Sheaves on Networks

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Sheaves I

- sheaves assign data to subsets of topological spaces
- came up in 1940s
- powerful in algebraic topology and geometry
- we consider special case: cellular sheaves
- further reading: chapter 4 of Justin Curry's PhD thesis

Notation (Finite graph)

$G = (V, E)$ with V, E finite sets, E contains ordered tuples (v_1, v_2).

Partial ordering on G by saying $v \leq e = (v_1, v_2)$ if $v = v_1$ or $v = v_2$

Sheaves II

Definition (Sheaf / cellular sheaf on a graph)

A **sheaf** assigns a finite-dimensional real vector space $\mathcal{F}(v)$ resp. $\mathcal{F}(e)$ to each vertex resp. edge and a linear map $\mathcal{F}_{v \leq e} : \mathcal{F}(v) \to \mathcal{F}(e)$ to each incidence $v \leq e$. (Abstractly: A sheaf on G is a functor $\mathcal{F}$ from poset G to finite-dimensional real vector spaces.)

$$
\begin{tikzcd}
\mathcal{F}(u) \arrow[r, "\mathcal{F}_{u \leq e}"] & \mathcal{F}(e) & \mathcal{F}(v) \arrow[l, "\mathcal{F}_{v \leq e}"'] \arrow[r, "\mathcal{F}_{v \leq f}"] & \mathcal{F}(f) & \mathcal{F}(w) \arrow[l, "\mathcal{F}_{w \leq f}"'] \\
u \arrow[r, dash] & e \arrow[r, dash] & v \arrow[r, dash] & f \arrow[r, dash] & w
\end{tikzcd}
$$

Example of a sheaf

Example (Constant sheaf on G)

$\underline{\mathbb{R}}_n$ for any G and $n \in \mathbb{N}$. All stalks are $\mathbb{R}^n$, all restriction maps $\mathbb{I}_n$.

Sections I

Definition

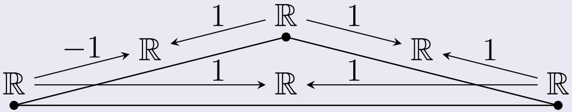
Let $W \subset V$ and $\mathcal{F}$ be sheaf on G. A **section** of $\mathcal{F}$ over W is an assignment of $s(w) \in \mathcal{F}(w)$ to each vertex $w \in W$ s.t for all edges $e = (w_1, w_2)$ between points in W:

$$\mathcal{F}_{w_1 \leq e}\, s(w_1) = \mathcal{F}_{w_2 \leq e}\, s(w_2).$$

Sections of $\mathcal{F}$ over W form the vector space $\Gamma(W, \mathcal{F})$. If $W = V$, the section s is called **global**.

Example

The following sheaf has only the zero-section as global section:



Constant sheaves have locally constant global sections.

Sections II

Sections encode linear local consistency conditions:

Example (Conservation law)

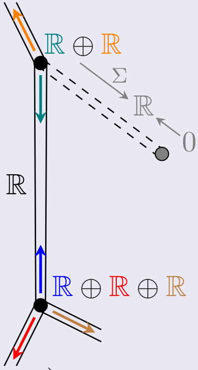
Let G represent system of water pipes.
Define flow sheaf on G by:

$$\mathcal{F}(e) = \mathbb{R} \ \ \forall e \in E \qquad \mathcal{F}(v) = \mathbb{R}^{\deg(v)} \ \ \forall v \in V$$

and $\pm\mathcal{F}_{v \leq e}$ the corresponding projection.
Sections represent flow values s.t. pipes don't leak.

To ensure the same for nodes, extend $\mathcal{F}$ to sheaf $\mathcal{F}'$ on G':
for each v, define v' with edge e' to v and let

$$\mathcal{F}'(v') = 0 \qquad \mathcal{F}'(e') = \mathbb{R} \qquad \mathcal{F}'_{v \leq e'} = (1, \dots, 1).$$



Sheaf Cohomology I

Defined similar to cellular cohomology.

