Progress on quantum complexity growth conjectures

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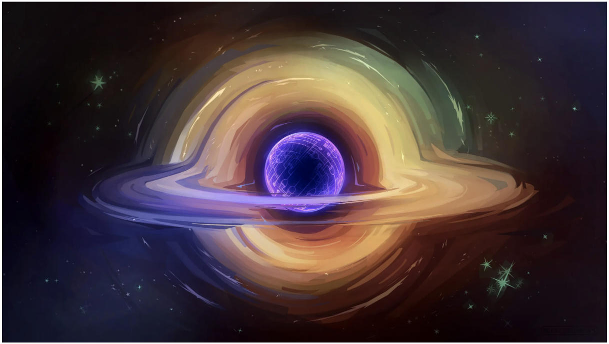
UT Austin

June 06, 2023

Quantum Information Theory in
Quantum Field Theory and Cosmology
Banff International Research Station

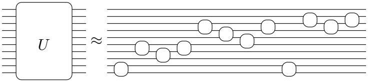
Based on:
[Brandão, Chemissany, NHJ, Kueng, Preskill], PRX Quantum, 1912.04297
[Oszmaniec, Horodecki, NHJ], 2205.09734
and work in progress
also mentioning some results from:
[NHJ], 1905.12053, [Haferkamp, NHJ], PRA, 2012.05259, [Cotler, NHJ, Ranard], PRA, 2010.11922

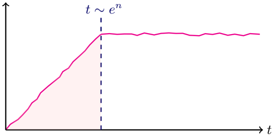


[Quanta Magazine, this morning]

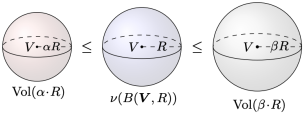
Based on:

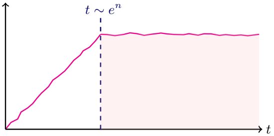
1) with F. Brandão, W. Chemissany, R. Kueng, J. Preskill
"Models of quantum complexity growth"



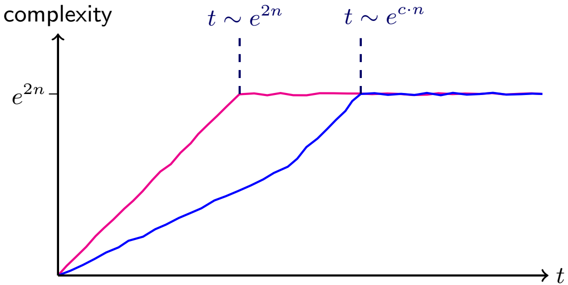


2) with M. Oszmaniec, M. Horodecki, "Saturation and recurrence of quantum complexity for random quantum circuits"

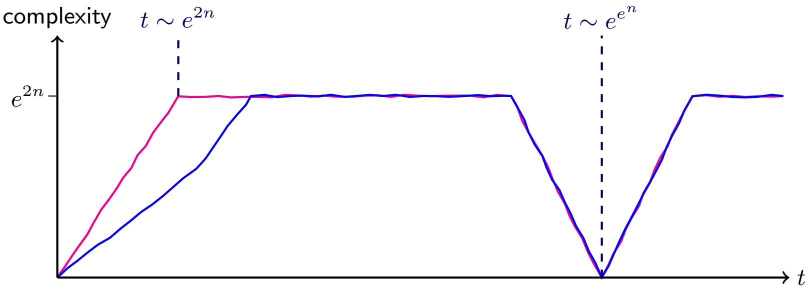




RQC complexity growth



RQC complexity growth

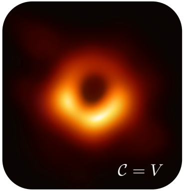


Quantum complexity

Quantum complexity is an important and well-established notion in QI

Recent interest in quantum many-body physics:

- distinguish topological phases of matter at zero temperature [Chen, Gu, Wen]
- describe regions behind black hole horizons in AdS/CFT [Susskind], [Stanford, Susskind]



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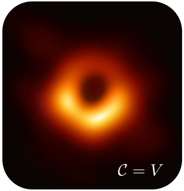
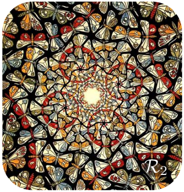


More generally, complexity growth is one universal aspect of real-time dynamics in strongly-interacting many-body systems

$\rightarrow$ relation to thermalization, quantum chaos, ...

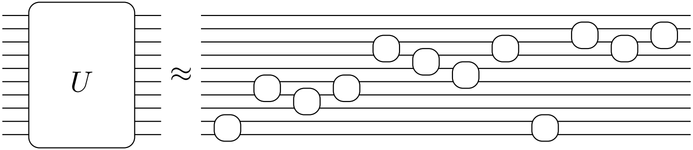


Complexity

some intuition

Circuit complexity is a somewhat intuitive notion

The traditional definition involves building a circuit with gates drawn from a universal gate set, which implements the state or unitary to within some tolerance δ



We are interested in the minimal size of a circuit that achieves this

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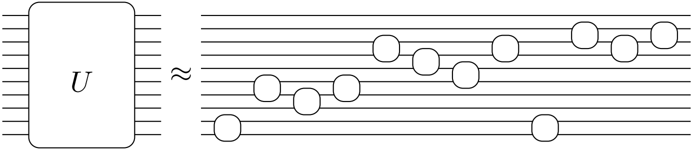


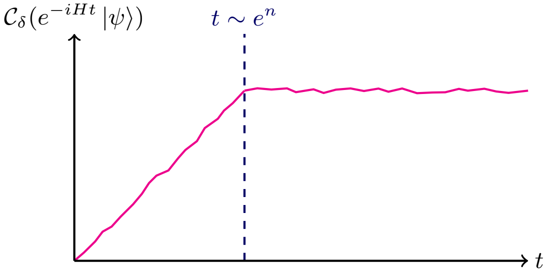
We are interested in the minimal size of a circuit that achieves this

Consider systems of n qudits (with local dim q), such that $d = q^n$

Complexity

some expectations

It is believed(/expected/conjectured) that the complexity of a simple initial state, grows (possibly linearly) under the time-evolution by a chaotic Hamiltonian



saturating after an exponential time

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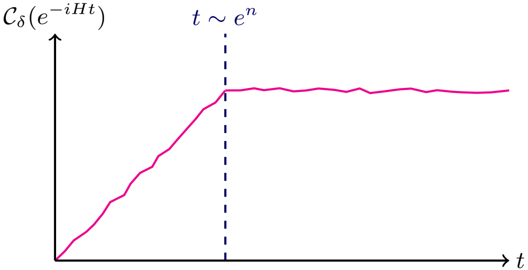
It is believed(/expected/conjectured) that the complexity e^{-iHt} grows (possibly linearly) for a chaotic Hamiltonian H



