IKKT thermodynamics and early universe cosmology

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Cosmic Microwave Background

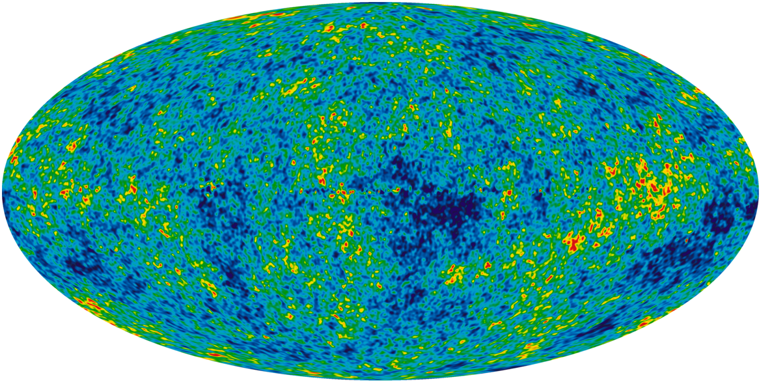


Figure: Wilkinson Microwave Anisotropy Probe (WMAP) heat map of temperature fluctuations in the cosmic microwave background.

Temperature Anisotropies

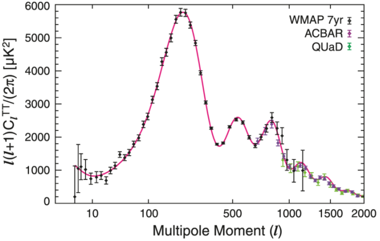


Figure: Angular power spectrum of CMB temperature fluctuations.

Harmonic expansion

$$\Theta(\hat{n}) = \frac{\Delta T(\hat{n})}{T_0} = \sum_{lm} a_{lm} Y_{lm}(\hat{n})$$

Angular power spectrum

$$C_l^{TT} = \frac{1}{2l+1} \sum_m \langle a_{lm}^* a_{lm} \rangle$$

Relation to the power spectrum of scalar fluctuations

$$C_l^{TT} = \frac{2}{\pi}\int k^2 dk \quad \underbrace{P_{\mathcal{R}}(k)}_{\text{Power spectrum}} \quad \underbrace{\Delta_{Tl}(k)\Delta_{Tl}(k)}_{\text{Anisotropy transfer functions}}$$

Large scale transfer function: $\quad \Delta_{Tl} = \frac{1}{3} j_l(k[\eta_0 - \eta_{rec}])$

Angular power spectrum on large scales

$$C_l^{TT} \propto \underbrace{k^3 P_{\mathcal{R}}(k)|_{k\approx l/(\eta_0-\eta_{rec})}}_{\Delta_s^2(k)|_{k\approx l/(\eta_0-\eta_{rec})}} \underbrace{\int d\ln x j_l^2(x)}_{\propto l(l+1)}$$

Dimensionless angular spectrum

$$\mathcal{C}_l \equiv \frac{l(l+1)}{2\pi} \propto \Delta_s^2(k)|_{k\approx l/(\eta_0-\eta_{rec})} \propto l^{n_s-1}$$

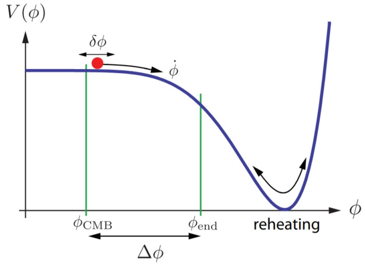


Figure: Example of inflaton potential. The inflaton slowly rolls down the potential until the conditions for inflation are broken.

