Toric reduction & Applications

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§1. Symplectic reduction.

(X, ω) symplectic manifold

$H : X \to \mathbb{R}$ smooth autonomous Hamiltonian

$$\boxed{dH(-) = \omega(X_H, -)}$$

Let $\varphi_t^H : X \to X$ denote its flow.

Def: (X, ω, H) is called a Hamiltonian circle action if $\varphi_1^H = \mathrm{id}$.

$$\boxed{\psi^H_\bullet : S^1 \times X \longrightarrow X}$$ Hamiltonian circle action.

<u>Question</u>: Is there a reasonable way of "taking a quotient"?

<u>Thm.</u>: (Marsden - Weinstein)

If $S^1 \circlearrowright H^{-1}(c)$ free then the quotient $X_c := H^{-1}(c)/S^1$ carries a symplectic form for which:

$$\begin{tikzcd}
H^{-1}(c) \arrow[r, hook, "\iota"] \arrow[d, "\pi"] & (X, \omega) \\
(X_c, \omega_c) &
\end{tikzcd}$$

$$\boxed{\iota^* \omega = \pi^* \omega_c}$$

Example: $X = \mathbb{C}^{n+1} = \{(z_0, \dots, z_n)\}$

$$H = \pi |z_0|^2 + \dots + \pi |z_n|^2$$

generates diagonal circle action:

$$\varphi_t^H(z_0, \dots, z_n) = \left(e^{2\pi i t} z_0, \dots, e^{2\pi i t} z_n\right)$$

$$\begin{array}{ccc} H^{-1}(c) = S^{2n+1}(c) & \hookrightarrow & (\mathbb{C}^{n+1}, \omega_{std}) \\ \big\downarrow & & \\ (\mathbb{CP}^n, c\,\omega_{FS}) & & \end{array}$$

i.e. induces a natural symplectic form on $\mathbb{CP}^n$.

Lagrangian submanifolds:

Recall: $L^n \subset (X^{2n}, \omega)$ is called Lagrangian

if $$\boxed{\omega|_{TL} = 0}$$

There is a one-to-one correspondence of Lagrangians in $H^{-1}(c)$ and Lagrangians in X_c.

$$\begin{array}{ccccc} L & \subset & H^{-1}(c) & \hookrightarrow & X \\ & & \downarrow & & \\ L_c & \subset & X_c & & \end{array}$$

Hamiltonian systems:

Let $F : X \to \mathbb{R}$ be another Hamiltonian s.th.

$$\boxed{\{F, H\} = 0}$$

Then get $F_c : X_c \to \mathbb{R}$ s.th.

$$\begin{tikzcd}
\varphi_t^{F} \circlearrowright H^{-1}(c) \arrow[r, hook] \arrow[d] & X \\
\varphi_t^{F_c} \circlearrowright X_c &
\end{tikzcd}$$

commutes.

["Solve" F on a space of smaller dimension.]

§2. Toric manifolds

Def : $(X^{2n}, \omega, \underbrace{\mu : X \to \mathbb{R}^n}_{\text{"moment map"}})$ is called toric if

1) $\mu_i : X \to \mathbb{R}$ generates Ham. S^1-action,
2) $\{\mu_i, \mu_j\} = 0$,

$\ulcorner$ 1) & 2) $\Rightarrow$ $\exists\, T^n$-action

$$\varphi_{(t_1 \ldots t_n)}(x) = \varphi^{\mu_1}_{t_1} \circ \ldots \circ \varphi^{\mu_n}_{t_n}(x) \lrcorner$$

3) induced T^n-action is effective.

Key facts about toric manifolds:

Thm.: (Atiyah / Guillemin – Sternberg '82)

The image $\mu(X) \subset \mathbb{R}^n$ is a convex rational polytope (= "moment polytope")

⌈ this is true for any Hamiltonian torus action, i.e. not necessarily toric. ⌋

Thm: (Delzant '88)

Toric manifolds are classified by their polytope.

$$\text{cpct toric manifolds} / \text{equiv. symp.} \xleftrightarrow{1:1} \text{Delzant polytopes} / GL(n;\mathbb{Z}) \times \mathbb{R}^n$$

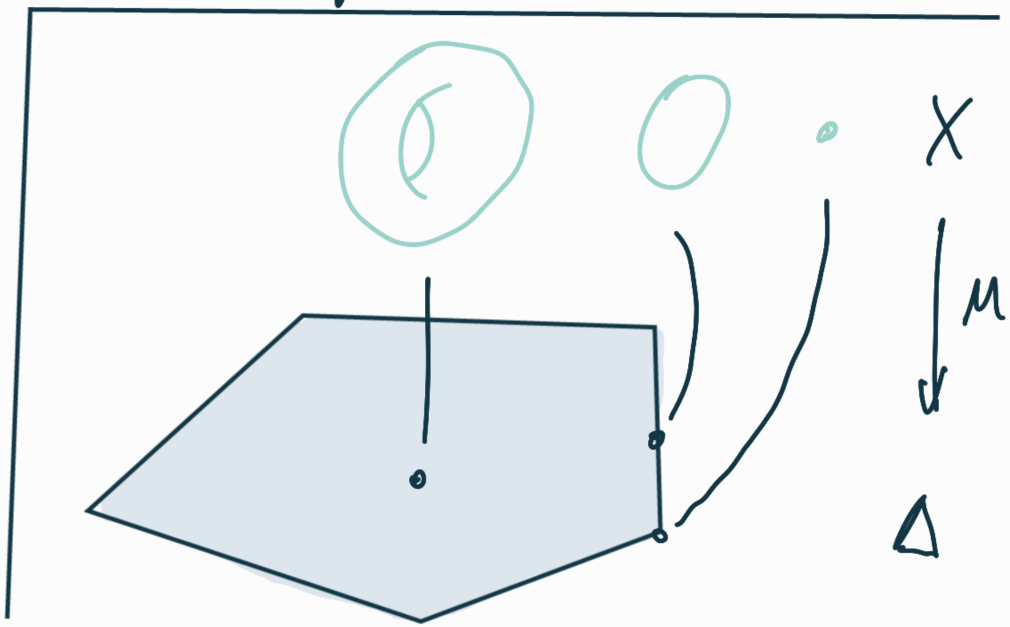
Delzant's thm. gives rise to the following "philosophy":

$$\boxed{\text{Symplectic topology of } (X,\omega) \longleftrightarrow \text{Combinatorial/geom. features of } \mu(X)=\Delta.}$$

Fibration structure of $\mu : X \longrightarrow \Delta$:

1) Orbits of T^n-action $\overset{1:1}{\longleftrightarrow}$ points in Δ

2) Orbit type is determined by the geometry of Δ.