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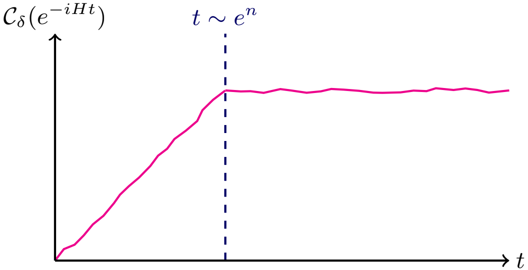
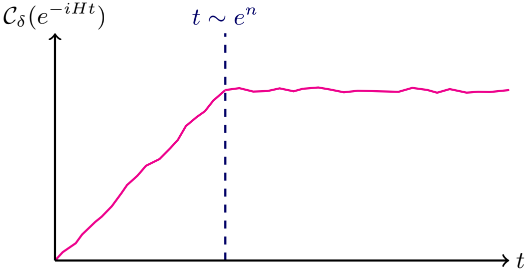


saturating after an exponential time

computing the quantum complexity analytically is very hard (especially for a fixed H)

Complexity

some expectations



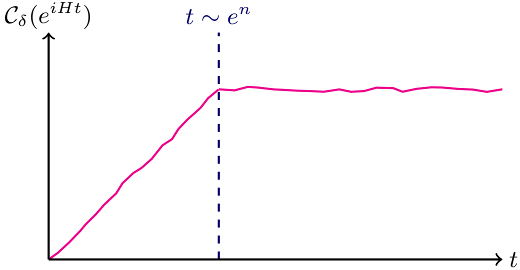
Why?

polynomial/linear growth: early time collisions should be rare; upper bounds on growth from Hamiltonian simulation algorithms

saturation: counting δ-balls in $U(d)$, doubly exp $(\sim (1/\delta)^{2^{2n}})$ 'distinct' unitaries, and thus can reach any unitary with a depth $t \sim e^{2n}$ circuit

Complexity

some expectations



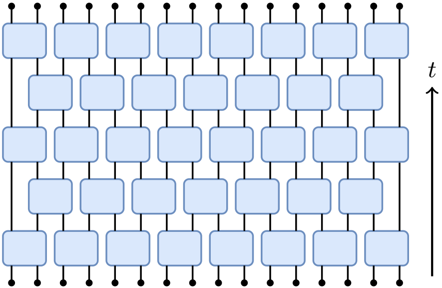
To make progress:

$\rightarrow$ use complexity theoretic assumptions to make statements about the complexity of a particular Hamiltonian evolution at exponentially long times [Aarsonson], [Susskind], [Bohdanowicz, Brandão]

$\rightarrow$ focus on ensembles of time-evolutions (RQCs)

Our goal

Consider random quantum circuits, on n qudits of local dimension q, evolving with staggered layers of 2-site unitaries, each drawn randomly from a gate set G

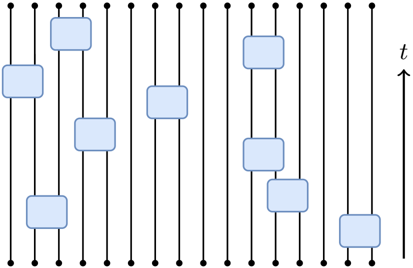


where evolution to time t is given by $U_t = U^{(t)} \dots U^{(1)}$

and try to prove the growth of complexity in this model

Our goal

Consider random quantum circuits, on n qudits of local dimension q, evolving with random nearest-neighbor 2-site unitaries, each drawn randomly from a gate set G



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and try to prove the growth of complexity in this model

Complexity growth in RQCs

Specifically, it has been conjectured that

Conjecture [Brown, Susskind], [Susskind]

Most local random quantum circuits of depth t have a complexity that scales *linearly* in t for an exponentially long time.

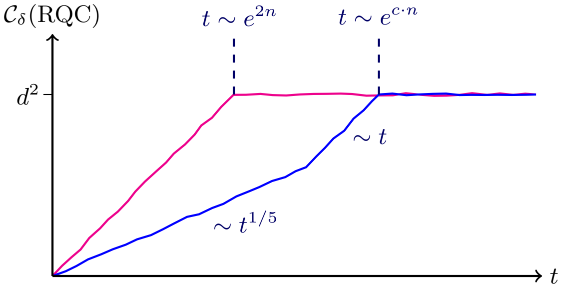
This sounds reasonable, but is hard to prove: one needs to show that collisions between circuits of subexponential size are rare.

Complexity growth in RQCs

(some results)

We expect that complexity grows linearly in time, saturating after an exponential time

What we can prove for RQCs on n qubits



Complexity growth in RQCs

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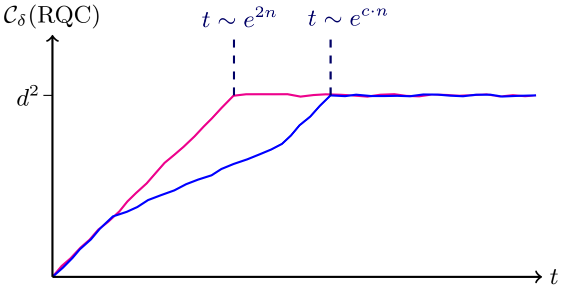
We expect that complexity grows linearly in time, saturating after an exponential time

What we prove for RQCs on n qudits (large q)



Overview

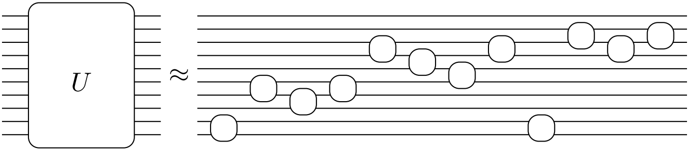
- Define complexity
- Complexity by design
- Complexity of local random quantum circuits
- Complexity saturation and recurrence for RQCs

Unitary complexity

Consider a system of n qudits with local dimension q, where $d = q^n$.