Single field slow-roll action

$$S = \frac{1}{2}\int dx^4 \sqrt{-g}\left[R - (\partial_\mu \phi)^2 - 2V(\phi)\right]$$

Equation of state parameter

$$\omega_\phi \equiv \frac{p_\phi}{\rho_\phi} = \frac{\dot{\phi}^2 - 2V}{\dot{\phi}^2 + 2V}$$

Slow roll conditions

$$\dot{\phi}^2 \ll V(\phi)$$
$$|\ddot{\phi}| \ll |3H\dot{\phi}|, |V,\phi|$$

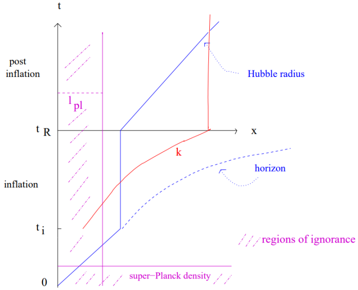


Figure: Propagation of modes during inflation.

Second order expansion of the action (comoving gauge)

$$S = \frac{1}{2}\int dx^4 a^3 \frac{\dot{\phi}^2}{H^2}\left[\dot{\mathcal{R}}^2 - a^{-2}(\partial_i \mathcal{R})^2\right]$$

$$\delta\phi = 0 \quad , \quad g_{ij} = a^2[(1-2\mathcal{R})\delta_{ij}]$$

Relation to the power spectrum

$$\langle \mathcal{R}_k \mathcal{R}_{k'} \rangle = (2\pi)^3 \delta(k+k') P_{\mathcal{R}}(k)$$

Inflationary power spectrum

$$\Delta^2_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2} P_{\mathcal{R}}(k) = \frac{H_*^2}{(2\pi)^2}\frac{H_*^2}{\dot{\phi}_*^2}$$

Swampland Program

Recent conjectures from the Swampland program put inflation under tight constraint:

- Refined de Sitter conjecture : The scalar potential of a theory coupled to gravity must satisfy one of the conditions

$$|\nabla V| \geq cV \quad , \quad \min(\nabla_i \nabla_j V) \leq -c' V$$

  where $c, c' > 0$ are of order one, to be consistent with string theory.

- Trans-Planckian Censorship Conjecture: Sub-Planckian quantum fluctuations should remain quantum.

$$T \leq H^{-1} \ln(H^{-1})$$

  This imposes a bound on the duration of inflation (see above).

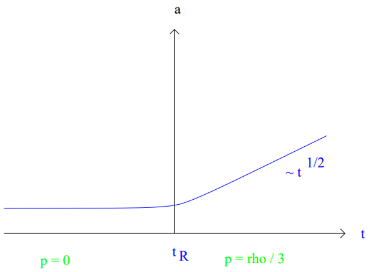


Figure: Evolution of the scale factor in an emergent scenario.

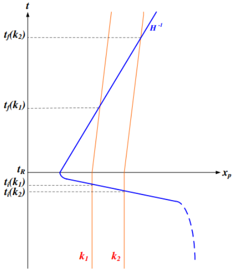


Figure: Propagation of modes during an emergent scenario

Examples of emergent scenarios:

1. String gas cosmology
2. Emergent universe from the BFSS matrix model
3. Emergent universe from the IKKT matrix model

The IKKT model is a non-perturbative formulation of type IIB theory in ten dimensions. The action of the IKKT model is given by

$$S_{IKKT} = -\mathrm{Tr}\left(\frac{1}{4g^2}[A_\mu, A_\nu][A^\mu, A^\nu] + \frac{1}{2g^2}\bar{\psi}\Gamma^\mu[A_\mu, \psi]\right)$$

where A_μ ($\mu = 0, \dots, 9$) and ψ are $N \times N$ Hermitian matrices.

Here, space-time is described by the matrix elements A_0 and A_i, which hold information about time and space respectively.

[N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, , Nucl. Phys. B **498**, 467-491 (1997)]

The IKKT action can be derived from the schild action of a Type IIB string

$$S_{Schild} = \int d\sigma^2 \sqrt{g} \left[\alpha \left(\frac{1}{4} \{X^\mu, X^\nu\}^2 - \frac{i}{2} \bar{\psi} \Gamma^\mu \{X_\mu, \psi\} \right) + \beta \right] ,$$

by replacing the Poisson brackets by commutators and the integral by a trace

$$\{,\} \implies -i[,] \quad , \quad \int d^2\sigma \sqrt{g} \implies Tr\,.$$

We obtain the IKKT action

$$S_{IKKT} = \alpha \left(-\frac{1}{4} Tr[A_\mu, A_\nu]^2 - \frac{1}{2} Tr \left(\bar{\psi} \Gamma^\mu [A_\mu, \psi] \right) \right) + \beta\, Tr 1$$

plus a constant β term that can be neglected.

