Complexity and BMS

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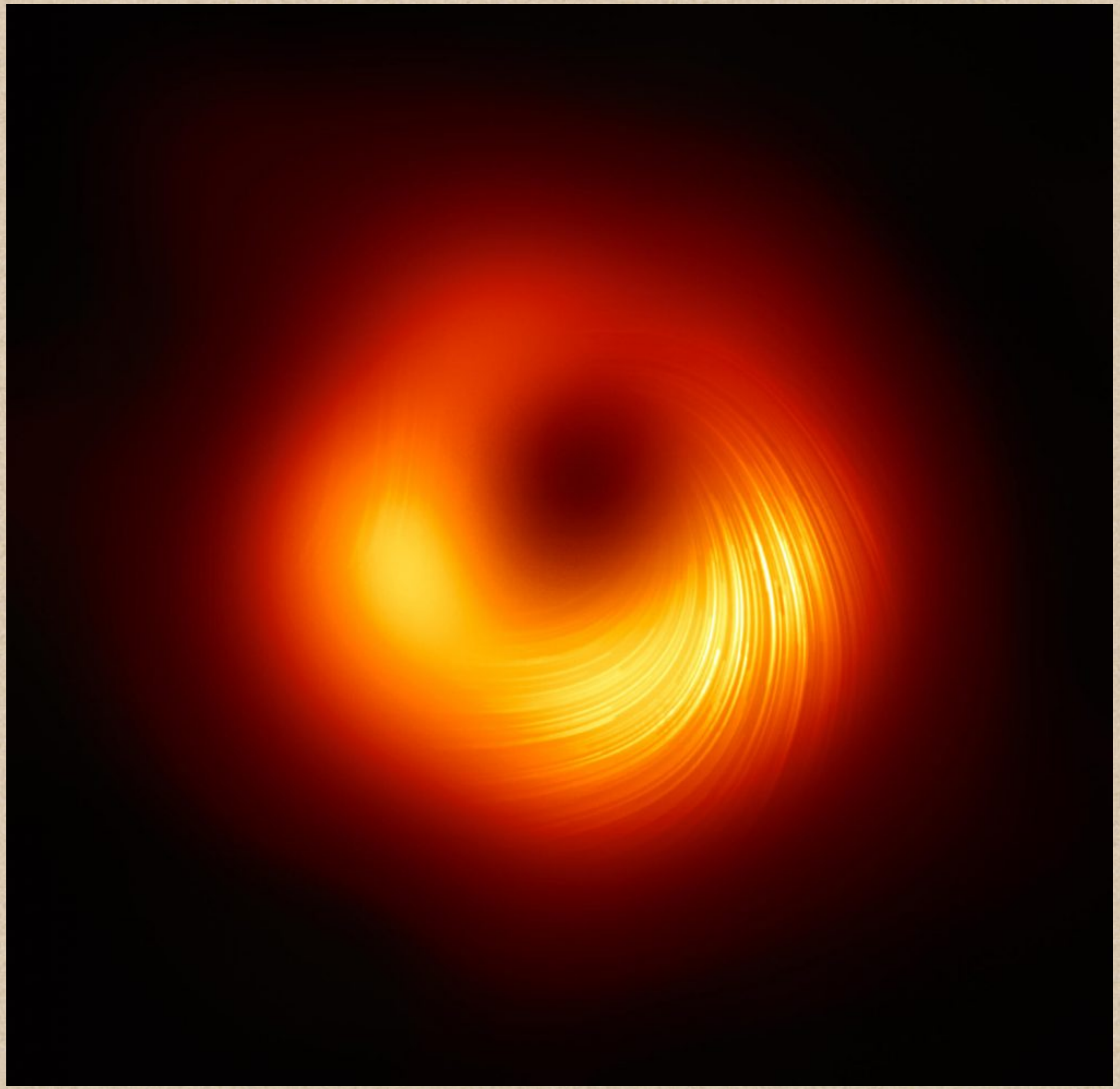
