Edge Augmentation Beyond Uncrossable Families

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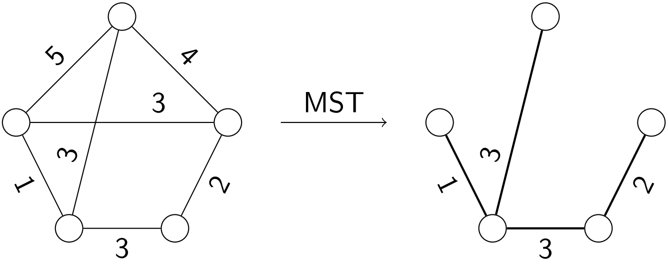
Network Design

Given a network with edge costs, find a cheapest subgraph satisfying given connectivity requirements.

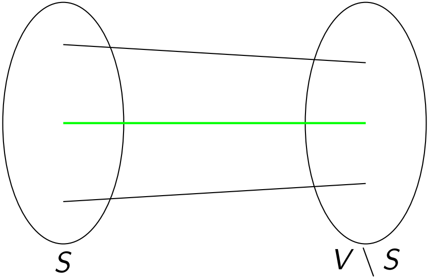
- Uniform edge connectivity,
- Survivable network design,
- Capacitated network design,
- Flexible graph connectivity...



Edge Augmentation: A subroutine

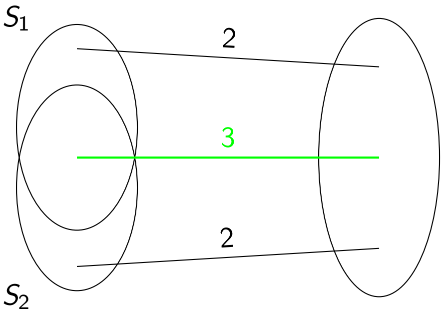
Given a family of cuts $\mathcal{F}$, find a cheapest subgraph that covers every cut i.e. $F \cap \delta(S) \neq \emptyset$ for all $S \in \mathcal{F}$.

- Tree Augmentation,
- Cactus Augmentation,
- Steiner Tree Augmentation,
- Matching Augmentation...



Edge augmentation needs structure

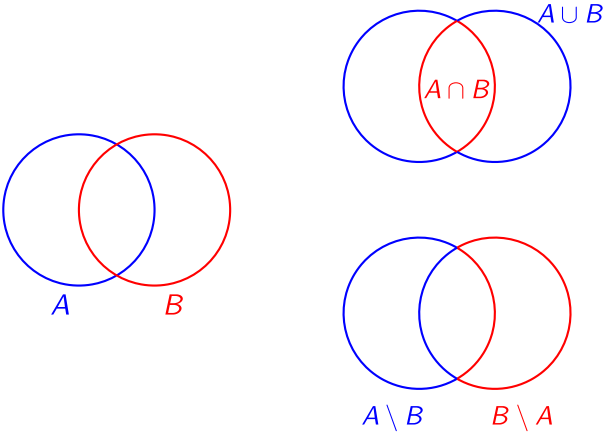
- Generalizes set cover.
- $O(\log |E|)$ hardness of approximation algorithm in general.
- Can do better if $\mathcal{F}$ has structure.
- Williamson, Goemans, Mihail, Vazirani (WGMV) in 1995 considered families $\mathcal{F}$ that are *uncrossable* and provided a 2-approximation algorithm.



Uncrossable Families

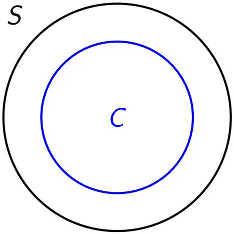
- A family of sets $\mathcal{F} \subseteq 2^V$ is called *uncrossable* if

$$A, B \in \mathcal{F} \implies (A \cup B \in \mathcal{F} \text{ AND } A \cap B \in \mathcal{F}) \textbf{ OR } (A \setminus B \in \mathcal{F} \text{ AND } B \setminus A \in \mathcal{F})$$

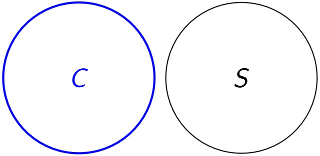


Augmentation of Uncrossable Families

- Williamson et al. provided a 2-approximation primal-dual algorithm for augmenting uncrossable families.
- They observed and leveraged two key properties:
  i) **Non-crossing minimal sets:** Any inclusion-wise minimal set in $\mathcal{F}$ does not "cross" any other set in $\mathcal{F}$.