Complexity of a unitary: the minimal size of a circuit, built from elementary 2-local gates, that approximates the unitary U

We assume the circuits are built from 2-local gates chosen from a universal gate set G. Let G_r denote the set of all circuits of size r



where  $\in G$

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Complexity of a unitary

We say that a unitary $U \in U(d)$ has δ-complexity $\mathcal{C}_\delta(U) = r$ if and only if

$$r = \min\left\{r' : \exists V \in G_{r'} \text{ s.t. } \|U - V\| \leq \delta\right\}$$

(where the distance used is $\|\mathcal{U} - \mathcal{V}\|_\diamond$ and $\mathcal{U} = U(\rho)U^\dagger$)

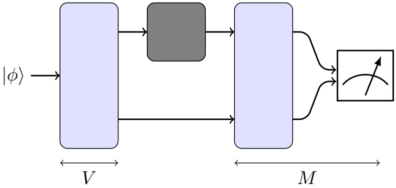
Complexity from measurements

We can consider an alternative (stronger) definition of the complexity of a state or unitary, in terms of an optimal distinguishing measurement

Roughly, the strong complexity of U is the minimal circuit required to implement an ancilla-assisted measurement capable of distinguishing $\mathcal{U}$ from the completely depolarizing channel $\mathcal{D}$

Task is to distinguish the channels with restricted state preparation and measurements as

$$\begin{aligned} \text{maximize} \quad & \left|\mathrm{Tr}\Big(M\big((\mathcal{U}\otimes\mathcal{I})|\phi\rangle\langle\phi| - (\mathcal{D}\otimes\mathcal{I})|\phi\rangle\langle\phi|\big)\Big)\right| \\ \text{subject to} \quad & M \in M_{r'}, \;\; |\phi\rangle = V\,|0\rangle\,, \;\; V \in G_r \end{aligned}$$



Complexity by design

We are interested in the complexity of random quantum circuits

To make progress we can derive some general statements about the complexity of unitary k-designs

But first, we need to define the notion of a unitary design

Unitary k-designs

Haar: (unique L/R invariant) measure on the unitary group $U(d)$

k-fold channel: $\Phi_{\mathcal{E}}^{(k)}(\mathcal{O}) \equiv \sum_i p_i U_i^{\otimes k}(\mathcal{O}) U_i^{\dagger \otimes k}$

exact k-design: $\Phi_{\mathcal{E}}^{(k)}(\mathcal{O}) = \Phi_{\text{Haar}}^{(k)}(\mathcal{O})$

but for general k, few exact constructions are known

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Approximate k-design

For $\epsilon > 0$, an ensemble $\mathcal{E}$ is an ϵ-approximate k-design if the k-fold channel obeys

$$\left\| \Phi_{\mathcal{E}}^{(k)} - \Phi_{\text{Haar}}^{(k)} \right\|_{\diamond} \leq \epsilon$$

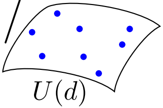
$\rightarrow$ designs are powerful

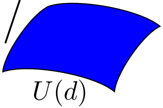
Unitary k-designs

If an ensemble of unitaries $\mathcal{E}$ forms an approximate k-design

the average over $\mathcal{E}$ is close to the average over the full unitary group up to the k-th moment

$$\left\langle \cdots \underbrace{U \cdots U^\dagger}_{k \text{ of these}} \cdots \right\rangle_{\text{(ensemble)}} \approx \left\langle \cdots \underbrace{U \cdots U^\dagger}_{k \text{ of these}} \cdots \right\rangle_{\text{(full group)}}$$

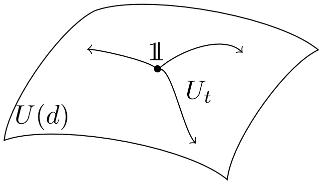




Intuition for k-designs
(eschewing rigor)

How random is the time-evolution of a system compared to the full unitary group $U(d)$?

Consider an ensemble of time-evolutions at a time t: $\mathcal{E}_t = \{U_t\}$



when does $\mathcal{E}_t$ form a k-design?

Complexity by design

an exercise in enumeration

Consider an approximate unitary k-design $\mathcal{E} = \{p_i, U_i\}$

Can we say anything about the complexity of U_i's?

The structure of a design is sufficiently restrictive, can bound the complexity of design elements

Can prove that:

Complexity for unitary designs

With high prob, a unitary U drawn from an ϵ-approx k-design $\mathcal{E}$ has complexity

$$\mathcal{C}_\delta(U) \gtrsim nk$$

Complexity by design

an exercise in enumeration

Consider an approximate unitary k-design $\mathcal{E} = \{p_i, U_i\}$

Can we say anything about the complexity of U_i's?

The structure of a design is sufficiently restrictive, can bound the complexity of design elements

Theorem (Complexity for unitary designs)

With probability $\geq 1 - e^{-nk}$, a unitary $U \sim \mathcal{E}_k$ drawn from an ϵ-approximate k-design has

$$\mathcal{C}_\delta(U) \geq \frac{1}{\log n|G|}\big(nk \log q - \log(1+\epsilon) + k \log(1+\delta^2)\big)$$

RQCs and randomness

Consider local RQCs on n qudits, with gates drawn randomly from a universal gate set G

Now we need a powerful result from [Brandão, Harrow, Horodecki]

RQCs form approximate designs

For $k \leq \sqrt{d}$, the set of local random quantum circuits of depth t forms an ϵ-approximate unitary k-design if

$$t \geq ck^{11}(n + \log(1/\epsilon))$$

where c is a constant

i.e. RQCs of depth $t = O(nk^{11})$ form k-designs

Complexity by design

We now combine these two results to say something about the complexity of local random circuits

With very high probability, a local RQC of depth t, has complexity

$$C_\delta(U_t) \gtrsim n(t/n)^{1/11}$$

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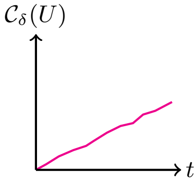
The k^{11} has been incrementally improved, the current best known bounds are $t = O(nk^{5+o(1)})$, which implies a $t^{1/5}$ complexity growth

$\rightarrow$ but what we really want is linear growth



RQCs and $t \sim k$

an appeal for linearity

To get a linear growth in complexity we need a linear growth in design

$$\text{complexity} \sim k \sim t$$

best known is $t = O(nk^5)$, but would need $t = O(nk)$

A lower bound on the k-design depth for these RQCs is $\Omega(nk)$

Can we prove that RQCs saturate this lower bound? (and are thus optimal implementations of k-designs)