Infinitely long static D-string solutions can be fund by solving the Schild/IKKT equations of motion and setting $\psi = 0$.

Schild equation of motion

$$\{X_\mu, \{X^\mu, X^\nu\}\} = 0$$

Static D-string in X^1 direction

$$X^0 = \tau$$
$$X^1 = \sigma$$
$$X^\mu = 0 \quad \text{Otherwise}$$

Here, τ and σ are continuous parameters that parametrize the worldsheet.

IKKT equation of motion

$$[A_\mu, [A^\mu, A^\nu]] = 0$$

Static D-string in X^1 direction

$$A^0 = q$$
$$A^1 = p$$
$$A^\mu = 0 \quad \text{Otherwise}$$

Here, q and p are matrix operators that describe the string geometry and satisfy $[q, p] = i\mathbf{1}$.

Higher dimensional objects can be found by generalizing the string solution. Consider solutions of the EoM's which satisfy

$$[A_\mu, A_\nu] = ic_{\mu\nu}\mathbf{1}\,,$$

where $c_{\mu\nu}$ is a 10 by 10 antisymmetric matrix of constants. $c_{\mu\nu}$ can always be expressed in the Jordan canonical form

$$c_{\mu\nu} = \begin{bmatrix} 0 & \omega_1 & & & \\ -\omega_1 & 0 & & & \\ & & \dots & & \\ & & & 0 & \omega_5 \\ & & & -\omega_5 & 0 \end{bmatrix}.$$

Solutions with $(p+1)/2$ non zero coefficients out of five ω_k take the form

$$A_\mu = (Q_1, P_1, ..., Q_{(p+1)/2}, P_{(p+1)/2}, 0, ..., 0)\,,$$

where each Q_k's and P_K's satisfy

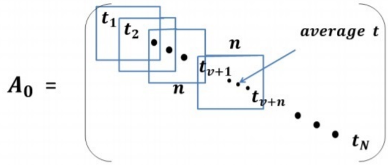
$$[Q_k, P_k] = i\omega_k\,.$$

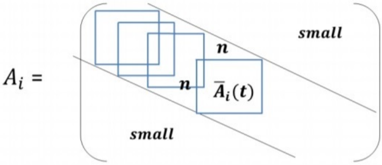
This class of solution describe $(p+1)$-dimensional static objects that are similar to D_p branes. When $p=1$, we recover the string solution

$$A_\mu = (Q, P, 0, ..., 0)\,,$$

that we described before.

To find dynamical solutions, we must minimize the IKKT action (including fermions) using Monte-Carlo methods. When A^0 is chosen to be diagonal, A^i has a band diagonal structure (see below).





Time parameter

$$t \equiv \frac{1}{n} \sum_{a=1}^{n} t_{\nu+a}$$

Time evolving matrix element

$$\bar{A}_i^{ab}(t) \equiv \langle t_{\nu+a} | A_i | t_{\nu+b} \rangle$$

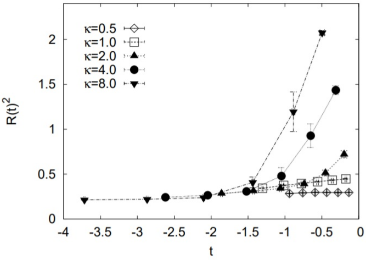


Figure: The extent of space $R(t)^2$ becomes large at a critical time t_c.

Extent of space parameter

$$R(t)^2 \equiv \frac{1}{n}\mathrm{Tr}\bar{A}_i(t)^2$$

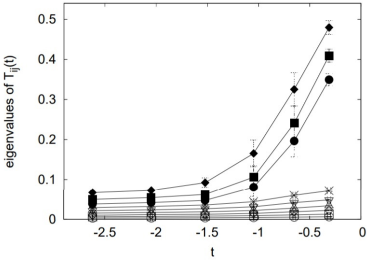


Figure: 3 out of 9 eigenvalues of T_{ij} become large at a critical time t_c.