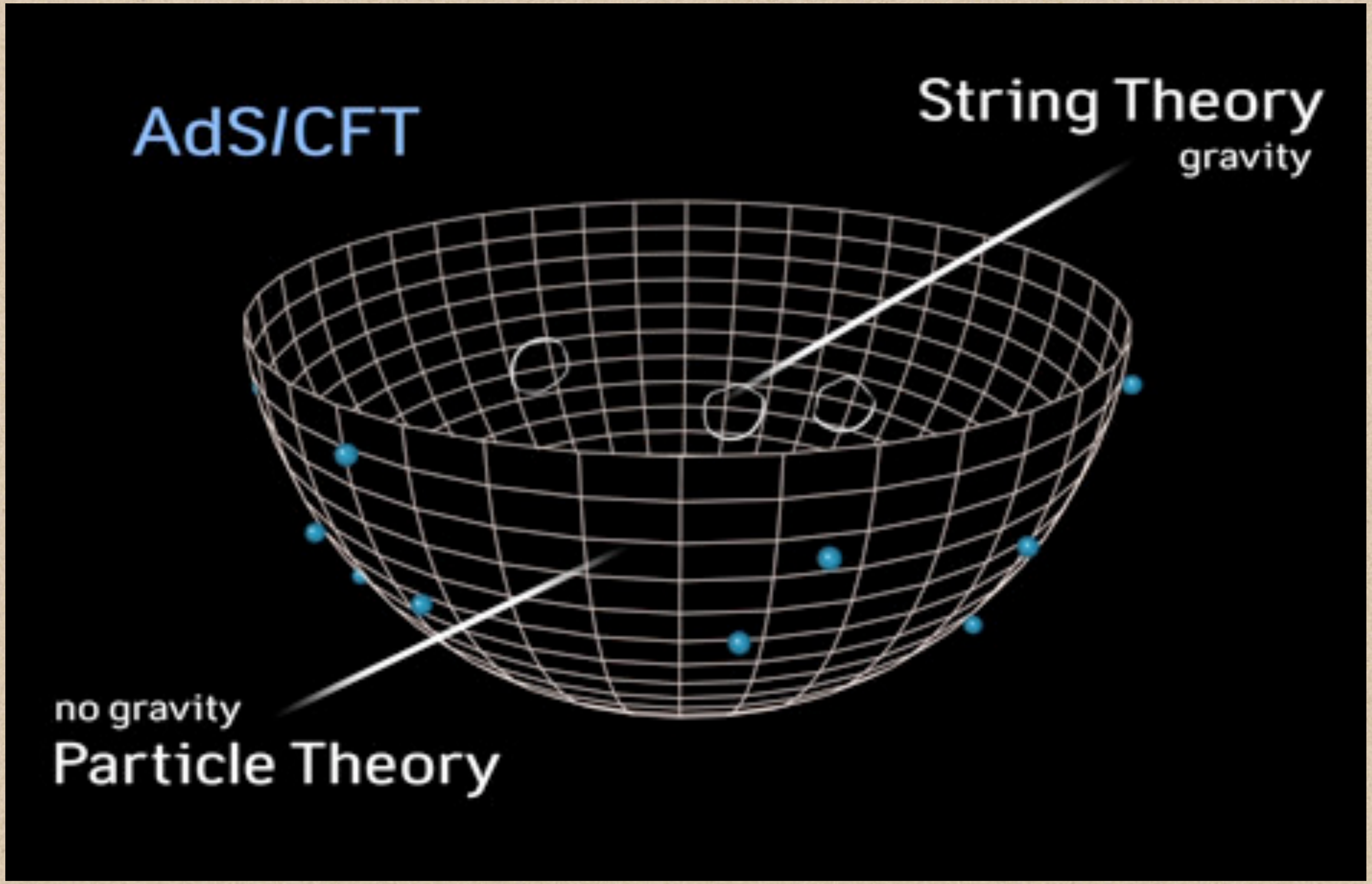
Based on 2301.12845 with Arpan Bhattacharya