Design growth in RQCs

Theorem (Design growth at large q) [NHJ]

RQCs on n qudits form ϵ-approximate k-designs when

$$t \geq 4nk + \log 1/\epsilon \quad \rightarrow \quad t = O(nk)$$

for some $q \geq q_0$, where q_0 depends on the size of the circuit

Theorem (Design growth for $q = \Omega(k^2)$) [Haferkamp, NHJ]

RQCs on n qudits with $q \geq 6k^2$ form ϵ-approximate k-designs when

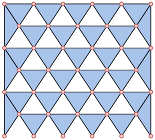
$$t \geq 18(2nk \log q + \log 1/\epsilon) \quad \rightarrow \quad t = O(nk \log k)$$

Designs from domain walls and gaps

Two approaches to computing the design depth for RQCs:

1) Partition function of a lattice model

$$Z = \sum_{\{\sigma\}}$$



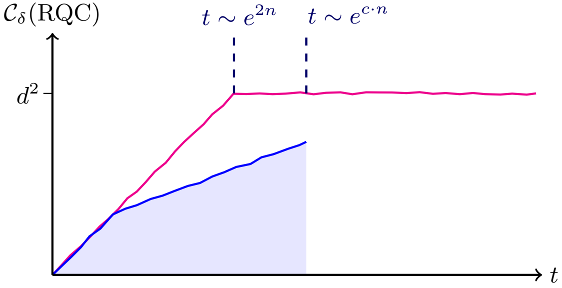
$$\leq \quad ?$$

2) Spectral gap of a local Hamiltonian

$$\Delta(H_{n,k}) \quad \geq \quad ?$$

Towards linear complexity growth

This makes some progress on the conjecture for local random circuits with large local dimension q



i.e. complexity is growing linearly in time t

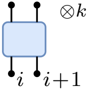
Linear growth from small gaps

For RQCs, the spectral gap enters as [Brown, Viola], [Brandão, Horodecki]

$$\text{(distance to forming a design)} \leq d^{2k}\left(1 - \frac{\Delta(H_{n,k})}{n}\right)^t$$

where $H_{n,k}$ is a frustration-free Hamiltonian

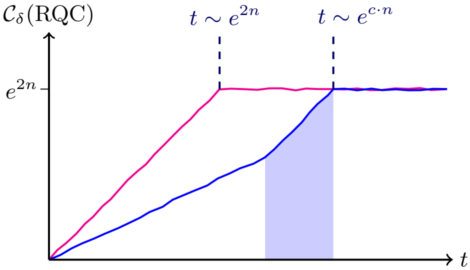
$$H_{n,k} = \sum_{i=1}^{n}\left(\mathbb{I} - \right.$$



$$\left.\right)$$

An exponentially-small, but k-ind, gap allows us to prove a linear complexity growth at late times

$(\Delta(H_{n,k}) \geq \Omega(e^{-c\cdot n}))$



Complexity saturation

How do we prove that complexity has saturated?

Haar random unitaries have maximal complexity, $C_\delta(U) \approx d^2$, but RQCs only approach Haar when $t \to \infty$

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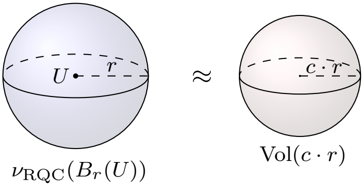
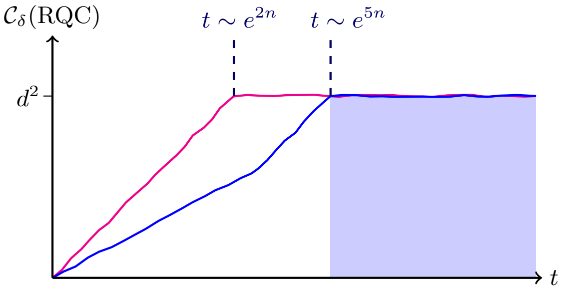
At exponential times ($t \sim e^{5n}$) RQCs equidistribute



(more formally, the measure assigned to balls by the ensemble of RQCs $\nu_{\mathrm{RQC}}(B_r(U)) \approx \mathrm{Vol}_{\mathrm{Haar}}(c \cdot r)$ for all $U \in U(d)$)

Complexity saturation

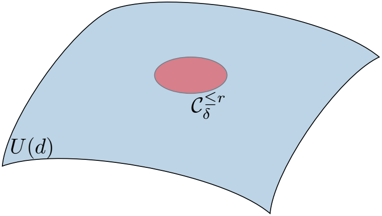
This allows us to show that

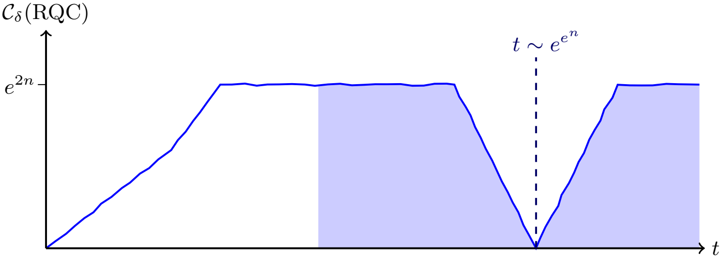


(can also prove that recurrences happen at doubly-exp times)

Explicit recurrence times

Once we achieve equidistribution, the probability of ‘walking’ to a particular unitary becomes ≈ that as prescribed by the Haar measure

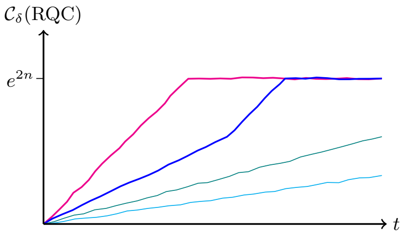




Where do we go from here?

Getting closer and closer to the Brown-Susskind conjecture for RQCs!