Example: $X = S^2 \quad (\subseteq \mathbb{R}^3)$

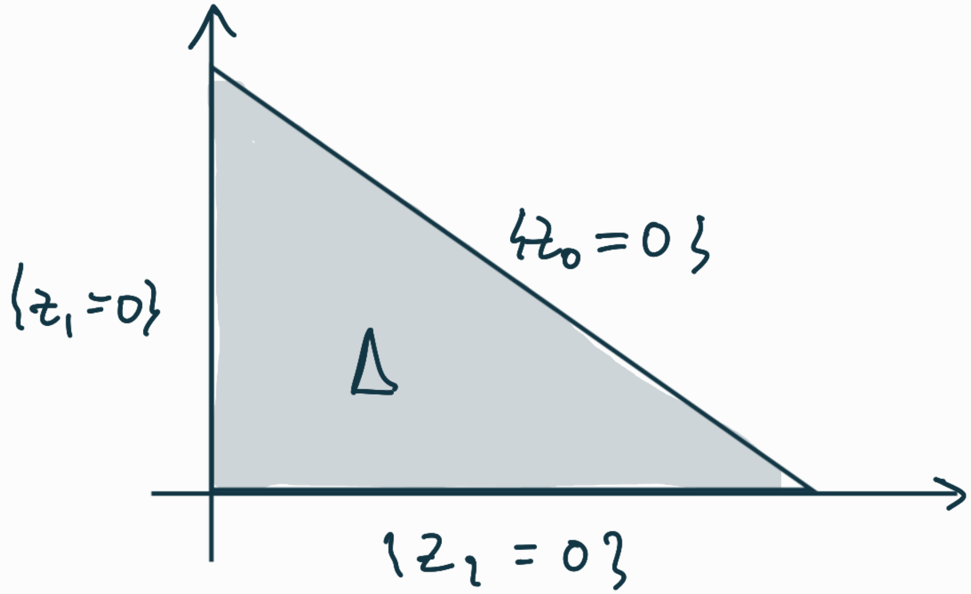
$\mu(x,y,z) = z$

ψ rotates S^2.

Example: $X = \mathbb{CP}^2$

$$\mu([z_0 : z_1 : z_2]) = \left(\frac{\pi |z_1|^2}{\|z\|^2}, \frac{\pi |z_2|^2}{\|z\|^2} \right)$$

$$\psi_{(t_1,t_2)}(z) = [z_0 : e^{2\pi i t_1} z_1 : e^{2\pi i t_2} z_2]$$



Another way to see the toric structure from the previous example:

$$\begin{array}{ccc} H^{-1}(1) = S^5 & \hookrightarrow & \underbrace{\mathbb{C}^3 \circlearrowleft T^3}_{\text{generated by:}} \\ \downarrow & & \\ \mathbb{CP}^2 & & \end{array}$$

$$\nu(z_1, z_2, z_3) = (\pi|z_1|^2, \pi|z_2|^2, \pi|z_3|^2)$$

with $H = \nu_1 + \nu_2 + \nu_3$, therefore $\{H, \nu_i\} = 0$

and get: $\mathbb{CP}^2 \circlearrowleft T^3$

This action is not effective, but the one by T^3/S^1 is effective $\rightsquigarrow$ $\boxed{\text{toric structure on } \mathbb{CP}^2.}$

diagonal circle.

§ 3. Toric reduction.

In the same vein, let

$$S^1(\xi) = \{\exp(t\xi) \mid t \in S^1\} \subset T^n,$$

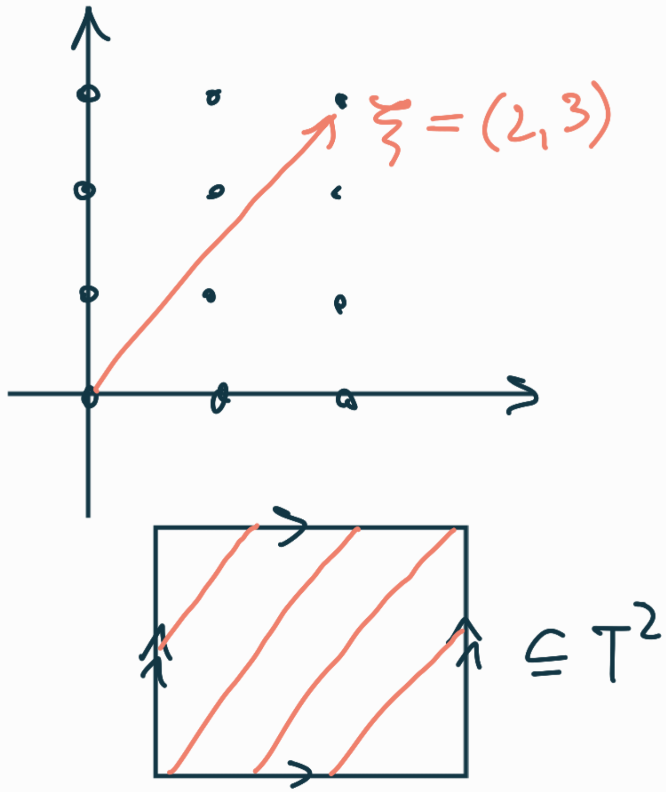
$\xi = (\xi_1, \dots, \xi_n) \in \mathbb{Z}^n$ primitive