Introduction: Holography



$$S_{\mathrm{BH}} = \frac{Ac^3}{4G\hbar}$$



Most prominent example of the Holographic Principle:
The AdS/CFT Correspondence

AdS/CFT

Condensed Matter Physics (Superconductivity, Optics, ...)

Quantum Information (Entanglement entropy, Quantum Complexity,...)

Hydrodynamics

But, our world is clearly Not AdS!

For many astrophysical purposes, the universe is well approximated by Flat spacetimes

Complexity in Flatspace holography?

Outline of the rest of talk:

- Quantum Computational Complexity: General Notion
- Complexity in AdS/CFT Holography: 2d CFT Revisited
- BMS_3 in Flat Space Holography
- Complexity in Flat Holography
  - Limiting approach: From CFT to BMS
  - Intrinsic Analysis
- Summary and Discussions

Complexity in AdS/CFT Holography

Quantum Computational Complexity

- The notion of computational complexity is borrowed from quantum information. [Nielsen et al 2006, 2007]
- Describes the minimum number of operations (gates) to reach from one reference state $|\psi_R\rangle$ to the target state $|\psi_T\rangle$. [Watrous 2008]
- For discrete system: an extremal circuit consisting of quantum gates starting from $|\psi_R\rangle$ to $|\psi_T\rangle$.
- For field theories: a geodesic distance in the manifold of unitary (group) operators.
- Also characterises quantum chaos and grows linearly with time in chaotic systems, and responds to perturbations distinctly. [See Dymarsky's talk and later talks today]
- Similarity with the behaviour of the volume behind the event horizon of a black hole.

Complexity in Holography

- Conjectured primarily in the context of AdS/CFT. [Nice review in Chapman, Policastro 2021]
- New addition to the holographic dictionary through AdS/CFT, connecting several aspects of black hole physics, e.g. the interior and information processing inside the black hole.
  - Complexity=Volume (CV) [Stanford, Susskind 2014]
  - Complexity=Action (CA) [Brown, Roberts, Susskind, Swingle, Zhao 2015]
  - Complexity= Geometric action [Caputa, Megan 2018, Erdmenger et al 2020]
- There have been different notions of complexity and successful applications in many condensed matter and quantum field theories.
- Very little progress in extending the computation of holographic complexity for non-Lorentz invariant field theories.

Complexity for 2d CFT: Review

- The symmetry generators for 2d CFTs give two copies of the Virasoro algebra.

$$[\mathscr{L}_m, \mathscr{L}_n] = (m-n)\mathscr{L}_{m+n} + \frac{c}{12}(n^3-n)\delta_{m+n,0},\ [\bar{\mathscr{L}}_m, \bar{\mathscr{L}}_n] = (m-n)\bar{\mathscr{L}}_{m+n} + \frac{\bar{c}}{12}(n^3-n)\delta_{m+n,0}.$$

- Group elements of Virasoro $(f(\sigma), \mathfrak{a})$: $f(\sigma)$ diffeos on $\mathbb{S}^1$, $\mathfrak{a} \in \mathbb{R}$ from central extension.
- $\sigma \to f(\sigma)$: conformal transformations in 2d ($\sigma \in \mathbb{S}^1$).
- Geometric notion of Nielsen's complexity defines an extremised path (geodesic) in the manifold of the group transformations from $|\psi_R\rangle$ to $|\psi_T\rangle$.
- Use infinitesimal symmetry transformations as gates, $|\psi_T\rangle = U_{f(T)}|\psi_R\rangle$.
- Group elements are related by instantaneous group velocities: $f(t+dt,\sigma) = e^{\epsilon(t,\sigma)dt} \cdot f(t,\sigma)$

Circuit complexity functional = Geometric action on Virasoro co-adjoint orbit

[also for Kac-Moody]

[Caputa, Magan 2018; Erdmenger, Gerberhagen, Weigel 2020]

Circuit Complexity for Virasoro: More Details

[Caputa, Magan 2018; Erdmenger, Gerberhagen, Weigel 2020]

- $|\psi_T\rangle = U_{f(T)}|\psi_R\rangle$, T is the time to reach from $|\psi_R\rangle$ to $|\psi_T\rangle$. Unitary representations of the group elements: U_f.
- Decomposing in terms of infinitesimal transformations $U_{f(T)} = U_{\epsilon(T)}U_{\epsilon(T-dt)}\dots U_{\epsilon(dt)}1$; initial state is the reference state $U_{f(t=0)} = 1$.
- In 2d CFT, the conserved energy-momentum tensor $T(\sigma)$ is used to write the gates, $Q(t) = \frac{1}{2\pi}\int d\sigma\, \epsilon(t,\sigma)\, T(\sigma)$.
- $\epsilon(t,\sigma)$ describes the infinite symmetry transformations applying at a given time, denotes time-dependent group velocity.
- Choice of cost function: $\mathcal{F} = |tr[\rho(t)Q(t)]| = |\langle\psi_R|\,U_f^\dagger(t)\,Q(t)\,U_f(t)\,|\psi_R\rangle|$
- Density matrix $\rho(t) = U(t)\rho_0 U^\dagger(t)$ is obtained from the initial density matrix $\rho_0 = |\psi_R\rangle\langle\psi_R|$.