- Prove linear designs conjecture $\to$ linear complexity growth (seems hard, but continued progress)
- Forgo designs, look directly at specific moment quantities (nice ideas in recent work [Haferkamp, 2303.16944])



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- Can prove a linear growth for the exact complexity
  [Haferkamp, Faist, Kothakonda, Eisert, Yunger Halpern], [Li]

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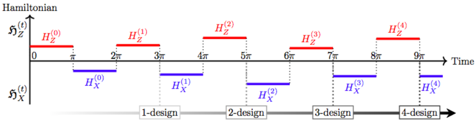
- Can prove a linear growth for the exact complexity
  [Haferkamp, Faist, Kothakonda, Eisert, Yunger Halpern], [Li]
- Other time-dependent evolutions (Brownian spin systems, Brownian SYK)
  [Onorati, Buerschaper, Kliesch, Brown, Werner, Eisert], [Nakata, Hirche, Koashi, Winter], [Jian, Bentsen, Swingle]

$$H_{\mathrm{BSS}}(t) = \sum_{j<k} \sum_{\alpha,\beta} \mathcal{J}_{jk}^{\alpha\beta}(t)\, \sigma_j^{\alpha} \sigma_k^{\beta} \qquad H_{\mathrm{BSYK}}(t) = \sum_{i<j<k<\ell} \mathcal{J}_{ijk\ell}(t)\, \chi_i \chi_j \chi_k \chi_\ell$$



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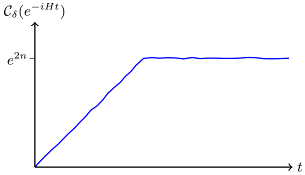
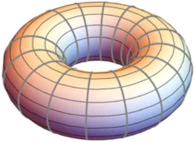
- Can prove a linear growth for the exact complexity
  [Haferkamp, Faist, Kothakonda, Eisert, Yunger Halpern], [Li]
- Other time-dependent evolutions (Brownian spin systems, Brownian SYK)
  [Onorati, Buerschaper, Kliesch, Brown, Werner, Eisert], [Nakata, Hirche, Koashi, Winter], [Jian, Bentsen, Swingle]
- Time-independent Hamiltonian evolution
  [Kotowski, Oszmaniec, Horodecki], [work in progress]





Where do we go from here?

- Can prove a linear growth for the exact complexity
  [Haferkamp, Faist, Kothakonda, Eisert, Yunger Halpern], [Li]
- Other time-dependent evolutions (Brownian spin systems, Brownian SYK)
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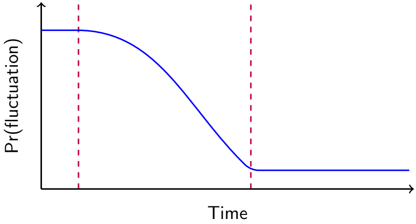
- Time-independent Hamiltonian evolution
  [Kotowski, Oszmaniec, Horodecki], [work in progress]
- Connections to entropies
  [Cotler, NHJ, Ranard]

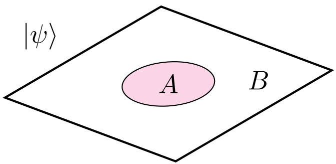


Subsystem entropy fluctuations

(a potential avatar of complexity)

Entropy fluctuations

Consider an n qubit system, initially in an unentangled state $|\psi\rangle$, which undergoes some unitary evolution $U_t = e^{-iHt}$ (e.g. by a chaotic H)



Consider the vN entropy ($S(\rho) = -\mathrm{tr}\,\rho\log\rho$) of a subsystem

$$\rho_A(t) = \mathrm{tr}_B\, U_t|\psi\rangle\langle\psi|U_t^\dagger$$

we expect the subsystem entropy to go like

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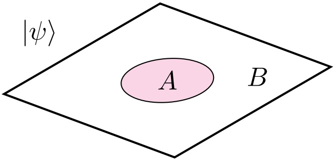

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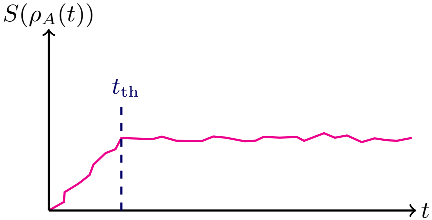

How often does the subsystem entropy fluctuate?

Entropy fluctuations

- How rare are entropy fluctuations after thermalization?
- How long must we wait (post-eq) to see an $O(1)$ fluctuation in the subsystem entropy $S(\rho_A(t))$?

Entropy fluctuations

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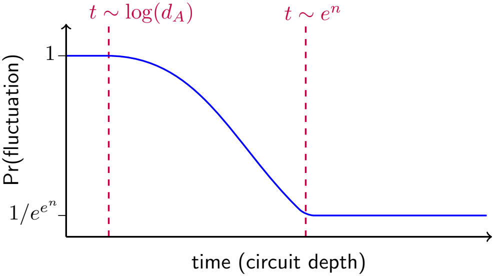
For RQCs, we prove ([Cotler, NHJ, Ranard])



Need to wait a doubly-exp long time to see a fluctuation

Future science

- Can we prove anything about $\mathcal{C}_\delta(e^{-iHt})$ for a fixed Hamiltonian? or for an ensemble of Hamiltonians?
- Can we prove a linear design growth at small q (e.g. some constant local dimension) for an exponentially long times?
- Improved RQC gaps? would give closer to linear growth and earlier saturation time
- Connections between (the rarity of) subsystem entropy fluctuations and complexity growth in many-body systems?
- Study the pseudorandomness properties of other RQCs (e.g. charge conserving circuits [Khemani, Vishwanath, Huse], [Rakovszky, Pollmann, von Keyserlingk])
- Explore implications of strong definition of complexity (in terms of an optimal measurement) in holography and for many-body physics?

Thanks!

Entropy fluctuations

Entropy fluctuations

The (informal) theorem statements are

For 1D RQCs on n qubits of depth t, the entropy of the evolved state on the subsystem $\rho_A(t)$ obeys

$$\Pr\left(S(\rho_A(t)) \leq \log(d_A) - \delta\right) \lesssim \begin{cases} e^{-t} & t \leq e^n \\ e^{-e^n} & t > e^n \end{cases}$$

Let N_A^{ent} be the number of times t that a subsystem A satisfies $S(\rho_A(t)) \leq \log(d_A) - \delta$ for all times from $t = c_{\text{th}} \log(d_A)$ up to $t = e^{c_{\text{rec}} d}$, where $c_{\text{th}} > 1$ and $c_{\text{rec}} < 1$

For 1D RQCs on n qubits, and $n \geq \Omega(c_{\text{th}} \log(d_A))$, the probability of an entropy fluctuation is bounded as

$$\Pr\left(N_A^{\text{ent}} > 0\right) \lesssim \frac{1}{e^{\delta}} \frac{1}{d_A^{c_{\text{th}}}}$$