Moment of inertia tensor

$$T_{ij}(t) \equiv \frac{1}{n}\mathrm{Tr}\{\bar{A}_i(t)\bar{A}_j(t)\}$$

[S. W. Kim, J. Nishimura and A. Tsuchiya, Phys. Rev. Lett. **108**, 011601 (2012)]

Background evolution

$$ds^2 = (1+2\Phi)dt^2 - a(t)^2[(1-2\Phi)\delta_{ij} + h_{ij}]dx^i dx^j$$

Equations of motion for the perturbations (at linear order)

$$\nabla^2\Phi = 4\pi G \delta T^0_0 \quad , \quad \nabla^2 h_{ij} = -4\pi G \delta T^i_j$$

Power spectrum of perturbations

$$P_\Phi(k) = k^3 \langle |\Phi(k)|^2 \rangle = 16\pi^2 G^2 k^{-4} \langle \delta T^0_0(k) T^0_0(k) \rangle_R$$
$$P_h(k) = k^3 \langle |h(k)|^2 \rangle = 16\pi^2 G^2 k^{-4} \langle \delta T^i_j(k) T^i_j(k) \rangle_R$$

Correlation functions for matter in a thermal state

$$\langle \delta T^0_0(k) T^0_0(k) \rangle_R = \langle \rho^2 \rangle_R - \langle \rho \rangle_R^2 = \frac{T^2}{R^6} C_V$$
$$\langle \delta T^i_j(k) T^i_j(k) \rangle_R = \langle (T^i_j)^2 \rangle_R - \langle T^i_j \rangle_R^2 = \alpha \frac{T}{R^2} \frac{\partial \tilde{p}}{\partial R} \quad i \neq j$$

String action (Euclidean target space)

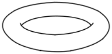
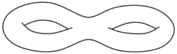
$$S = -T \int d\tau d\sigma \, \left(\partial_\alpha X_\mu \partial^\alpha X^\mu - i \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \right) \quad , \quad g_{\mu\nu} = \delta_{\mu\nu}$$

Conditions to obtain a theory at finite temperature

1. Space-time bosons are periodic under $X^0 \to X^0 + \beta$
2. Space-time fermions are anti-periodic under $X^0 \to X^0 + \beta$

Free energy of the system

$$F = -\beta \ln Z \quad , \quad Z = \int DXD\psi e^{-S}$$

$$= -\beta \ln \Bigg[$$

$$+$$

$$+ \dots \Bigg]$$

Euclidean IKKT action ($A^0 \to iA^0$, $\Gamma^i \to i\Gamma^i$)

$$S_{IKKT} = -\mathrm{Tr}\left(\frac{1}{4g^2}[A_\mu, A_\nu][A^\mu, A^\nu] + \frac{i}{2g^2}\bar{\psi}\Gamma^\mu[A_\mu, \psi]\right)$$

Constraint equation for compactification on S_1 + anti-periodic boundary conditions for the fermions

$$U^{-1}A^0U = A^0 + 2\pi\beta$$
$$U^{-1}A^iU = A^i$$
$$U^{-1}\psi U = -\psi\,.$$

$U \equiv$ some unitary matrix that generates a translation of magnitude $2\pi\beta$