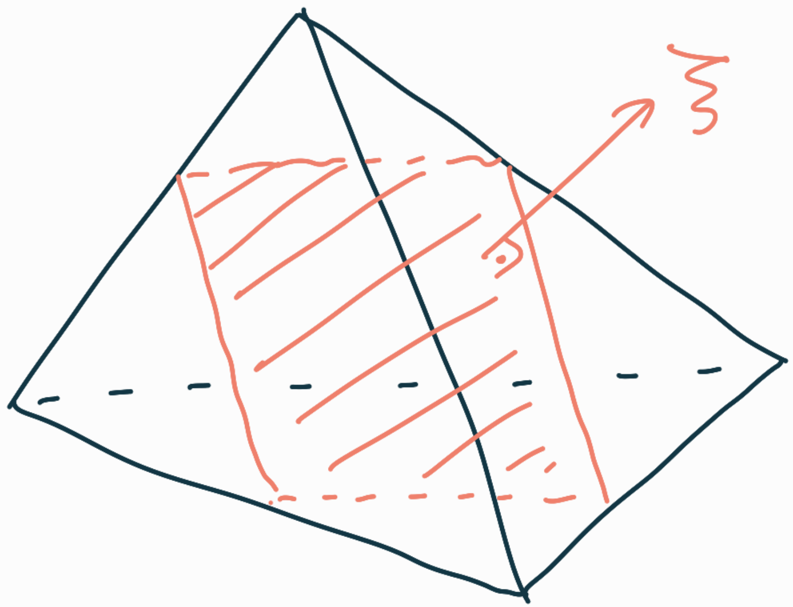
be a circle in T^n.

$$S^1(\xi) \subset \overbrace{T^n \circlearrowright X}^{\text{toric}}$$

generated by:

$$H = \xi_1 \mu_1 + \dots + \xi_n \mu_n .$$





$\Delta = \mu(X)$

$H^{-1}(c)$ is equal to $\mu^{-1}(\square)$.

Since $\{H, \mu_i\} = 0$, get a moment map

$$\mu_c : X_c \longrightarrow \mathbb{R}^n$$

from the reduced space.

Get an effective action by $T^{n-1} \cong T^n / S^1(\xi)$ on the quotient X_c

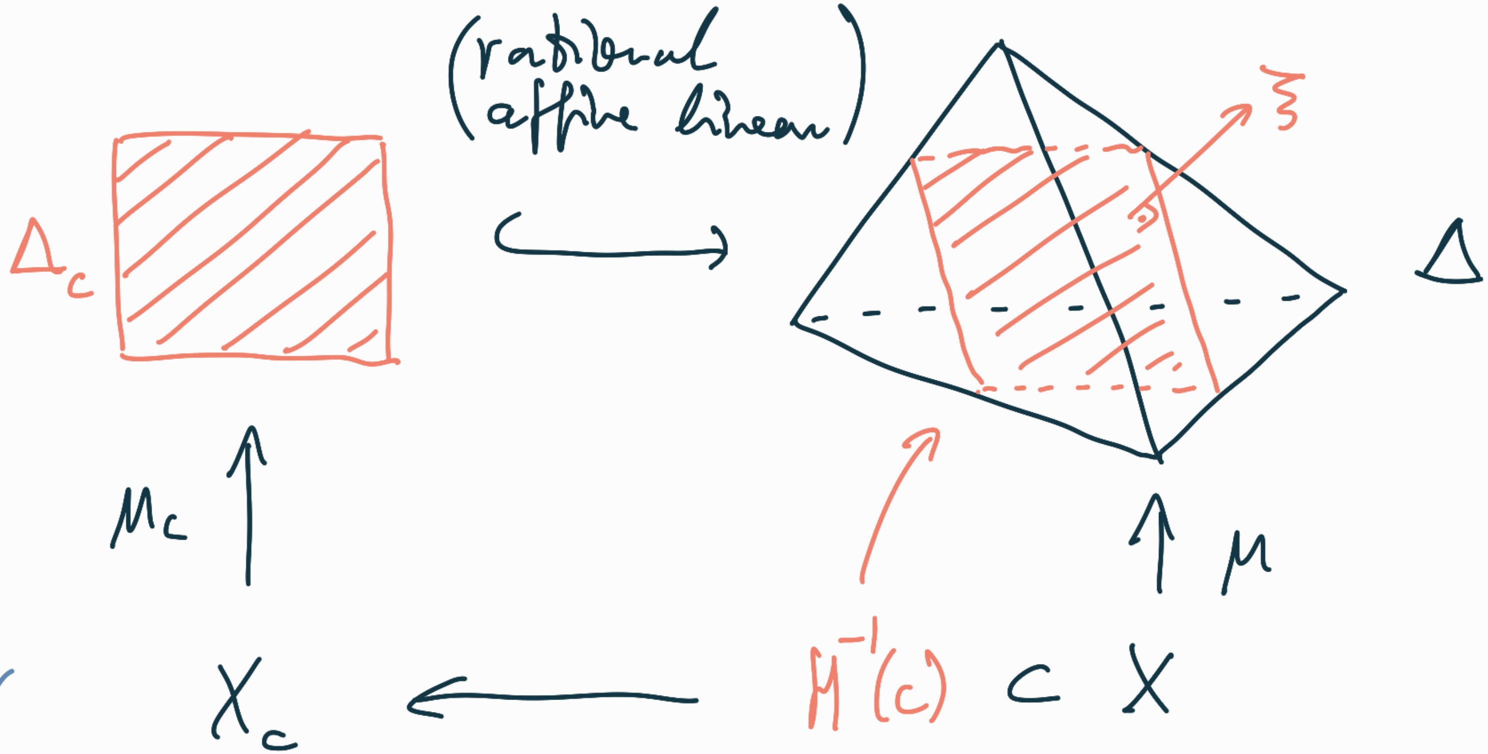
$\rightsquigarrow$ toric structure with moment map image 

Conclusion:

inclusion of moment polytopes

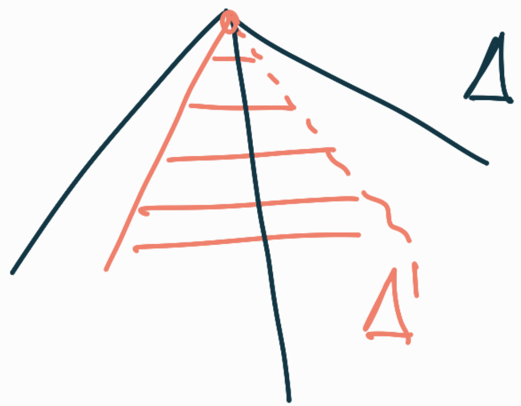
$\Updownarrow$

symplectic reduction by subgroup of T^n.



