenstein algebra ($n = 3$) with Hilbert function $(1, 3, 6, 6, 3, 1)$ has the WLP. The argument involved the interaction of several different areas of mathematics. It was also shown that to prove WLP for *all* artinian Gorenstein algebras with $n = 3$, it is enough to prove a surprisingly simple-looking special case (although it remains open). And it was conjectured that *all* artinian Gorenstein algebras with $n = 3$ have WLP.

When $n = 5$, the "smallest" artinian Gorenstein algebra failing WLP has Hilbert function $(1, 5, 5, 1)$ [6]. So it is natural to turn to $n = 4$ and look for Hilbert functions that force WLP. This has been studied in general [10], but these results do not apply to "small" Hilbert functions and especially not to the specific situation of artinian Gorenstein algebras that we focus on.

Specifically, it is known that the Hilbert function $(1, 4, 7, 7, 4, 1)$ does not necessarily force WLP to hold (see for instance [3]), and it is not hard to show that any artinian Gorenstein algebra with Hilbert function $(1, 4, 5, 5, 4, 1)$ does necessarily have WLP, as does any Hilbert function with $n = 4$ and ending in degree ≤ 4. This leads naturally to the open question whether $(1, 4, 6, 6, 4, 1)$ forces WLP or not. If it does, it can lay claim to being the smallest Hilbert function with $n = 4$ that forces WLP. This latter question arose in conversation at the beginning of our week in Banff, and it became the focus of our research. It proved to be a difficult problem indeed.

3 Presentation Highlights

Being a Research in Teams project rather than a workshop, no presentations were given.

4 Scientific Progress Made

Let $R = k[x_1, x_2, x_3, x_4]$ (with k of characteristic zero and algebraically closed) and let R/I be an artinian Gorenstein algebra with Hilbert function $(1, 4, 6, 6, 4, 1)$. Let ℓ be a general linear form. Our approach to the stated problem centers on the short exact sequence of artinian graded algebras

$$0 \to R/(I : \ell)(-1) \xrightarrow{\times \ell} R/I \to R/(I, \ell) \to 0.$$

We showed that for R/I to fail to have the WLP there is only one possible Hilbert function for $R/(I : \ell)$, namely $(1, 4, 5, 4, 1)$, and only one possible Hilbert function for $(R/(I, \ell)$, namely $(1, 3, 2, 1)$. We needed to show that even these Hilbert functions lead to a contradiction (namely a contradiction to the assumption that A is Gorenstein), so WLP is forced. The idea was to show that even these impose such stringent conditions on the ideals I, $(I : \ell)$ and (I, ℓ) that we necessarily reach an impossibility.

Our main tools were Macaulay's theorem, Green's theorem, and properties of the generic initial ideal of I with respect to the lexicographic order and with respect to the reverse lexicographic order.

Along the way we used Macaulay inverse systems, and made a careful study of the possible Hilbert polynomials of the scheme defined by the degree 2 component of I in $\mathbb{P}^3$. We also used very recent work of Abdallah [1] and others.

By the end of the week we did not carefully eliminate all obstacles, but it seems that we see a way to the finish line.

We also plan to make a related study of the possible Betti diagrams of our artinian Gorenstein algebras, following work of Abdallah and Schenck [2]. We did not discuss this aspect very deeply while we were at Banff, so we will see if it is a feasible addition to the project or not.

5 Outcome of the Meeting

As noted above, we believe that we have all the strings in our hands, and we have only to tie together loose ends. A preprint has already been started on Overleaf, and the team members expect it to be finished in the next few months. The idea is to bring together all of the facts that we have been able to establish, then we will see what is missing, and thus complete the proof based on the ideas we carefully analyzed during our stay in Banff.

The four of us feel that the week was very productive. We are very grateful to the Banff scientific committee for giving us the opportunity to carry out our research in this beautiful and inspirational setting, and to the many BIRS staff members and employees at all levels at the Banff Centre, for making our stay so enjoyable.

References

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