Definition

Define vector spaces of 0- resp. 1-**cochains** by

$$C^0(G,\mathcal{F}) := \bigoplus_{v\in V} \mathcal{F}(v) \qquad C^1(G,\mathcal{F}) := \bigoplus_{e\in E} \mathcal{F}(e).$$

The linear **coboundary map** $\delta : C^0(G,\mathcal{F}) \to C^1(G,\mathcal{F})$ is defined by acting on stalks as

$$x_v \in \mathcal{F}(v) \mapsto \sum_{e=(v_k,v)} \mathcal{F}_{v\leq e}\, x_v - \sum_{e=(v,v_l)} \mathcal{F}_{v\leq e}\, x_v.$$

The **sheaf cohomology** is defined by

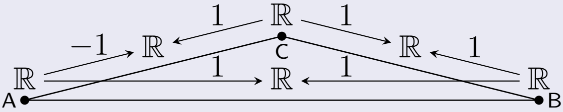
$$H^0(G,\mathcal{F}) := \ker(\delta) \qquad H^1(G,\mathcal{F}) := \mathsf{coker}(\delta) = C^1(G,\mathcal{F})/\mathsf{im}(\delta).$$

Sheaf Cohomology II

Example

In example from before:

$$\delta = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$



has rank 3. So $H^0(G,\mathcal{F}) = 0 = H^1(G,\mathcal{F})$.

- δ is block matrix
- choice of orientation irrelevant for cohomology
- $H^0(G,\mathcal{F}) = \Gamma(V,\mathcal{F})$
- setting $\mathcal{F} = \underline{\mathbb{R}}$ gives cellular cohomology and $\delta = B^\top$ for incidence matrix $B \in \{0,\pm 1\}^{|V|\times|E|}$

Relative Cohomology

Let $A = (V', E')$ be subgraph of G.

Definition (Relative cohomology)

Define the relative cochain complex by

$$C^0(G, A; \mathcal{F}) := \bigoplus_{v \in V \setminus V'} \mathcal{F}(v) \qquad C^1(G, A; \mathcal{F}) := \bigoplus_{e \in E \setminus E'} \mathcal{F}(e)$$

and the coboundary map $\delta_{(G,A)}$ by restricting δ_G to these spaces.

Get SES of chain complexes

$$0 \to C^*(G, A) \to C^*(G) \to C^*(A) \to 0$$

and LES of cohomology:

$$0 \to H^0(G, A) \to H^0(G) \to H^0(A) \to H^1(G, A) \to \dots$$

Example: Sampling

$$0 \to H^0(G, A) \to H^0(G) \to H^0(A) \to H^1(G, A) \to \dots$$

Interpretation: $H^*(G, A)$ is obstruction for global sections of G being in bijection with global sections of A.

Example (Water pipes revisited)

Recall the example with the water pipes.

What are optimal nodes to measure the system with few measurements?

Look for subgraph $A \subset V$ such that

$$H^0(G, A; \mathcal{F}) = 0 = H^1(G, A; \mathcal{F}).$$

The Sheaf Laplacian

Recall the (unweighted) graph Laplacian: $L_G = BB^\top : V \to V$ and that for the sheaf $\underline{\mathbb{R}}$, $B = \delta^\top$. Set

$$L_{\mathcal{F}} := \delta^\top \delta : C^0(G, \mathcal{F}) \to C^0(G, \mathcal{F}).$$

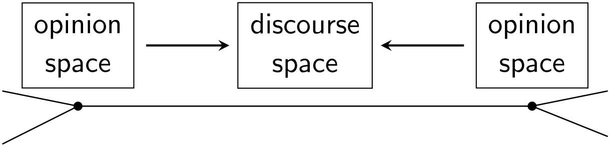
- $L_{\mathcal{F}}$ has block structure with

$$(L_{\mathcal{F}})_{uv} = \begin{cases} \sum_{v \leq e} \mathcal{F}_{v \leq e}^\top \mathcal{F}_{v \leq e} & u = v \\ -\mathcal{F}_{v \leq e}^\top \mathcal{F}_{u \leq e} & u \overset{e}{\sim} v \\ 0 & \text{otherwise} \end{cases}$$

