Polynomial and Rational Convexity

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Definitions

Definition

If $K \subset \mathbb{C}^n$ is compact, the *polynomially convex hull* of K is the set

$$\widehat{K} = \{z \in \mathbb{C}^n : |P(z)| \leq \max_{w \in K} |P(w)| \text{ for any polynomial } P\}.$$

K is called *polynomially convex* if $K = \widehat{K}$. Similarly, the *rationally convex hull* of a compact K is

$$\mathcal{R}\text{-hull}(K) = \{z : |R(z)| \leq \max_{w \in K} |R(w)|, \ \forall R \text{ rational, poles off } K\}.$$

K is *rationally convex* if $K = \mathcal{R}\text{-hull}(K)$.

Polynomial convexity is holomorphic convexity with respect to $\mathbb{C}^n$, i.e., $\widehat{K} = \widehat{K}_{\mathcal{O}(\mathbb{C}^n)}$ for all compacts $K \subset \mathbb{C}^n$.

Further, $K \subset \mathcal{R}\text{-hull}(K) \subset \widehat{K}$, and $\widehat{K}$ and $\mathcal{R}\text{-hull}(K)$ are compact for any compact K.

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Example 1

Some examples:
(1) A compact set $K \subset \mathbb{C}$ is polynomially convex iff $\mathbb{C} \setminus K$ is connected. Any compact $K \subset \mathbb{C}$ is rationally convex.

In higher dimensions Forstneric (Michigan Math. J.,1993) proved the following: if $K \subset \mathbb{C}^n$ is polynomially convex, $n \geq 1$, then for $j = 0, 1, \ldots, n-1$ we have $\pi_j(\mathbb{C}^n \setminus K) = 0$. In particular, $H_j(\mathbb{C}^n \setminus K, \mathbb{Z}) = 0$.

In general, rationally or polynomially convex compacts cannot be identified by topological properties only. For example, Izzo (Proc. of AMS, 2019) proved the following: *let $K \subset \mathbb{R}^n$ be an arbitrary uncountable compact, then there exists $X \subset \mathbb{C}^{n+4}$ homeomorphic to K such that X has nontrivial polynomially and rationally convex hulls; the inclusion $K \subset \mathbb{R}^n \subset \mathbb{C}^n$ is polynomially convex.*

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Example 2

(2) Let M be a compact real submanifold of $\mathbb{C}^n$, and $f : \mathbb{D} \to \mathbb{C}^n$ be a holomorphic map, $f \in C^0(\overline{\mathbb{D}})$, and $f(b\mathbb{D}) \subset M$ (i.e., f is a holomorphic disc attached to M). Then $f(\mathbb{D}) \subset \widehat{M}$.

One may wonder if the hull always contains a holomorphic disc, this is known as "analytic structure" in the hull. For example, this trivially holds for compacts in $\mathbb{C}$. Also, if $\gamma \subset \mathbb{C}^n$ is a closed rectifiable curve, then either γ is polynomially convex, or $\widehat{\gamma} \setminus \gamma$ is a purely one-dimensional subvariety of $\mathbb{C}^n \setminus \gamma$. (Any rectifiable closed curve in $\mathbb{C}^n$ is rationally convex.)

Further, the following was proved by Alexander (Michigan Math. J. 1977), Basener (Pros. of AMS, 1975), and Sibony (LNM 512, 1976): *If $K \subset \mathbb{C}^n$ is such that $\widehat{K} \setminus K$ has finite two-dimensional Hausdorff measure, then $\widehat{K} \setminus K$ is a one-dimensional subvariety of $\mathbb{C}^n \setminus K$.* (However, $\widehat{K}$ may contain nonempty interior in $\mathbb{C}^n$ even for a Cantor set K.)

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Example 2 cont'd

But in general the answer is NO. Stolzenberg (J. Math. Mech., 1963) was the first to construct a counterexample: there exists a compact $K \subset \mathbb{C}^2$ such that $\widehat{K} \setminus K \neq \varnothing$ but contains no holomorphic disc. By now there exist dozens of various examples of this phenomenon, see, e.g., a survey by Levenberg (Kyoto 1996).