(similar statements for the distance to the max mixed state)

Early time fluctuations

Theorem (Fluctuation bound at early times)

Assume A is a contiguous subsystem. For depth t RQCs on a periodic 1D chain of qudits, and for some $\delta > 0$, the entropy of the evolved state on the subsystem $\rho_A(t)$ obeys

$$\Pr\left(S(\rho_A(t)) \leq \log(d_A) - \delta\right) \leq \frac{1}{e^{\delta} - 1}\left(\frac{d_A}{d_B} + d_A\left(\frac{2q}{q^2+1}\right)^{2(t-1)}\right)$$

and the trace distance to the maximally mixed state obeys

$$\Pr\left(\left\|\rho_A(t) - \mathbb{I}_A/d_A\right\|_1 \geq \delta\right) \leq \frac{1}{\delta^2}\left(\frac{d_A}{d_B} + d_A\left(\frac{2q}{q^2+1}\right)^{2(t-1)}\right).$$

Fluctuations for designs

Theorem (Fluctuation bound for approximate designs)

For an approximate unitary $4k$-design $\mathcal{E}$, the entropy $S(\rho_A)$ of $\rho_A = \mathrm{tr}_B(U|\psi\rangle\langle\psi|U^\dagger)$, where U is drawn from $\mathcal{E}$, obeys

$$\Pr\left(S(\rho_A) \le \log(d_A) - \delta\right) \le 2\left(k! + \frac{1}{d^k}\right)\left(\frac{9\pi^3}{\gamma^2}\frac{d_A}{d_B}\right)^k,$$

where $\gamma := e^\delta - 1 - \frac{d_A}{d_B}$ and for $\delta \ge \frac{d_A}{d_B}$. Similarly, the distance between ρ_A and the maximally mixed state $\mathbb{I}_A/d_A$ obeys

$$\Pr\left(\left\|\rho_A - \mathbb{I}_A/d_A\right\|_1 \ge \delta\right) \le 2\left(k! + \frac{1}{d^k}\right)\left(\frac{9\pi^3}{\eta^2}\frac{d_A}{d_B}\right)^k,$$

where $\eta := \max\{\delta^2, e^{\delta^2/2} - 1\} - \frac{d_A}{d_B}$ and taking $\delta^2 > \frac{d_A}{d_B}$.

Counting subsystem fluctuations

Let $N_A^{\mathrm{ent}}(\delta)$ be the number of times t that a subsystem A satisfies $S(\rho_A(t)) \leq \log(d_A) - \delta$ for times $c_{\mathrm{th}} \log(d_A) \leq t \leq e^{c_{\mathrm{rec}} d}$, where $c_{\mathrm{th}} > 1$ and $c_{\mathrm{rec}} < 1$.

Theorem (Counting fluctuations)

For 1D brickwork RQCs on n qubits, for $n \geq \Omega(c_{\mathrm{th}} \log(d_A))$ and the constant $c_{\mathrm{rec}} = \gamma^2/(9\pi^3 d_A^2 e)$, the probability of an entropy fluctuation is bounded as

$$\Pr\left(N_A^{\mathrm{ent}}(\delta) > 0\right) \leq \frac{8}{e^\delta - 1} \left(\frac{1}{d_A}\right)^{\frac{2}{5}c_{\mathrm{th}} - 1} .$$

(similar statement for the distance to the max mixed state)

Unitary designs from domain walls

k-designs from stat-mech in RQCs

Using an exact stat-mech mapping, we can show that RQCs form k-designs in $O(nk)$ depth in the limit of large local dimension

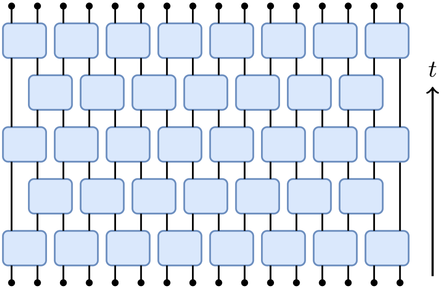
This is now for local random quantum circuits with Haar-random gates

Linear design growth in RQCs [NHJ]

Random quantum circuits on n qudits of local dimension q form approximate unitary k-designs when the circuit depth is $t = O(nk)$ for some $q > q_0$, where q_0 depends on the size of the circuit.

Random quantum circuits

Consider local RQCs on n qudits of local dimension q, evolved with staggered layers of 2-site unitaries, each drawn randomly from the Haar measure on $U(q^2)$



where evolution to time t is given by $U_t = U^{(t)} \dots U^{(1)}$

Study the convergence of random quantum circuits to **unitary k-designs**, i.e. depth where we start approximating moments of the unitary group

Our approach

- Focus on 2-norm and analytically compute the frame potential for random quantum circuits
- Making use of the ideas in [Nahum, Vijay, Haah], [Zhou, Nahum], we can write the frame potential as a lattice partition function
- We can compute the $k = 2$ frame potential exactly, but for general k we must sacrifice some precision
- We'll see that the decay to Haar-randomness can be understood in terms of domain walls in the lattice model

Frame potential

The frame potential is a tractable measure of Haar randomness, defined for an ensemble of unitaries $\mathcal{E}$ as [Gross, Audenaert, Eisert], [Scott]

$$k\text{-th frame potential}: \quad \mathcal{F}_{\mathcal{E}}^{(k)} = \int_{U,V\in\mathcal{E}} dU\,dV\, \left|\mathrm{Tr}(U^\dagger V)\right|^{2k}$$

For any ensemble $\mathcal{E}$, the frame potential is lower bounded as

$$\mathcal{F}_{\mathcal{E}}^{(k)} \geq \mathcal{F}_{\mathrm{Haar}}^{(k)} \quad \text{and} \quad \mathcal{F}_{\mathrm{Haar}}^{(k)} = k! \ \ (\text{for } k \leq d)$$

with $=$ if and only if $\mathcal{E}$ is a k-design

$$\boxed{\mathcal{F}_{\mathcal{E}}^{(k)} \geq k!}$$

Related to ϵ-approximate k-design as

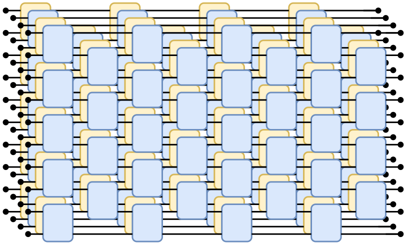
$$\left\|\Phi_{\mathcal{E}}^{(k)} - \Phi_{\mathrm{Haar}}^{(k)}\right\|_\diamond^2 \leq d^{2k}\left(\mathcal{F}_{\mathcal{E}}^{(k)} - \mathcal{F}_{\mathrm{Haar}}^{(k)}\right)$$