- $L_{\mathcal{F}}$ independent of chosen orientations
- $\ker(L_{\mathcal{F}}) = \ker(\delta) = H^0(G, \mathcal{F})$

The Discourse Sheaf

- represent social network by a graph G:
  V = persons, E = connections
- opinions and discourse will be modelled as sheaf $\mathcal{F}$
- person v has opinion space $\mathbb{R}^{n_v} = \mathcal{F}(v)$ and opinion x_v
- edges stand for discourse about topics in $\mathbb{R}^{n_e} = \mathcal{F}(e)$
- agents v project their opinion to the discussed topics on e via $\mathcal{F}_{v \leq e}$
- consensus along $u \overset{e}{\sim} v$ if $\mathcal{F}_{v \leq e}\, x_v = \mathcal{F}_{u \leq e}\, x_u$
- global sections have consensus in all discussions (harmonic situation)



Dynamics of Opinions

Simplest model: agents v change their opinion towards average opinion of their friends.

Consider v and friend u connected by e. Their disagreement (seen from v) is $\mathcal{F}_{u \leq e}\, x_u - \mathcal{F}_{v \leq e}\, x_v$. So given an orthonormal basis $\{e_i\}$ of $\mathcal{F}(v)$, v changes her opinion according to:

$$\begin{aligned} \frac{d}{dt} x_v(t) &= \alpha \sum_i \left\langle \mathcal{F}_{u \leq e}\, x_u - \mathcal{F}_{v \leq e}\, x_v, \mathcal{F}_{v \leq e}\, e_i \right\rangle e_i \\ &= \alpha\, \mathcal{F}_{v \leq e}^{\top} \left(\mathcal{F}_{u \leq e}\, x_u - \mathcal{F}_{v \leq e}\, x_v \right) \end{aligned}$$

That yields the heat equation

$$\frac{d}{dt} x(t) = \alpha \sum_{v \overset{e}{\sim} u} \mathcal{F}_{v \leq e}^{\top} (\mathcal{F}_{u \leq e}\, x_u - \mathcal{F}_{v \leq e}\, x_v) = -\alpha\, L_{\mathcal{F}}\, x \qquad (1)$$

Solution for Basic Model

Theorem (Hansen/Ghrist 2021)

Solutions $x(t)$ to eq. (1) converge exponentially to the orthogonal projection of $x(0) \in C^0(G, \mathcal{F})$ onto $H^0(G, \mathcal{F})$ as $t \to \infty$.

Proof.

$L_{\mathcal{F}} = \delta^\top \delta$ is symmetric and positive semi-definite. Therefore:

$$L_{\mathcal{F}} = ODO^\top.$$

Equation (1) has solution

$$x(t) = \exp(-\alpha L_{\mathcal{F}})\, x(0) = O \exp(-\alpha D t) O^\top x(0).$$

As $t \to \infty$, this becomes orthogonal projection to 0-eigenspace of $L_{\mathcal{F}}$, which is the kernel. □

Stubborn Agents

Assume, a subset $U \subset V$ is stubborn. Modified heat equation:

$$\left(\frac{dx}{dt}\right)_v = \begin{cases} -(\alpha\, L_{\mathcal{F}}\, x)_v & v \in V \setminus U \\ 0 & v \in U. \end{cases}$$

Theorem (Hansen/Ghrist 2021)

For every $y \in C^0(U)$, there is a harmonic extension of y to all of V, i.e. a 0-cochain x with $x|_U = y$ and $(L_{\mathcal{F}} x)_v = 0$ for all $v \in V \setminus U$.

If $H^0(G, U; \mathcal{F}) = 0$, this is unique.