Nevertheless, polynomially convex hulls exhibit some "attributes" of an analytic structure, for example, Rossi's Maximum Principle:

Theorem (Rossi, Ann. of Math., 1960)

Let $p \in \widehat{K} \setminus K$, and let V be a relatively compact neighbourhood of p that does not intersect K. Then for any holomorphic f, $|f(p)| \leq \sup\{|f(w)| : w \in \widehat{K} \cap bV\}$.

In general, $f(\mathbb{D})$ is not necessarily part of $\mathcal{R}$-hull(K), e.g., $S^1 \times S^1$ is rationally convex. But if K is a surface and $f(b\mathbb{D})$ bounds a domain in K, then $f(\mathbb{D}) \subset \mathcal{R}$-hull$(K)$.

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Example 3

(3) If K is any compact subset of $\mathbb{R}^n_x$, $z \in \mathbb{C}$, $z = x + iy$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n = \mathbb{R}^n_x$, then K is rationally and polynomially convex, and $\mathcal{R}(K) = \mathcal{P}(K) = \mathcal{C}(K)$.

$\mathbb{R}^n_x$ is an example of a totally real manifold (Recall that a real submanifold $M \subset \mathbb{C}^n$ is called *totally real*, if for any $p \in M$, the linear space T_pM, viewed as a linear subspace of $\mathbb{C}^n$, contains no complex linear subspaces of positive dimension).

One may wonder if the above statement holds for all compacts in totally real submanifolds. This is false: there are examples of totally real discs that are not rationally convex, or that are rationally convex but not polynomially convex. Also note that no smooth manifold $M \subset \mathbb{C}^n$ with $\dim_{\mathbb{R}} M > n$, is polynomially or rationally convex (such M always contains an open subset which is a CR manifold of positive CR-dimension).

We will discuss convexity properties of totally real submanifolds of $\mathbb{C}^n$ later in the talk.

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Characterizations of convexity 1

Formal definitions of polynomial and rational convexity are difficult to verify for a given compact $K \subset \mathbb{C}^n$, but several characterizations exist.

K is rationally convex iff for any $p \in \mathbb{C}^n \setminus K$ there exists a complex hypersurface $Z \subset \mathbb{C}^n$ such that $p \in Z$, but $Z \cap K = \varnothing$. The proof is elementary.

Analogous statement for polynomial convexity is more complicated and is known as Oka's Characterization.

Theorem (Oka's characterization of polynomial convexity)

A point $p \notin \widehat{K}$ iff there exists a continuous family $\{V_t\}_{t\in[0,1)}$ of complex hypersurfaces in $\mathbb{C}^n$ such that $p \in V_0$, $V_t \cap K = \varnothing$ for all t, and $\{V_t\}$ diverges to infinity as $t \to 1^-$.

Other characterizations also exist. A pseudoconvex domain $\Omega \subset \mathbb{C}^n$ is called a *Runge domain* if polynomials (or entire functions) are dense in $\mathcal{O}(\Omega)$.

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Theorem (Poletsky, Indiana Univ. Math. J., 1993)

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Here σ is normalized Lebesgue measure on $b\mathbb{D}$. The maps f_ϵ are known as *Poletsky discs*.

Theorem (Duval-Sibony, Duke Math. J., 1995)

For a compact $K \subset \mathbb{C}^n$, the following are equivalent: (a) $p \in \widehat{K}$; (b) there exists a positive $(1,1)$-current T in $\mathbb{C}^n$ such that $dd^c T = \mu - \delta_x$, for a probability measure μ supported in K.

Wold (J. Geom. Anal., 2011) showed that Poletsky discs can be used to construct the Duval-Sibony current T. Existence of T in the hull perhaps explains Rossi's Maximum Principle.

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Unions of balls

Connected components of polynomially convex compacts are polynomially convex. But the unions of polynomially convex compacts are surprisingly difficult to analyze.