Caveat: Not every inclusion $\Delta' \subset \Delta$ yields a symplectic reduction, since the action may not be free,

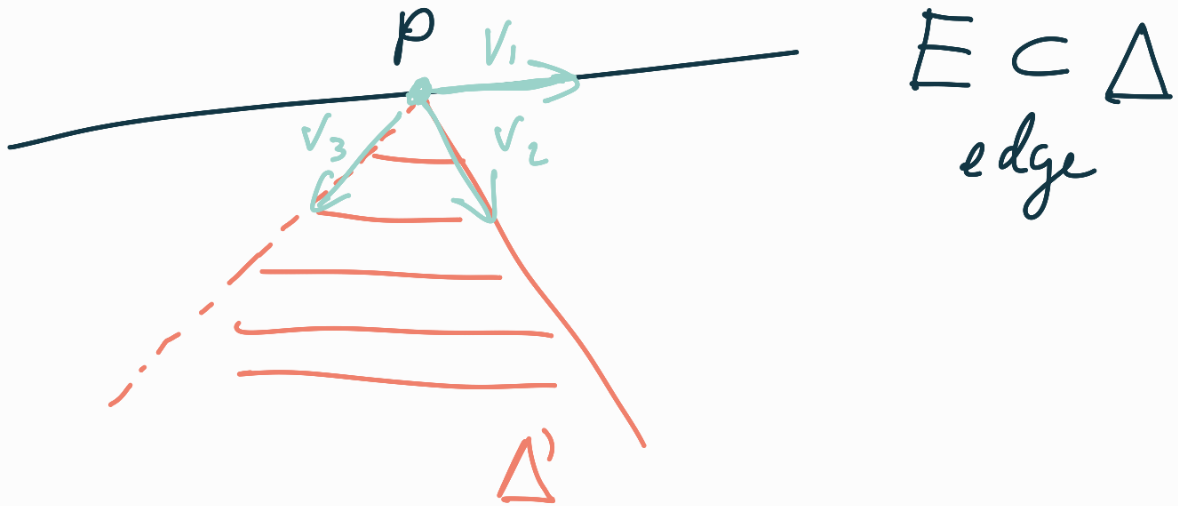
e.g.



If Δ' contains a $\underbrace{\text{vertex of } \Delta}_{=\text{ fixed point}}$,

then the action is not free.

Def.: An inclusion $\Delta' \subset \Delta$ of moment polytopes is called *admissible* if at every vertex $p = E \cap \Delta'$, the union $E \cup \Delta'$ contains a lattice basis,



with $\operatorname{span}_{\mathbb{Z}}\{v_1, v_2, v_3\} = \mathbb{Z}^3$.

<u>Thm</u>: Let (X, ω, μ) be toric and $\xi \in \mathbb{Z}^n$, $c \in \mathbb{R}$ a pair s.th.

$$\Delta' = \mu(H_\xi^{-1}(c)) \subset \Delta$$

admissible. Then

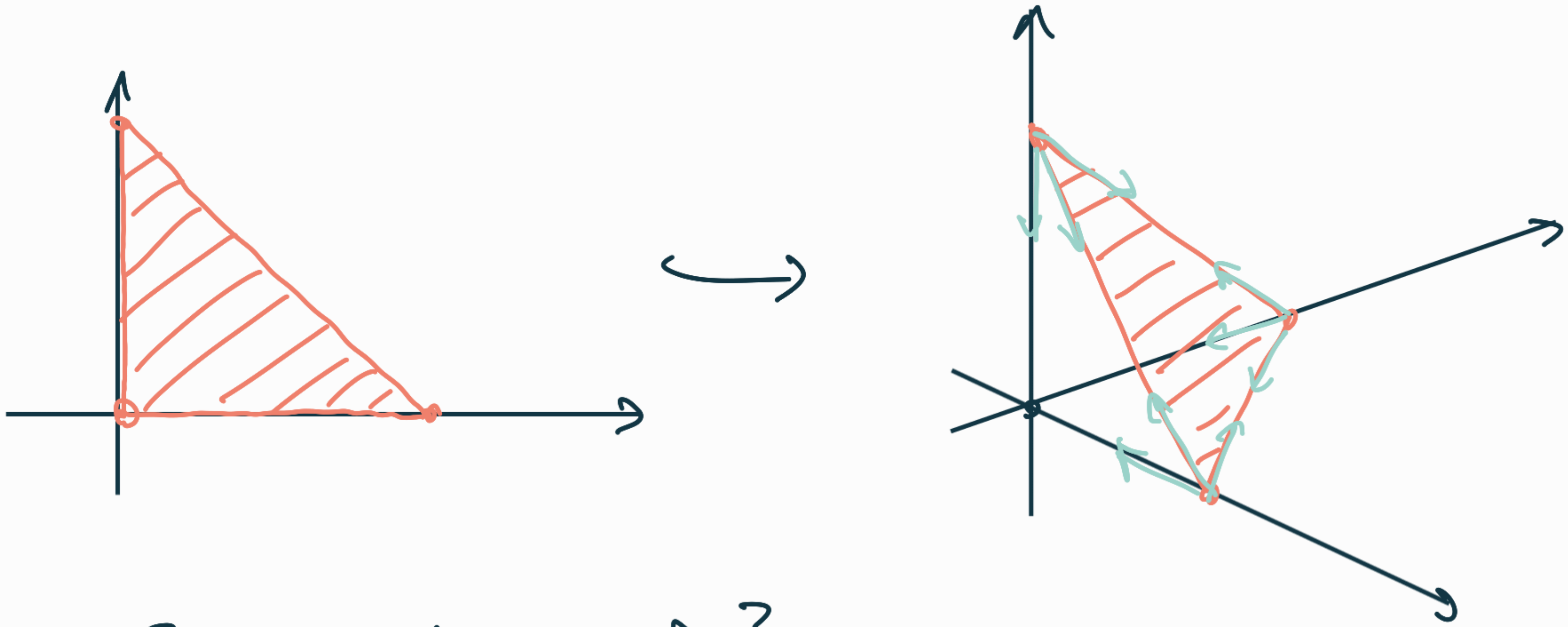
1) $S^1 \circlearrowright H_\xi^{-1}(c)$ is free ($\rightsquigarrow$ can perform symp. red.)
2) (X_c, ω_c) is toric with moment polytope Δ'.