  * Complexity functional: $C[f] = \int dt\mathcal{F} = \frac{1}{2\pi}\int dt \int d\sigma\, \epsilon(t,\sigma)\langle\psi_R|\,U^\dagger(t)T(\sigma)U(t)\,|\psi_R\rangle$.

- $C[f]$ gives total cost of a constructed path from $|\psi_R\rangle$ to $|\psi_T\rangle$ in the group manifold in terms of group element $f(t)$.

Circuit Complexity for Virasoro: More Details

[Caputa, Magan 2018; Erdmenger, Gerberhagen, Weigel 2020]

- Transformed current $U_f^\dagger \, T \, U_f$ is written in terms of inverse diffeomorphism F: $F(t, f(t,\sigma)) = \sigma$.
  $U_f^\dagger T U_f = F'^2 T(F) + \frac{c}{12}\{F, \sigma\}$.
- Thus, group velocity: $\epsilon(t,\sigma) = \frac{\partial f(t, F(t,\sigma))}{\partial t} = -\frac{\dot{F}(t,\sigma)}{F'(t,\sigma)} = \theta$. [$\theta$ resembles Maurer-Cartan form for Virasoro].
- Choose $|\psi_R\rangle = |h, \bar{h}\rangle$ (CFT primaries).
- The contribution due to the central extension modifies the cost function:
  $\mathcal{F} = c\beta(t) + \int d\sigma \, \epsilon(t,\sigma) \langle \psi_R | U^\dagger(t) T(\sigma) U(t) | \psi_R \rangle$. [$\beta(t)$ is the central extension of the Maurer-Cartan form].

$$C[F] = \int d\sigma \, dt \left[-j_0(F)\dot{F}F' + \frac{c}{48\pi} \frac{\dot{F}}{F'} \left(\frac{F'''}{F'} - 2\left(\frac{F''}{F'}\right)^2 \right) \right]$$

Complexity functional=Geometric action

- Minimise $C[f]$ in terms of the group path $f(t,\sigma)$ and solve the equations of motion.
- The solution gives the optimal circuit, and we put it back to $C[f]$ to find the complexity value.
- Simplest solution: $\frac{\dot{f}}{f'} = \text{const}$, $C[f](t) \propto |h - \frac{c}{24}| \, t$. [Caputa, Magan 2018]