Translation operator

$$U = e^{-i2\pi q} e^{-ip} \quad , \quad [q, p] = i$$

Solutions to the constraint equations

$$A^0 = \sum_{n\in\mathbb{Z}} A_n^0 e^{inp} + 2\pi\beta q$$

$$A^i = \sum_{n\in\mathbb{Z}} A_n^i e^{inp}$$

$$\psi = \sum_{r\in\mathbb{Z}+1/2} \psi_r e^{irp}$$

Compact IKKT action

$$
\begin{aligned}
S_{IKKT} = \frac{N}{2g^2}\mathrm{Tr}\Bigg(&\sum_n (2\pi\beta n)^2 A^i_{-n}A^i_n + i\sum_r 2\pi\beta r\psi_{-r}C_{10}\Gamma^0\psi_r \\
&+\sum_{nm} 4\pi\beta n[A^0_{-n-m},A^i_m]^2 A^i_n - \sum_{nml}[A^0_{-n-m-l},A^i_l][A^0_m,A^i_n] \\
&-\frac{1}{2}\sum_{nml}[A^i_{-n-m-l},A^j_l][A^i_m,A^j_n] - i\sum_{rn}\psi_{-r-n}C_{10}\Gamma^0[A^0_n,\psi_r] \\
&-i\sum_{rn}\psi_{-r-n}C_{10}\Gamma^i[A^i_n,\psi_r]\Bigg)
\end{aligned}
$$

Compact IKKT action

$$S_{IKKT} = \frac{\beta}{2g^2}\int d\tau \mathrm{Tr}\left((D_\tau X^i)^2 - \frac{1}{2}[X^i, X^j]^2 + \bar{\psi}\Gamma^0 D_\tau \psi - i\bar{\psi}\Gamma^i[X^i, \psi]\right)$$

Mode expansion

$$X^0 = \sum_n X_n^0 e^{i2\pi\beta nt} \quad , \quad X^i = \sum_n X_n^i e^{i2\pi\beta nt} \quad , \quad \psi = \sum_r \psi_r e^{i2\pi\beta rt}$$

BFSS model action (after $T \to 1/T$ and a redefinition of g^2)

$$S_{BFSS} = \frac{1}{2g^2}\int d\tau \mathrm{Tr}\left((D_\tau X^i)^2 - \frac{1}{2}[X^i, X^j]^2 + \bar{\psi}\Gamma^0 D_\tau \psi - i\bar{\psi}\Gamma^i[X^i, \psi]\right)$$

Partition function of the IKKT model at finite temperature

$$Z = \int dA d\psi e^{-S_{IKKT}} = \int dA d\psi e^{-S_0 - S_\omega - S_{int}}$$

Free energy of the system

$$F = -T \ln Z = -T \ln Z_0 - TM^2(D-2)\ln(\beta) - \frac{1}{2} TpM^2 \ln 2 + \left(\frac{D-2}{12} - \frac{p}{8}\right)\frac{D-2}{D-1} M^2 T^3 R_0^2 + \ldots$$

Energy of the system

$$\begin{aligned} E &= -\partial_\beta \ln Z \\ &= -M^2(D-2)T - 2\left(\frac{D-2}{12} - \frac{p}{8}\right)\frac{D-2}{D-1} M^2 T^3 R^2 + \ldots \end{aligned}$$

Power spectrum of scalar perturbations

$$P_\Phi(k) = 16\pi^2 G^2 k^2 T^2 C_V (kR)^{-6} \quad , \quad C_V = \left(\frac{\partial E}{\partial T}\right)_V$$
$$\approx 96\pi^2 G^2 (kR)^{-4} \left(\frac{p}{8} - \frac{D-2}{12}\right) \frac{D-2}{D-1} M^2 T^4$$

Power spectrum of tensor perturbations

$$P_\Phi(k) = 16\pi^2 G^2 k^2 T^3 \alpha \frac{\partial \tilde{p}}{\partial R} (kR)^{-2} \quad , \quad \tilde{p} = -\frac{1}{V} \frac{\partial F}{\partial \ln R}$$
$$= 32\pi^2 G^2 (kR)^{-4} \alpha \left(\frac{D-2}{12} - \frac{p}{8}\right) \frac{D-2}{D-1} M^2 T^4$$

- Inflation is not the only mechanism for large scale structure formation in the universe. There are alternate mechanisms (e.g. emergent scenarios).
- The IKKT model is a proposed non-perturbative description of type IIB string theory.
- A thermodynamic description of this model can be obtained by compactification on a torus where fermions acquire anti-periodic boundary condition.
- This model admits emergent universe solutions (Kim, Nishimura, Tsuchiya (2012)).
- If the universe begins in a thermal state of the IKKT model, then thermal fluctuations can source a scale invariant spectrum of cosmological perturbations.

Thank You

for your attention.