<u>Rk</u>: Works the same for $T^k \subset T^n$.

<u>Rk</u>: This is not really new. Special cases include:

1) Delzant construction (Delzant '88)
2) McDuff's probes (McDuff '11)
3) Abreu - Macarini ('13)

Back to the example:

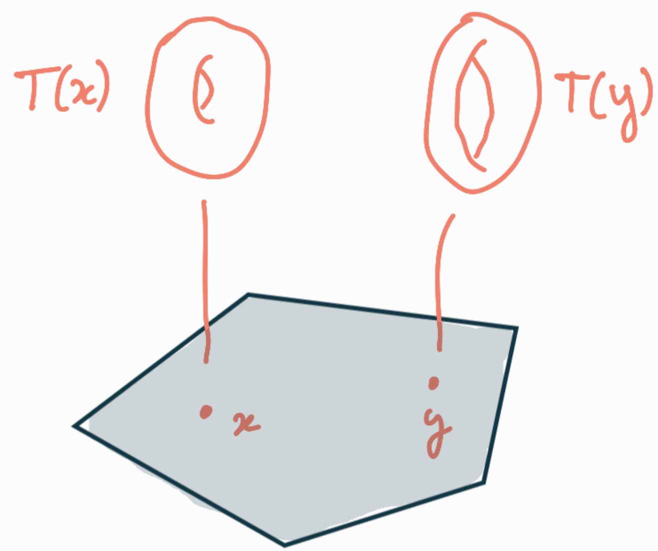


$X = \mathbb{C}^3$, $\Delta = \mathbb{R}^3_{\geq 0}$,

$X_c = \mathbb{CP}^2$, $\Delta' =$  .

§4. Application I : Constructing equivalences of toric fibres.

Recall : $x \in \mathrm{Int}\,\Delta \Rightarrow \mu^{-1}(x) =: T(x)$ Lagrangian torus. Call it toric fibre.



Question : For which $x, y \in \mathrm{Int}\,\Delta$ is there a $\phi \in \mathrm{Symp}(X, \omega)$ or $\phi \in \mathrm{Ham}(X, \omega)$ s.th. $\phi(T(x)) = T(y)$?

Question: For which $x, y \in \operatorname{Int} \Delta$ is there a $\phi \in \operatorname{Symp}(X, \omega)$ or $\phi \in \operatorname{Ham}(X, \omega)$ s.th. $\phi(T(x)) = T(y)$?

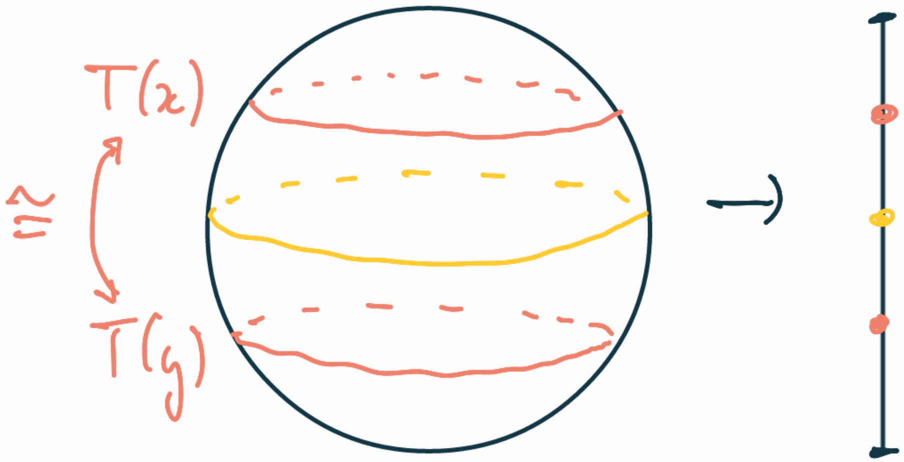
Answers:

*) Chekanov '96 for product tori in $\mathbb{R}^{2n}$,

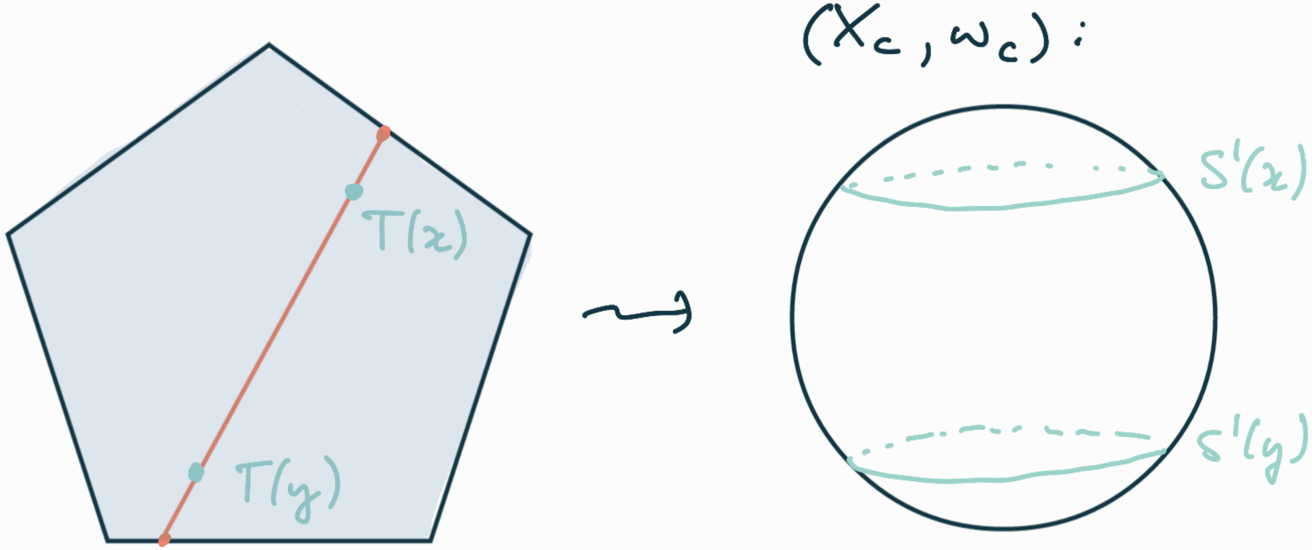
*) Shelukhin – Tonkonog – Vianna '19 for toric fibres in $\mathbb{CP}^2$,