BMS and Flat Space Holography

BMS in Flatspace Holography

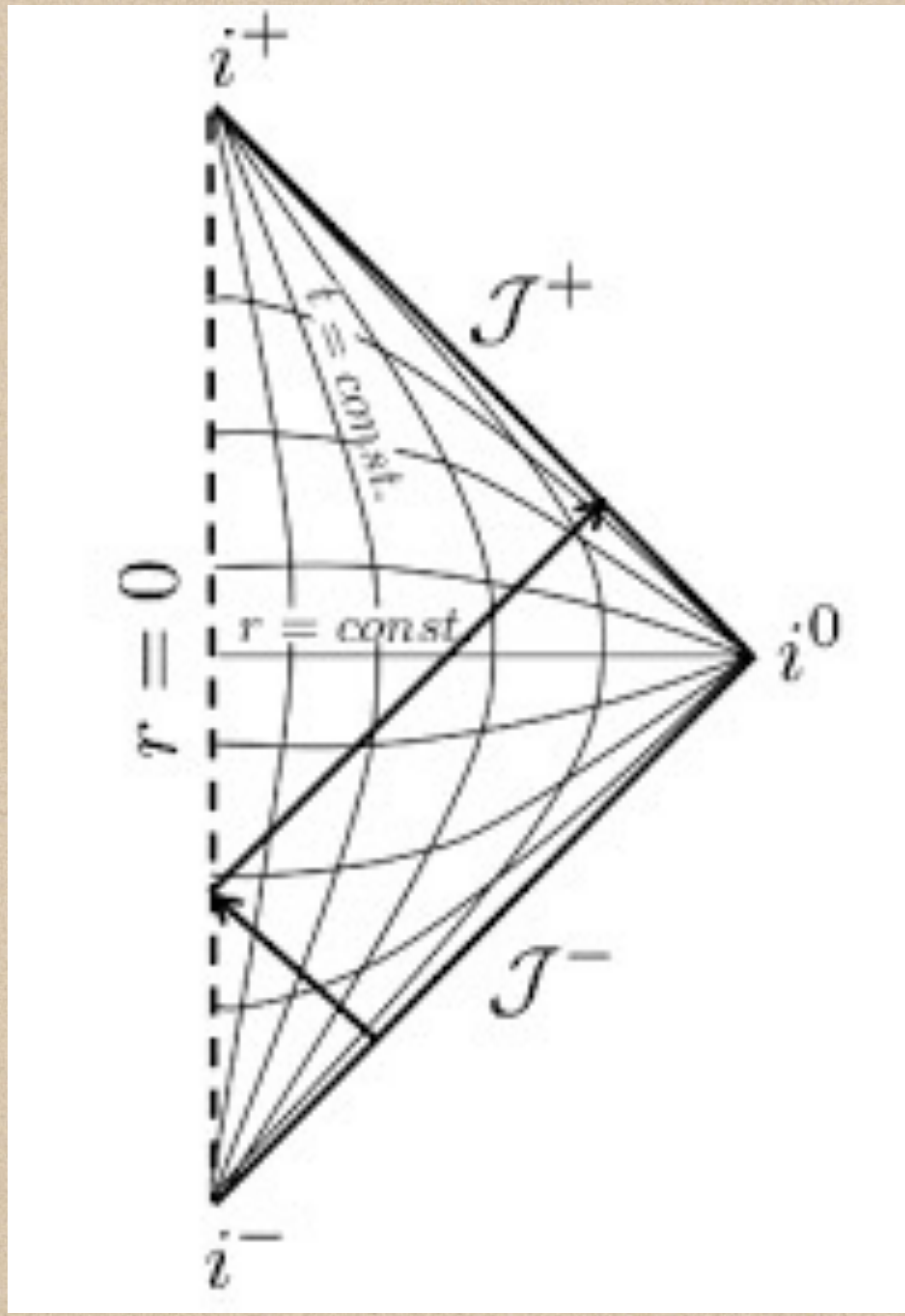
- Symmetry plays an important role in nature.
- Symmetries at the boundary of the spacetime are given by Asymptotic Symmetry Groups (ASG).
- The asymptotic symmetry of AdS_3 is enhanced to two copies of infinite dimensional Virasoro algebra. [Brown, Henneaux 1986]
- ASG of asymptotic flat spacetimes at the null boundary ($\mathscr{I}^{\pm}$) is Bondi-Metzner-Sachs (BMS) group. [Bondi, Burg, Metzner 1962, Sachs 1962]
- For Minkowski spacetime in bulk dimensions D=3, the dual field theory lives on its null boundary in d=2, and the ASG is BMS_3, also declared to be the symmetry of the dual field theory. [Bagchi, 2010]
- BMS group is also important for Soft graviton / Asymptotic symmetry correspondence, Symmetries on the black hole horizon. [Hawking, Perry, Strominger 2016].

BMS in Flatspace holography

- BMS group is infinite-dimensional in bulk dimension $D=3,4$.
- We concentrate on the asymptotic symmetry of 3D bulk flat spacetime, the BMS_3 algebra

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{c_L}{12}\delta_{n+m,0}(n^3-n)$$
$$[L_n, M_m] = (n-m)M_{m+n} + \frac{c_M}{12}\delta_{n+m,0}(n^3-n), \quad [M_n, M_m] = 0.$$

- The structure on null boundary $\mathscr{I}^+$ is $\mathbb{R}\times\mathbb{S}^1$.
- Here, M_n generates supertranslations: angle dependent translations on the null direction
- L_n generates super-rotations: Diffeomorphisms on $\mathbb{S}^1$ at null boundary
- For Einstein gravity, $c_L = 0,\ c_M = \frac{3}{G}$ [Barnich, Compere 2007]
- We call the dual BMS_3 invariant field theory in 2d boundary as BMSFT.



Penrose Diagram of Minkowski Spacetime

Flat space holography from AdS/CFT

- The asymptotic symmetry algebra for AdS_3 are given by 2 copies of the Virasoro algebra.