Frame potential for RQCs

The goal is to compute the FP for RQCs evolved to time t:

$$\mathcal{F}_{\mathrm{RQC}}^{(k)} = \int_{U_t, V_t \in \mathrm{RQC}} dU\, dV\, \left|\mathrm{Tr}(U_t^\dagger V_t)\right|^{2k}$$

Consider the k-th moments of RQCs, k copies of the circuit and its conjugate:

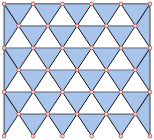


Lattice mappings for RQCs

Haar averaging the 2-site unitaries allows us to exactly write the frame potential as a partition function on a triangular lattice

The result is then that we can write the k-th frame potential as

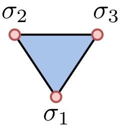
$$\mathcal{F}_{\mathrm{RQC}}^{(k)} = \sum_{\{\sigma\}} \prod_{\triangledown} J_{\sigma_2\sigma_3}^{\sigma_1} = \sum_{\{\sigma\}}$$



with $\sigma \in S_k$, width $n_g = \lfloor n/2 \rfloor$, depth $2(t-1)$, and pbc in time.

The plaquettes are functions of three $\sigma \in S_k$, written explicitly as

$$J_{\sigma_2\sigma_3}^{\sigma_1} =$$



$$= \sum_{\tau \in S_k} \mathcal{W}g(\sigma_1^{-1}\tau, q^2)\, q^{\ell(\tau^{-1}\sigma_2)} q^{\ell(\tau^{-1}\sigma_3)} \,.$$

Lattice mappings for RQCs

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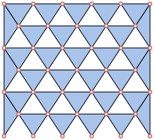
$$\mathcal{F}^{(k)}_{\rm RQC} = \sum_{\{\sigma\}} \prod_{\triangledown} J^{\sigma_1}_{\sigma_2\sigma_3} = \sum_{\{\sigma\}}$$ 

with $\sigma \in S_k$, width $n_g = \lfloor n/2 \rfloor$, depth $2(t-1)$, and pbc in time.

We can show that $J^{\sigma}_{\sigma\sigma} = 1$, and thus the minimal Haar value of the frame potential comes from the $k!$ ground states of the lattice model

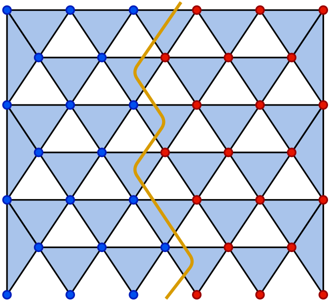
$$\mathcal{F}^{(k)}_{\rm RQC} = k! + \ldots$$

RQC domain walls

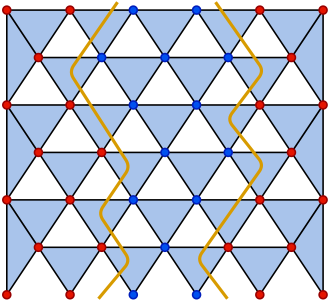
all non-zero contributions to $\mathcal{F}^{(k)}_{\mathrm{RQC}}$ are domain walls (which must wrap the circuit)

e.g. for $k = 2$ we have

a single domain wall configuration:



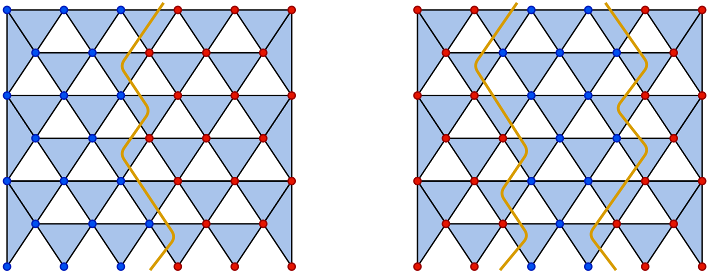
a double domain wall configuration:



k-designs from domain walls

To compute the k-design time, we simply need to count the domain wall configurations

$$\mathcal{F}_{\mathrm{RQC}}^{(k)} = k!\left(1 + \sum_{1\ \mathrm{dw}} wt(q,t) + \sum_{2\ \mathrm{dw}} wt(q,t) + \dots\right)$$



$\rightarrow$ decay to Haar-randomness from dws

RQC 2-design time

We have the $k = 2$ frame potential for random circuits

$$\mathcal{F}^{(2)}_{\rm RQC} \leq 2\left(1 + \left(\frac{2q}{q^2+1}\right)^{2(t-1)}\right)^{n_g - 1}$$

the circuit depth at which we form an ϵ-approximate 2-design is then

$$t_2 \geq C\big(2n\log q + \log n + \log 1/\epsilon\big) \quad \text{with} \quad C = \left(\log \frac{q^2+1}{2q}\right)^{-1}$$

where for $q = 2$ we have $t_2 \approx 6.2n$, and at large q we find $t_2 \approx 2n$

$$t_2 \sim n + \log 1/\epsilon$$

as is known [Harrow, Low]

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as is known [Harrow, Low]

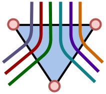
Can actually compute the $k = 2$ partition function exactly by solving the problem of p nonintersecting random walks [Fisher], [Huse, Fisher]

k-designs in RQCs
(a panoply of domain walls)

For general k, we can prove a simple contribution from the ground states and single domain wall sector, plus higher order contributions

$$\mathcal{F}^{(k)}_{\mathrm{RQC}} \leq k!\left(1 + (n_g - 1)\binom{k}{2}\binom{2(t-1)}{t-1}\left(\frac{q}{q^2+1}\right)^{2(t-1)} + \dots\right)$$

Moreover, the multi-domain wall terms are heavily suppressed and higher order interactions are subleading in $1/q$ as



$$\sim \frac{1}{q^{\#\text{dw's}}}$$

For some $q \geq q_0$, the single domain wall sector gives the ϵ-approximate k-design time:

$$t_k \geq 2nk + \log(1/\epsilon)$$

k-designs from stat-mech

RQCs form k-designs in $O(nk)$ depth at large q

As the lower bound on the design depth is nk, RQCs are then **optimal implementations of randomness**

we showed this in the large q limit, but this limit is likely not necessary

Conjecture (designs at small q)

The single domain wall sector of the lattice partition function dominates the multi-domain wall sectors for higher moments k and any local dimension q.