*) B. '23 : many 4-dimensional examples.

Observation: For S^2, toric fibres come in equivalent pairs (except for the equator.)



Combine this with toric reduction:

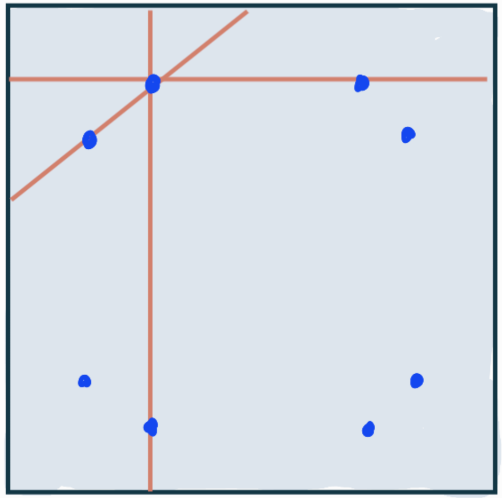


$$\boxed{\begin{array}{ccc} S^1(x) \cong S^1(y) & \Longrightarrow & T(x) \cong T(y) \\ \text{in } (X_c, \omega_c) & \text{lift Ham. isotopy.} & \text{in } (X, \omega) \end{array}}$$

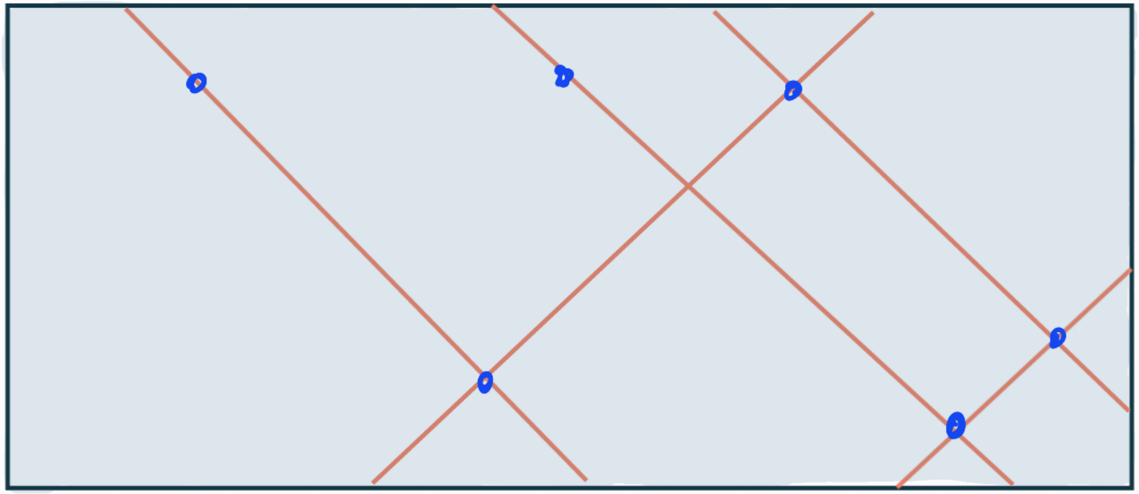
Def: Segments $\sigma \subset \Delta$ which are admissible are called symmetric probes.

(See McDuff '11 & Abreu - Borman - McDuff '15)

Example: monotone $S^2 \times S^2$:



non-monotone $S^2 \times S^2$:



Can have accumulation points!

Thm: For $S^2 \times S^2$, this yields the actual classification.

§5. Application II: Construction of exotic tori.

Let's return to $\mathbb{CP}^2$.

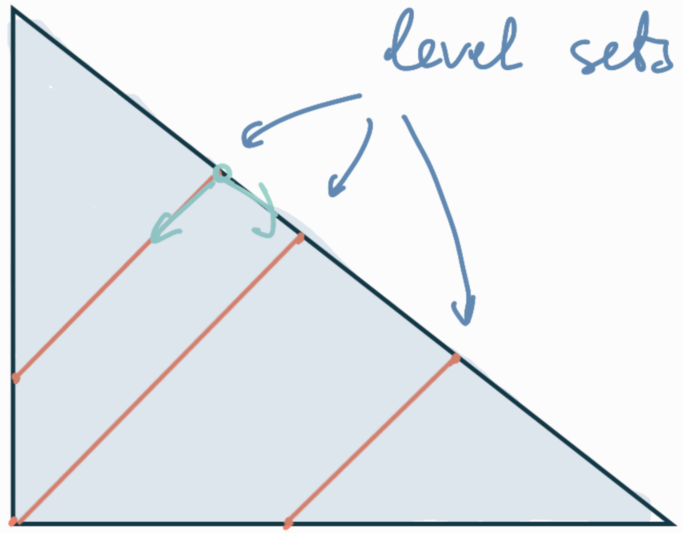
Question: Are there other Lag. tori in $\mathbb{CP}^2$?
(meaning which cannot be symplectically mapped to a toric fibre.)