$C \subseteq S$



$C \cap S = \emptyset$

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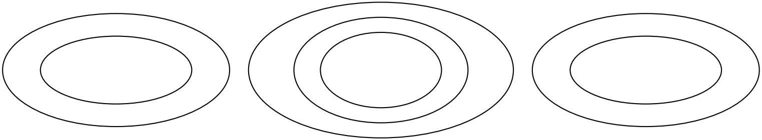
  - i) Non-crossing minimal sets
  - ii) **Dual laminarity:** There exists an optimal solution y^* to the dual linear program such that the sets S with $y^*_S > 0$ form a laminar family (no pair of sets cross).



A Laminar Family

Augmentation of Uncrossable Families

- Williamson et al. provided a 2-approximation.
- They observed and leveraged two key properties:
  i) Non-crossing minimal sets
  ii) Dual Laminarity
- For many years, it was believed that uncrossability and dual laminarity are essential and almost all problems in network design with $O(1)$-approximations use uncrossability and/or dual laminarity.
- Challenging open question whether $O(1)$-approximations can be obtained for families that are not uncrossable.

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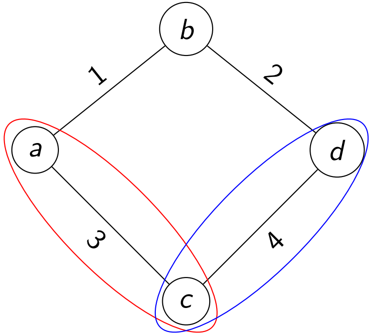
Generalizing Uncrossable Families

- There exist interesting and naturally arising families of cuts that are not captured by uncrossable families.
- For example near min-cuts: For a network with capacities on the edges and a threshold $\alpha \geq 0$, the family of cuts with total capacity at most α is not uncrossable.

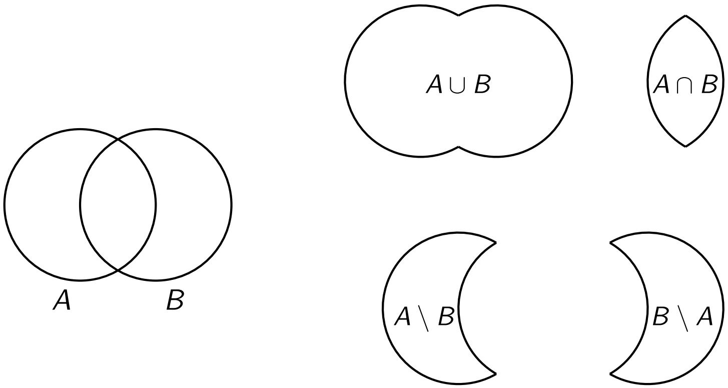


$\alpha = 5$

Pliable Families

A family of sets $\mathcal{F}$ is called *pliable* if

$$A,B \in \mathcal{F} \text{ implies at least two of the four sets } \{A \cup B, A \cap B, A \setminus B, B \setminus A\} \text{ also lie in } \mathcal{F}$$



Primal-Dual Method for Edge Augmentation

Primal LP

$$\min \sum_{e \in E} c_e x_e$$

$$\text{subject to: } \sum_{e \in \delta(S)} x_e \geq 1 \quad \forall S \in \mathcal{F}$$

$$x_e \geq 0$$

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Dual LP

$$\max \sum_{S \in \mathcal{F}} y_S$$

$$\text{subject to: } \sum_{S \in \mathcal{F}: e \in \delta(S)} y_S \leq c_e \quad \forall e \in E$$

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Phase 1: Starting from the empty set of edges,

- Increase uniformly the dual variables corresponding to the minimal sets of $\mathcal{F}$,
- Add edges to solution when dual constraint becomes tight,
- Repeat until feasible.

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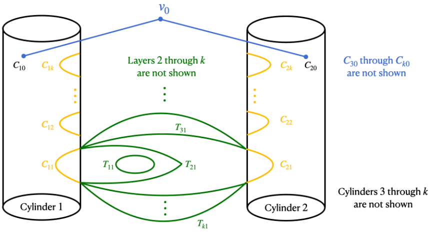
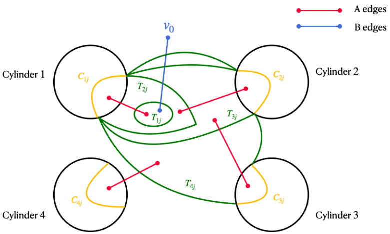
- Increase uniformly the dual variables corresponding to the minimal sets of $\mathcal{F}$,
- Add edges to solution when dual constraint becomes tight,
- Repeat until feasible.