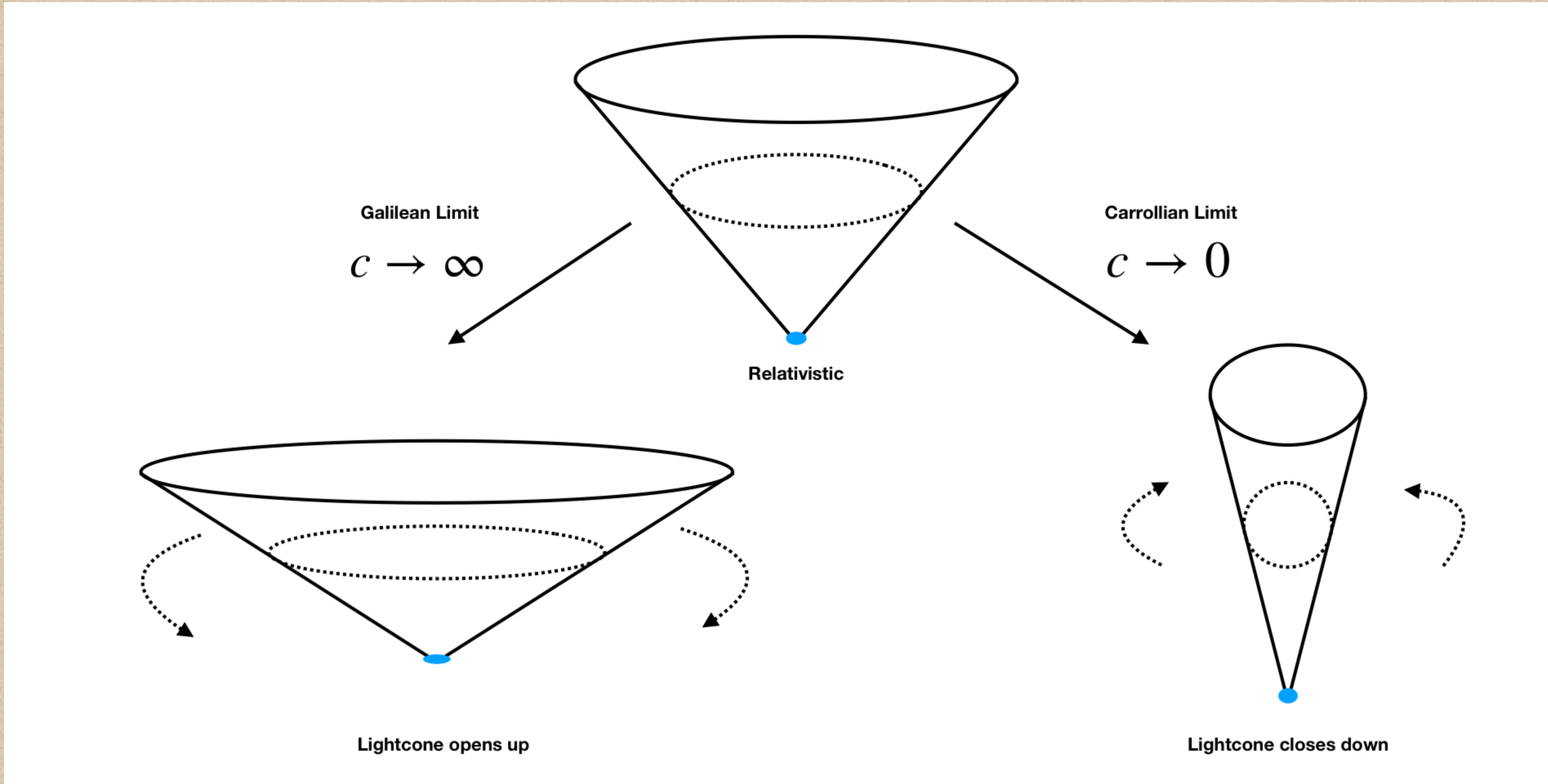
$$[\mathscr{L}_n, \mathscr{L}_m] = (n-m)\mathscr{L}_{n+m} + \frac{c}{12}\delta_{n+m,0}(n^3-n), \ [\bar{\mathscr{L}}_n, \bar{\mathscr{L}}_m] = (n-m)\bar{\mathscr{L}}_{n+m} + \frac{\bar{c}}{12}\delta_{n+m,0}(n^3-n)$$