Yes! Chekanov '96: in $\mathbb{C}^2$,
Chekanov-Schlenk '10: in $\mathbb{CP}^2$,
Vianna '16: ∞-many in $\mathbb{CP}^2$
(conjectured by Galkin-Usnich)

We will construct a Chekanov-type torus in $\mathbb{CP}^2$ by lifting a curve from red. space.

Let $\mu = (\mu_1, \mu_2) : \mathbb{C}P^2 \longrightarrow \mathbb{R}^2$ toric moment map and pick $\xi = (1,-1)$ to get

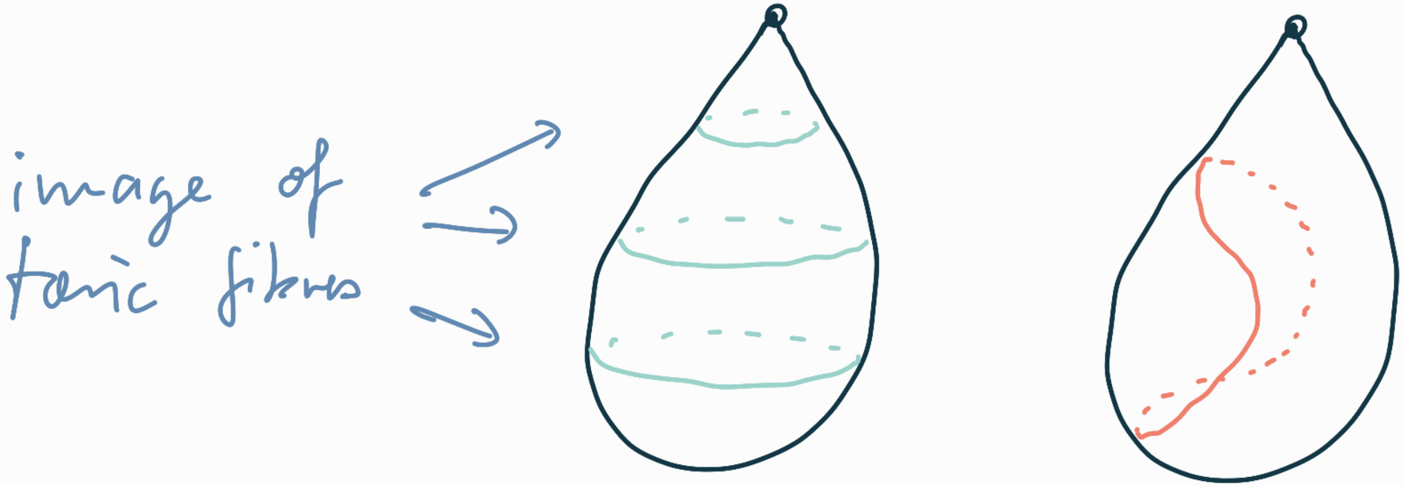
$$H = \mu_1 - \mu_2$$



What are the reduced spaces for $c \in]-1,1[$?

Segments are not admissible, get orbifolds.

for $c \neq 0$: X_c = 2-sphere with order 2 orbifold point for $c \neq 0$: X_c = 2-sphere with order 2 orbifold point

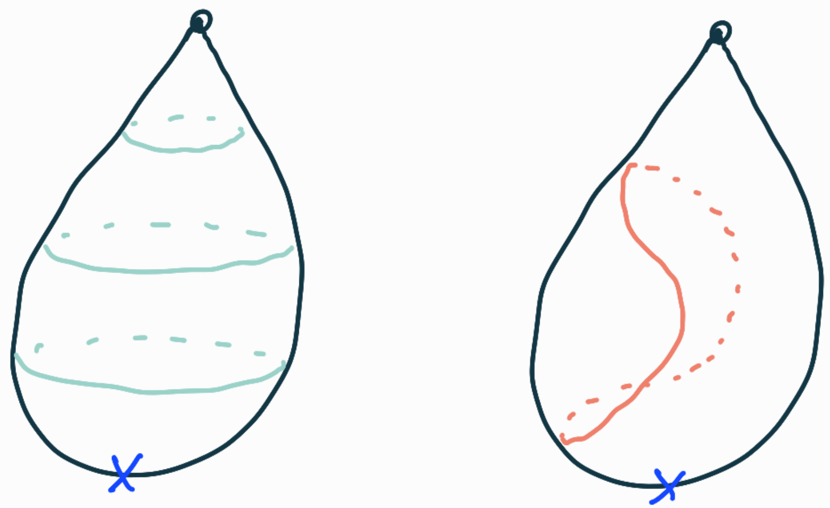


Every loop in X_c is Ham. isot. to one of the green circles.

$\rightsquigarrow$ no exotic tori!

for $c = 0$: $H^{-1}(0)$ contains the fixed point $p = [1:0:0] \in \mathbb{C}P^2$.

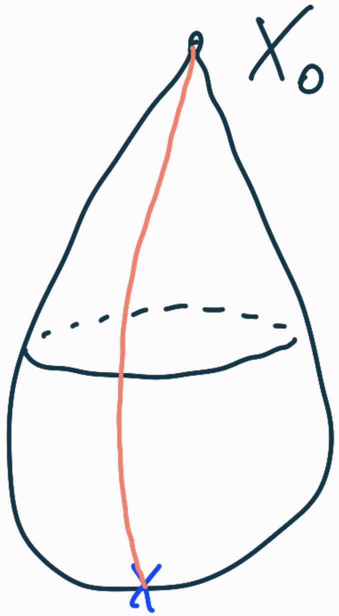
But can perform reduction on $H^{-1}(0) \setminus \{[1:0:0]\}$:



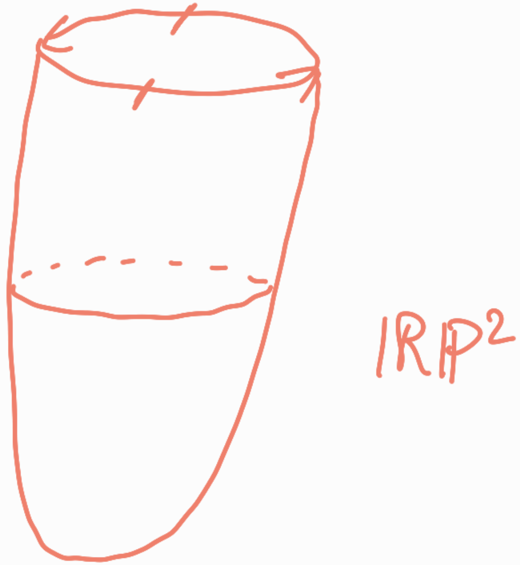
The red loop is not Hamiltonian isotopic to any of the green circles. Actually:

Thm: (Chekanov '96, E.-P. '95) Its lift is exotic.