Phase 2: In reverse order of edge additions, delete edges that are not required.

Primal-Dual Method for Pliable Families

- Does the primal-dual method for edge augmentation work for pliable families? **No**
- We show a counterexample where the primal-dual method provides a solution that is a factor $\Omega(\sqrt{|V|})$ worse than the optimal solution.
- A major issue seems to be that minimal sets of $\mathcal{F}$ start to cross other sets in $\mathcal{F}$.





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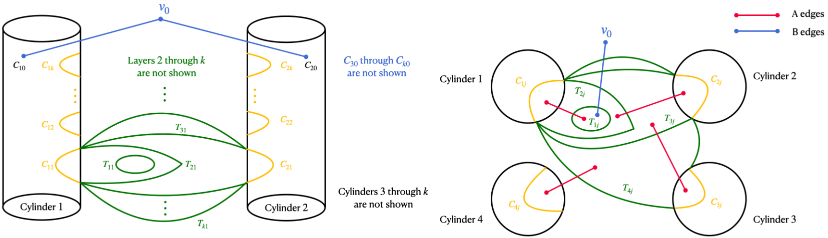
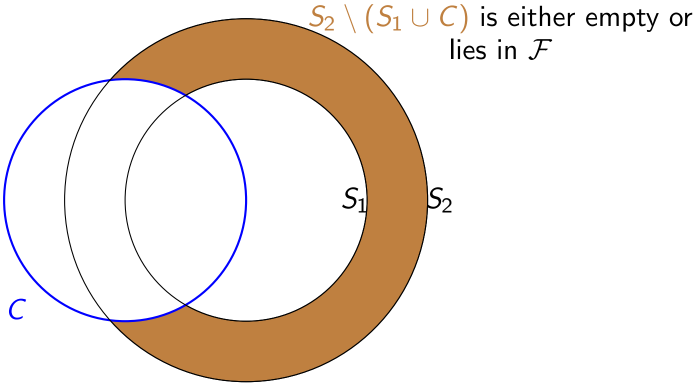


- Turns out pliable families that we know of (like near min-cuts) have an additional property that can be leveraged.

Property Gamma

The number of crossings between minimal sets of $\mathcal{F}$ and other sets of $\mathcal{F}$ is proportional to the total number of minimal sets of $\mathcal{F}$.

More formally,



Pliable Families with Property Gamma

Theorem (B., Cheriyan, Grout, Ibrahimpur)

The primal-dual algorithm for edge augmentation is a 16-approximation algorithm on pliable families satisfying property gamma.

- Pliable families satisfying property gamma need not have the two key properties that are typically used to obtain $O(1)$-approximation algorithms for network design problems:
  i) Non-crossing minimal sets.
  ii) Dual laminarity.

Proof Sketch

- Goal: In every iteration of the primal-dual, the average degree of minimal sets is bounded in our final solution F.

Primal LP

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e \\ \text{subject to:} \quad & \sum_{e \in \delta(S)} x_e \geq 1 \quad \forall S \in \mathcal{F} \\ & x_e \geq 0 \end{aligned}$$

Dual LP

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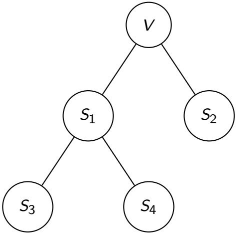
Witness Sets

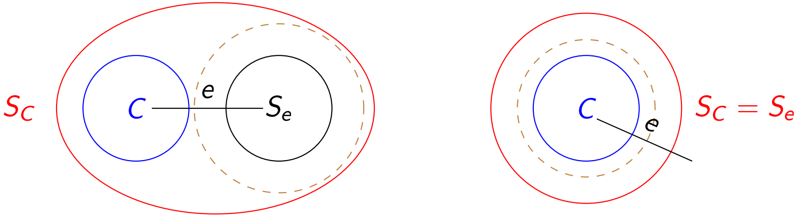
For every edge e incident to a minimal set, assign a witness set S_e such that

- $S_e \in \mathcal{F}$
- $\delta(S_e) \cap F = \{e\}$
- The family of witness sets $\{S_e\}$ is laminar



Bounding Average Degree: Uncrossable Case

- To every minimal set C, assign the smallest witness set S_C that contains it.
- For an edge $e \in \delta(C)$, the witness set S_e is either S_C or a child of S_C.