Theorem (Kallin, 1964)

The union of three mutually disjoint closed balls in $\mathbb{C}^n$ is polynomially convex.

Whether any disjoint union of 4 or more balls is polynomially convex is an open problem!!! This is false for unions of convex compacts or polydiscs. The corresponding problem of rational convexity of unions of balls was resolved by Nemirovski:

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Hulls and PSH functions

Underneath any complex-analytic object there are psh functions!

Definition

Given a compact $K \subset \mathbb{C}^n$, the hull of K with respect to the family of plurisubharmonic functions is

$$\text{psh-hull}(K) = \{p : u(z) \le \sup_K u, \text{ for all psh functions on } \mathbb{C}^n\}.$$

Similarly, this hull can be defined on any domain of $\mathbb{C}^n$ or on complex manifolds that admit nonconstant psh functions.

In the definition above psh functions can be taken to be continuous. The fundamental connection with polynomial convexity is the following

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This shows, in particular, that the notion of polynomial convexity can be generalized to manifolds where there are no holomorphic polynomials, but there are psh functions. For example, this can be done on an almost complex manifold (M, J): un upper semicontinuous function is called J-psh if its composition with any J-holomorphic disc $f : \mathbb{D} \to M$ is subharmonic on $\mathbb{D}$. Rosay (Michigan Math. J., 2006) proved that the hulls with respect to J-psh functions satisfy Rossi's Maximum Principle.

Returning to $\mathbb{C}^n$, we have the following

Theorem (Stout, Polynomial Convexity, 2006)

If $K \subset \mathbb{C}^n$ is polynomially convex, then there exists a nonnegative psh function v on $\mathbb{C}^n$ with $\lim_{z\to\infty} v(z) = \infty$ and $v^{-1}(0) = K$. Further, v can be chosen to be $\mathbb{C}^\infty$ smooth and spsh on $\mathbb{C}^n \setminus K$. Conversely, if v is a nonnegative psh function on $\mathbb{C}^n$ such that $\lim_{z\to\infty} = \infty$, then $\{v = 0\}$ is polynomially convex.

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If $K \subset \mathbb{C}^n$ is polynomially convex, then there exists a nonnegative psh function v on $\mathbb{C}^n$ with $\lim_{z\to\infty} v(z) = \infty$ and $v^{-1}(0) = K$. Further, v can be chosen to be $\mathbb{C}^\infty$ smooth and spsh on $\mathbb{C}^n \setminus K$. Conversely, if v is a nonnegative psh function on $\mathbb{C}^n$ such that $lim_{z\to\infty} = \infty$, then $\{v = 0\}$ is polynomially convex.

Hulls and PSH functions 3

What about psh functions and rational convexity?

Theorem (Duval-Sibony, Duke Math. J., 1995)

If $K \subset \mathbb{C}^n$ is a rationally convex compact, then there exists a smooth closed $(1,1)$ form ω_K which vanishes on K and is positive on $\mathbb{C}^n \setminus K$. The potential of ω_K is a psh function.

Theorem (Duval-Sibony, Duke Math. J., 1995)

Let M be a smooth compact totally real submanifold of $\mathbb{C}^n$. Then M is rationally convex iff there exists a Kähler form ω such that M is isotropic (Lagrangian) with respect to ω.

One direction can be derived by observing that ω can be obtained by "gluing" ω_K as above with the dd^c of the square-distance function to M (which is an spsh function on a neighbourhood of M). The other direction is a fairly technical construction of complex hypersurfaces in the complement of M using Hörmander's L^2-methods and solvability of $\overline{\partial}$.

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Duval-Sibony: examples

Another characterization of rational convexity that stems from the Duval-Sibony was explicitly formulated by Nemirovski.

Theorem (Duval-Sibony; Nemirovski, Russian Math. Surveys 2008)

Let ϕ be a strictly plurisubharmonic function on an open subset $U \subset \mathbb{C}^n$, such that $dd^c\phi$ extends to a positive d-closed $(1,1)$-form on the whole $\mathbb{C}^n$. If the set $\{z \in U : \phi(z) \le 0\}$ is compact, then it is rationally convex.