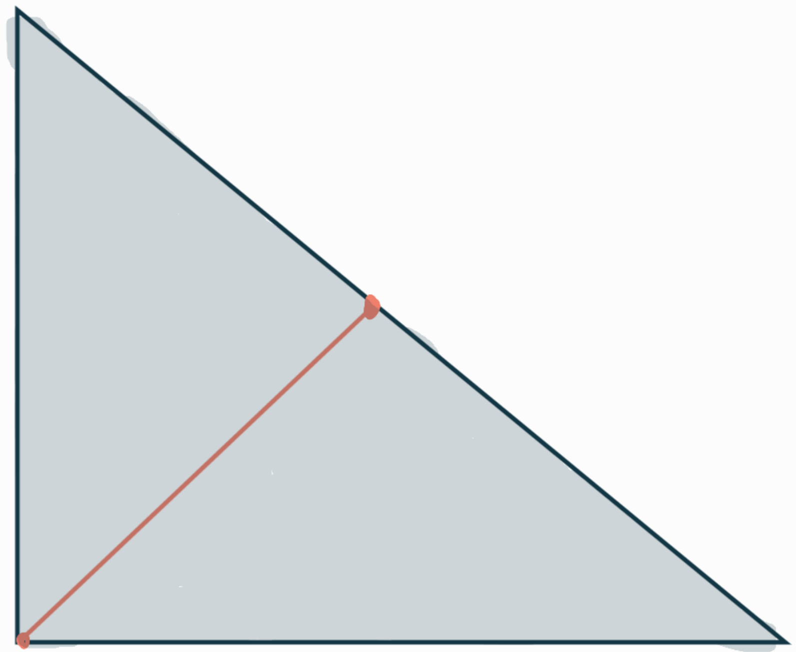
Another interesting lift: (see also Jonny's talk)



lifts to



Get a copy of $\mathbb{RP}^2$ fibering over the segment:



Other exotic tori that can be constructed using the same idea.

<u>Thm</u>: (B. '20)

Every monotone compact toric manifold X^{2n}, $n \geq 2$ contains a monotone exotic torus

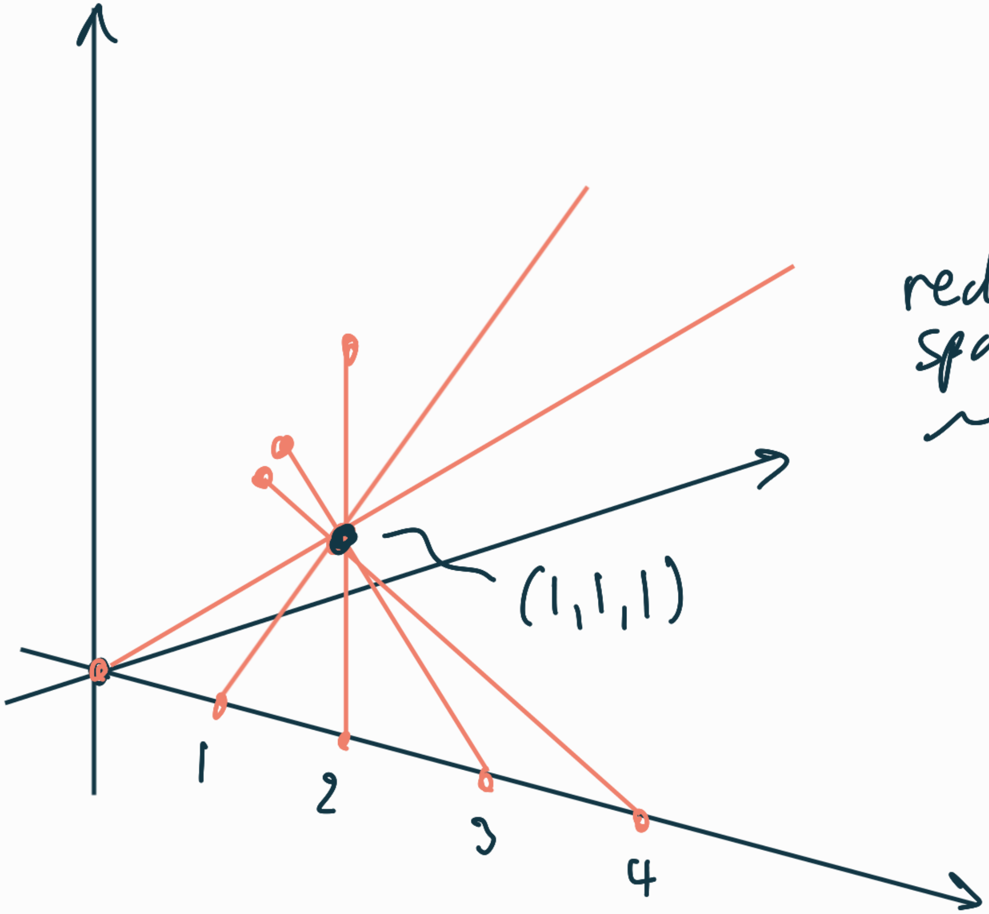
<u>Thm</u>: (Auroux '15 for monotone case
B. '23 non-monotone case)

There are infinitely many distinct 2-par. families of Lagrangian tori in $\mathbb{R}^6$.

Thm: (B. '23, Auroux '15 for monotone case)

There are infinitely many distinct 2-par. families of Lagrangian tori in $\mathbb{R}^6$.

Idea:



reduced space $\rightsquigarrow$

lift red curve.

Thank you !