Unitary designs from spectral gaps

A retreat to operator norms

[Brown, Viola], [Brandão, Horodecki], [Brandão, Harrow, Horodecki]

Another approach to compute the circuit depth required to form a design

$$\left\| M_{\mathcal{E}}^{(k)} - M_{\text{Haar}}^{(k)} \right\|_\infty$$

For depth t RQCs, the operator norm has two nice properties:

i) Amplification: $\left\| M_{\text{RQC}}^{(k)} - M_{\text{Haar}}^{(k)} \right\|_\infty = \left(\left\| M_{\text{layer}}^{(k)} - M_{\text{Haar}}^{(k)} \right\|_\infty \right)^t$

ii) Hamiltonian gap*: $\left\| M_{\text{layer}}^{(k)} - M_{\text{Haar}}^{(k)} \right\|_\infty \leq \dfrac{1}{\sqrt{\Delta(H_{n,k})/4 + 1}}$

where $H_{n,k} = \sum_i P_{i,i+1}$

and $P_{i,i+1} = \mathbb{I} - \mathbb{I} \otimes \left(\int dU\, U^{\otimes k,k} \right)_{i,i+1} \otimes \mathbb{I}$

Knabe bounds on the spectral gap

$H_{n,k} = \sum_{i=1}^{n} P_{i,i+1}$ is a sum of projectors, has g.s. energy 0, and is FF

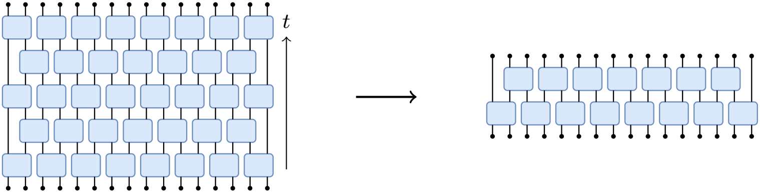
Theorem ([Knabe]). For a 1D translationally-invariant frustration-free Hamiltonian $H_{n,k} = \sum_i P_{i,i+1}$, the spectral gap obeys

$$\Delta(H_{n,k}) \geq 2\left(\Delta(H_{n=3,k}) - \frac{1}{2}\right).$$

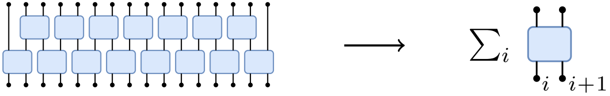
also [Gosset-Mozgunov], [Lemm-Mozgunov]

Rough recap:

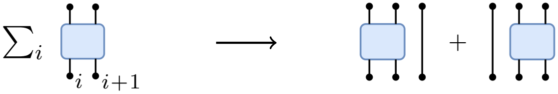
Amplification:



Reinterpret as spectral gap (+detectability lemma):



Knabe bound:



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also [Gosset-Mozgunov], [Lemm-Mozgunov]

Can exactly compute the second moment gap

$$\Delta(H_{n=3,k=2}) = \frac{3}{5}$$

Moreover, using almost-orthogonality of g.s. can show that for $q \geq 6k^2$

$$\Delta(H_{n=3,k}) \geq \frac{3}{4}$$

(Almost) Linear designs from spectral gaps

These lower bounds on the $n = 3$ gap allow us to conclude:

Theorem

RQCs on n qubits form ϵ-approximate 2-designs when

$$t \geq 20(4n \log q + \log 1/\epsilon)$$

and RQCs on n qudits with local dim $q \geq 6k^2$ form ϵ-approximate k-designs when

$$t \geq 18(2nk \log q + \log 1/\epsilon) \quad \rightarrow \quad t = O(nk \log k)$$

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$$t \geq 18(2nk \log q + \log 1/\epsilon) \quad \rightarrow \quad t = O(nk \log k)$$

More importantly for near-term applications of RQCs: find good constants from analytically and numerically computing the gaps

Can also improve design depths for non-local RQCs

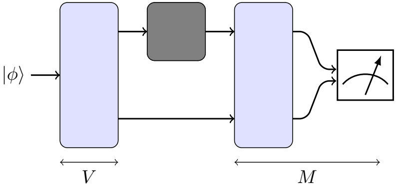
Complexity from measurements

We can consider an alternative (stronger) definition of the complexity of a state or unitary, in terms of an optimal distinguishing measurement

Roughly, the strong complexity of U is the minimal circuit required to implement an ancilla-assisted measurement capable of distinguishing $\mathcal{U}$ from the completely depolarizing channel $\mathcal{D}$

Task is to distinguish the channels with restricted state preparation and measurements as

$$\begin{aligned} &\text{maximize} \quad \left|\mathrm{Tr}\Big(M\big((\mathcal{U}\otimes\mathcal{I})|\phi\rangle\langle\phi| - (\mathcal{D}\otimes\mathcal{I})|\phi\rangle\langle\phi|\big)\Big)\right| \\ &\text{subject to} \quad M \in M_{r'},\ \ |\phi\rangle = V\,|0\rangle\,,\ V \in G_r \end{aligned}$$



Complexity from measurements

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Definition (strong δ-unitary complexity)

A unitary $U \in U(d)$ has strong δ-complexity of at most r if

$$\beta(r, U) \geq 1 - \frac{1}{d^2} - \delta$$

which we denote as $\mathcal{C}_\delta(U) \leq r$ and where the optimal bias to distinguish the channels with restricted state preparation and measurements is

$$\begin{aligned} \beta(r, U) = \text{maximize} \quad & \left|\mathrm{Tr}\Big(M\big((\mathcal{U} \otimes \mathcal{I})|\phi\rangle\langle\phi| - (\mathcal{D} \otimes \mathcal{I})|\phi\rangle\langle\phi|\big)\Big)\right| \\ \text{subject to} \quad & M \in M_{r'},\ \ |\phi\rangle = V\,|0\rangle\,,\ V \in G_{r''},\ r = r' + r'' \end{aligned}$$