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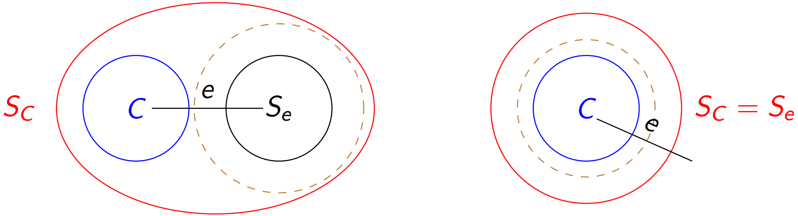
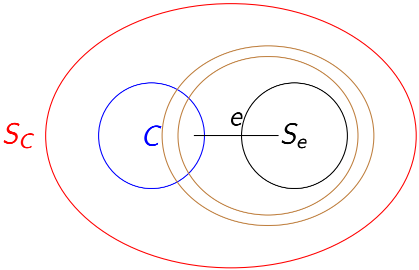
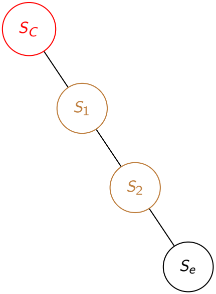


- The degree of C is paid for by the degree of S_C in the witness tree.
- Handshaking lemma shows that the average degree of minimal sets is at most 2.

Bounding Average Degree: Pliable Case

- For an edge $e \in \delta(C)$, the witness set S_e could be a distant descendant of S_C.





- Hence, primal-dual does not work for pliable families in general.

Bounding Average Degree: Property Gamma

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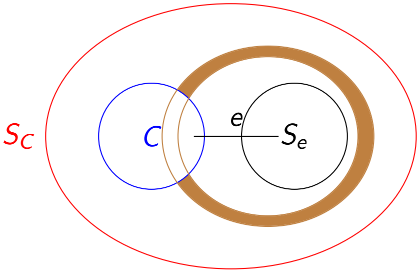

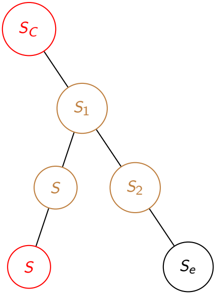

- Property gamma ensures that the shaded region contains a minimal set.
- This minimal set can 'pay' for the degree of set C.

Applications: Flexible Graph Connectivity

- Introduced by Adjiashvili, Hommelsheim, Mühlenthaler (2022).
- Given a graph with edge costs and a partition of the edge set into safe and unsafe edges, find a cheapest p-edge connected subgraph tolerant to q unsafe edge failures.
- When $q = 2$,
  i) Boyd, Cheriyan, Haddadan, Ibrahimpur (2023) showed an $O(\log |V|)$ approximation algorithm (essentially set cover).
  ii) Chekuri, Jain (2023) showed an $O(p)$ approximation algorithm (problem splits into augmenting p uncrossable families)

Theorem (B., Cheriyan, Grout, Ibrahimpur)

The $(p, 2)$-flexible graph connectivty problem admits a 20-approximation algorithm.

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The $(p, 2)$-flexible graph connectivty problem admits a 20-approximation algorithm.

- Nutov (2023) improved the factor to $7 + \epsilon$.
- We showed that a constant factor can be obtained for $(p, 3)$-flexible graph connectivity as well.

Applications: Capacitated Edge Connectivity

Theorem (B., Cheriyan, Grout, Ibrahimpur)

The problem of augmenting near-min cuts of a graph (with arbitrary thresholds) admits a 16-approximation algorithm.

- Given a graph with edge costs and capacities, find a cheapest k-edge connected subgraph (with capacities).
- Goemans et al. (1994) provided a $2k$-approximation algorithm.
- Boyd et al. (2023) provided a k-approximation algorithm.

Theorem (B., Cheriyan, Grout, Ibrahimpur)

The capacitated edge connectivity problem admits an $O(k/u_{min})$ approximation algorithm where u_{min} is the minimum capacity of an edge.

Takeaways and Open Questions

- The folklore belief for around 28 years that uncrossability is essential for primal-dual algorithms to work in the context of network design is not true!
- Constant factor approximations can be obtained for network design problems even when laminar supported dual optimal solutions do not exist!

Takeaways and Open Questions

- The folklore belief for around 28 years that uncrossability is essential for primal-dual algorithms to work in the context of network design is not true!
- Constant factor approximations can be obtained for network design problems even when laminar supported dual optimal solutions do not exist!
- Can we get rid of property gamma using techniques other than primal dual?
- Is there an exact characterization of a property that is necessary and sufficient for primal-dual to work on pliable families?
- Can we cover a general pliable family using $O(1)$ pliable families satisfying property gamma?