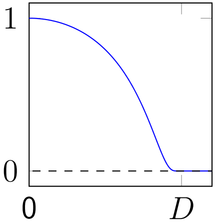
For every starting configuration x_0, the modified heat equation converges exponentially to a harmonic extension of $x_0|_U$.

Consequence: control over set U with $H^0(G, U) = 0$ gives control over the network

Bounded Confidence

Idea: connections are ignored if difference in opinion along this edge is too large.

For each edge e, consider a threshold D_e and smooth bump function $\phi_e : [0, \infty] \to \mathbb{R}$ that strictly falls on $[0, D_e)$ and vanishes on $[D_e, \infty]$.



Modify dynamics to:

$$\frac{d}{dt}x(t) = \alpha \sum_{v \overset{e}{\sim} u} \phi_e\left(||\mathcal{F}_{u \le e}\, x_u - \mathcal{F}_{v \le e}\, x_v||\right) \mathcal{F}_{v \le e}^{\top}(\mathcal{F}_{u \le e}\, x_u - \mathcal{F}_{v \le e}\, x_v)$$

Hansen and Ghrist showed that this vanishes if on each edge e

$$||\mathcal{F}_{u \le e}\, x_u - \mathcal{F}_{v \le e}\, x_v|| \in \{0\} \cup [D_e, \infty]$$

and starting sufficiently close to a stable configuration, the system converges to that configuration.

Changing Expression of Opinions

Assume on each edge $v \overset{e}{\sim} u$ the $\mathcal{F}_{v\leq e}$ change according to

$$\frac{d}{dt}\mathcal{F}_{v\leq e} = \beta(\mathcal{F}_{u\leq e}\, x_u - \mathcal{F}_{v\leq e}\, x_v)\frac{x_v^\top}{||x_v||^2}.$$

Example (Learning to lie)

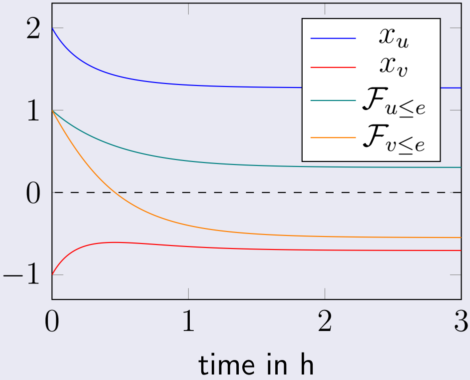
Consider the network

$$\begin{array}{ccccc} \mathbb{R} & \xrightarrow{\mathcal{F}_{u\leq e}} & \mathbb{R} & \xleftarrow{\mathcal{F}_{v\leq e}} & \mathbb{R} \\ u & \bullet\!\!-\!\!-\!\!-\!\!-\!\!-\!\!- & e & -\!\!-\!\!-\!\!-\!\!-\!\!-\!\!\bullet & v \end{array}$$

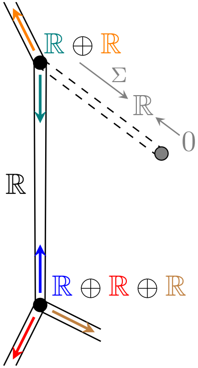
and the (initial) data

$$\mathcal{F}_{u\leq e}(0) = \mathcal{F}_{u\leq e}(0) = 1$$
$$x_u(0) = 2 \quad x_v(0) = -1$$
$$\alpha = \beta = \mathrm{h}^{-1}$$

to obtain the dynamics shown:



Conclusion

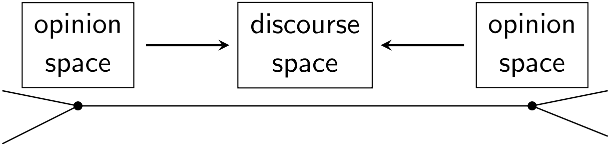


Sheaves, sections and (relative) cohomology can be generalised to cellular complexes

We saw an application to water pipes and sampling,

the sheaf Laplacian,

and how to use sheaves to model opinion dynamics



Sources

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