Examples:

1) $T^n = S^1 \times \cdots \times S^1$ is Lagrangian in $\mathbb{C}^n$ and so is rationally convex.
2) Let $S \subset \mathbb{C}^3$ be a totally real embedded 3-sphere. Then S is not rationally convex. Proof: if it were, then S would be Lagrangian by Duval-Sibony, and by Gromov it would admit a nontrivial holomorphic disc attached to S. But the boundary of the disc bounds a domain in S, and so the whole disc has to be in the rationally convex hull of S. Contradiction.

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Generalizations of Duval-Sibony 2

So rational convexity appears to be related to the values of the Levi form dd^c of psh functions rather than the values of the functions themselves.

Several generalizations of Duval-Sibony exist:

* Gayet (Ann. Sci. Ecole Norm. Sup., 2000): Lagrangian immersions with transverse double self-intersection points are rationally convex. However, rationally convex immersions are not necessarily Lagrangian: there exists a union of two totally real planes in $\mathbb{C}^2$ intersecting at the origin that are rationally convex when intersected with any large ball, but the union is not Lagrangian with respect to any Kähler form in $\mathbb{C}^2$, see Mitrea (J. Geom. Anal., 2020).
* Guedj (Math Ann., 2000): Closed isotropic embeddings into projective manifolds.
* Auroux, Gayet, Mohsen (Math. Ann., 2001): Closed Kähler manifolds – rational convexity is proved using Donaldson's method (J. Diff. Geom., 1996).

Generalizations of Duval-Sibony 2

* Mitrea (J. Geom. Anal., 2020): Rational convexity of immersions in terms of Kähler forms that are degenerate near singular points.
* Boudreaux - RS (Bull. London Math Soc., 2023) Rational convexity of totally real sets (zeros of spsh functions).

Conjecture: An immersed totally real submanifold $M \subset \mathbb{C}^n$ is rationally convex if and only if there exists a psh function ϕ on $\mathbb{C}^n$ such that $dd^c\phi > 0$ outside the self-intersections of M, and M is isotropic (Lagrangian) with respect to $dd^c\phi$.

Convex embeddings

A natural question that arises: what are the (closed) smooth manifolds that admit rationally convex (or polynomially convex) embeddings into $\mathbb{C}^n$?

Theorem (Browder, 1961; Duchamp-Stout, 1981)

No topological closed submanifold of $\mathbb{C}^n$ of dimension n is polynomially convex.

The proof can be derived from the homology result mentioned above: If $M \subset \mathbb{C}^n$ is polynomially convex, then $H_{n-1}(\mathbb{C}^n \setminus M) = 0$; by the Alexander duality, $\check{H}^n(M, \mathbb{Z}) = H_{n-1}(\mathbb{C}^n \setminus M) = 0$; but $\check{H}^n(M, \mathbb{Z}) = \mathbb{Z}$ or $\mathbb{Z}_2$. This still leaves a possibility of rationally convex embedding of n-manifolds in $\mathbb{C}^n$.

Theorem (Sukhov-RS, Enseign. Math., 2016)

Let S be a closed surface, $S \neq S^2, \mathbb{RP}_2$. Then there exists a topological rationally convex embedding of S into $\mathbb{C}^2$.

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Rationally convex embeddings of surfaces 1

If one considers only smooth totally real embeddings, then a rationally convex surface $S \subset \mathbb{C}^2$ is necessarily Lagrangian according to Duval-Sibony. This clearly limits the possibilities for S, for example, of the orientable surfaces only the torus admits a Lagrangian (or even totally real) embedding. To overcome this we use the following result.

Theorem (Givental, Funk. Anal. i Prilozh., (1986))

Any closed surface admits a Lagrangian inclusion into $\mathbb{C}^2$, i.e., a smooth map $\iota : S \to \mathbb{C}^2$ which is a local Lagrangian embedding (i.e., $\iota^* \omega_{st} = 0$) except a finite set of singular points that are either transverse double self-intersections (or simply double points) or the so-called open Whitney umbrellas.

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Rationally convex embeddings of surfaces 2

The *standard open Whitney umbrella* is the map

$$\pi : \mathbb{R}^2_{(t,s)} \ni (t,s) \mapsto (ts, (2/3)t^3, t^2, s) \in \mathbb{R}^4_{(x,u,y,v)} \cong \mathbb{C}^2. \tag{1}$$

The map π is a homeomorphism, smooth away from the origin, onto a semialgebraic set with an isolated singularity. General open Whitney umbrellas are defined as images of the standard umbrella under a local symplectomorphism.

If S is orientable then any inclusion satisfies (Audin, JGP, 1990)

$$-\chi(S) + 2 \cdot d - m = 0, \tag{2}$$

(counting index), and if S is nonorientable, then

$$\chi(S) + 2 \cdot d - m = 0 \mod 4. \tag{3}$$

Here $\chi(S)$ is the Euler characteristic of S, d is the number of double points, and m is the number of umbrella points.

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Audin showed that any admissible combination of $\chi(S)$, d, and m is realizable in the orientable case. So if $\chi(S) \leq 0$, then we may choose $d = 0$, and $m = -\chi(S)$, i.e., any orientable surface, except S^2, admits a singular Lagrangian embedding (inclusion without double points), while the map

$$\mathcal{W} : \mathbb{R}^3 \ni (t, s, \tau) \to (t + it\tau, s + is\tau) \in \mathbb{C}^2 \tag{4}$$

is a Lagrangian immersion of S^2 with one double point (Whitney sphere).

In the nonorientable case Givental showed that if $\chi(S) \leq -2$, then one may take $d = 0$, i.e., such surfaces admit a singular Lagrangian embedding into $\mathbb{C}^2$. Further, Nemirovski and Siegel (Invent. Math., 2016) gave all possibilities for the number of umbrellas that may appear in a singular Lagrangian embedding of an arbitrary closed surface S. In particular, all nonorientable surfaces except $\mathbb{R}P_2$ admit a singular Lagrangian embedding in $\mathbb{C}^2$. E.g., the Klein bottle requires 4 umbrellas.

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Rationally convex embeddings of surfaces 4

For rationally convex embedding of surfaces, we prove that any Lagrangian inclusion into $\mathbb{C}^2$ (with Whitney umbrellas and double points) is rationally convex. The proof is a modification of the proof of Duval-Sibony and Gayet (Ann. Sci. Ecole Norm. Sup, 2000). An important ingredient of the proof is the following local result

Theorem (Sukhov-RS (IMRN 2013), Mitrea-Sh (Proc. of AMS, 2016))

Whitney umbrellas are locally polynomially convex.

The standard umbrella is contained in the singular real hypersurface $M = \{x^2 - yv^2 + (9/4)u^2 - y^3 = 0\}$, note that the defining function is spsh. Polynomial convexity can be proved by studying the characteristic foliation on the umbrella with respect to M. The foliation is defined by a system of ODE

$$\dot{t} = -3t^3 - ts^2 - 3t^5, \quad \dot{s} = s^3 + 4t^2 s + 7st^4$$

Theorem (Sukhov-RS (Trans. of AMS 2016))

The union of two Lagrangian submanifolds in $\mathbb{C}^n$ is locally polynomially convex at the transverse double point.

Local polynomial convexity gives us a nonnegative psh function that vanishes on the Lagrangian inclusion near singular points; this function is used to modify the argument of Duval-Sibony and Gayet to work near the singularity.

It is an open question whether S^2 or $\mathbb{RP}_2$ admit a rationally convex embedding into $\mathbb{C}^2$. Such an embedding, if exists, is unlikely to be smooth.

Little is known about rationally convex embeddings of closed n-dimensional manifolds into $\mathbb{C}^n$ for $n \geq 3$.

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Thank you! Merci!