$$[\mathscr{L}_n, \bar{\mathscr{L}}_m] = 0, \qquad c = \bar{c} = \frac{3l}{2G}$$

- Here, l is the AdS radius and G is Newton's constant.
- Next, we take the limit. $L_n = \mathscr{L}_n - \bar{\mathscr{L}}_{-n}, \ M_n = \epsilon(\mathscr{L}_n + \bar{\mathscr{L}}_{-n}),$ and $\epsilon = \frac{1}{l} \to 0$ [Bagchi 2010]
- The contracted generators L_n, M_n generate BMS_3 algebra, the asymptotic symmetry algebra of 3D flat spacetime.
- The central terms for BMS_3 become $c_L = c - \bar{c} = 0$ and $c_M = \epsilon(c + \bar{c}) = \frac{3}{G}$.
- The $\epsilon \to 0$ limit corresponds to the flat space limit on bulk AdS spacetime. [Bagchi, Fareghbal '12]
- The flat space limit in the bulk corresponds to taking Ultra Relativistic (UR) limit (also known as Carrollian limit) on the boundary dual field theory.

BMS Invariant (Carrollian Conformal) Field Theories

- Carrollian limit : $x^i \to x^i,\ t \to \epsilon t,\ \epsilon \to 0$ Then , $\frac{v}{c} = \frac{1}{c}\frac{x}{t} \to \infty,\ \implies\ c \to 0.$
- Relativistic CFT $\xrightarrow{\text{Carrollian limit}}$ Carrollian CFT (CCFT). [Bagchi, Mehra, Nandi 2019, Bagchi, Basu, Mehra, Nandi 2020]
- The (pseudo) Riemannian structure fails on the null surface as the metric degenerates and the light cone closes up: emergence of Carrollian Structure. [Levy-Leblond 1965, Sengupta 1966]
- CCFTs live on the null manifold (event horizon, $\mathscr{I}^{\pm}$ of the asymptotically flat spacetimes), non-Lorentzian in nature.
- BMS group and Carrollian Conformal Group are isomorphic. $BMS_{d+1} = CCFT_d$.
- For dual field theory in 2d, isomorphism arises between the Carrollian and the Galilean limit of relativistic CFTs.
- The Galilean limit is the opposite to the UR limit and here $c \to \infty$ and the light cone opens up.
- This is the Non-Relativistic (NR) limit realised in terms of spacetime contraction as: $x^i \to \epsilon x^i,\ t \to t,\ \epsilon \to 0.$
- The isomorphism between NR and UR limit exists only in 2d, as only one of the directions (spatial or time) gets contracted. (We use this isomorphism during our calculation.)



Light Cones opening/collapsing due to Galilean/Carrollian limits [Pic: Bagchi, Banerjee, Muraki 2022]

Circuit Complexity in BMS_3

Circuit Complexity: From CFT_2 to BMS_3

- The structure on $\mathscr{I}^+$, for 3d asymptotically flat spacetime, is $\mathbb{R} \times \mathbb{S}^1$.
- $\text{BMS}_3 = \underbrace{\text{Diff}(S^1)}_{\text{Super-rotation}} \ltimes \underbrace{\text{Vec}(S^1)}_{\text{Supertranslation}}$. Elements are denoted by $(f, c_1, \alpha, c_2) : f(\sigma + 2\pi) = f(\sigma) + 2\pi,\ f'(\sigma) > 0,\ \ \alpha(\sigma + 2\pi) = \alpha(\sigma)$.
- Take both holomorphic and anti-holomorphic contributions to the complexity calculation for 2d CFT.
- Start with the cost functional defined for CFT_2: $C_1 = \int dt\, \mathscr{F} = \frac{1}{2\pi} \int dt \int d\sigma\, \epsilon(t,\sigma) \langle \psi_R \big| U^\dagger(t) \left(T + \bar{T} \right) U(t) \big| \psi_R \rangle$.
- Take the Non-Relativistic (NR) limit: $T_1 = T^c + \bar{T}^c, \quad T_2 = \lim_{\epsilon \to 0} \epsilon (T^c - \bar{T}^c)$
- T_1, T_2 are the BMS_3 stress tensors, and on the cylinder: $T_1(t,\sigma) = \sum_n (L_n - in\sigma M_n) e^{-int} + \frac{c_L}{12},\ \ T_2(t,\sigma) = \sum_n M_n e^{-int} + \frac{c_M}{12}$
- Choice: BMS primaries in the highest weight representation: $|\psi_R\rangle = |\Delta, \xi\rangle$.
- BMS primaries: $L_0 |\Delta, \xi\rangle = \Delta |\Delta, \xi\rangle, M_0 |\Delta, \xi\rangle = \xi |\Delta, \xi\rangle$. $L_n |\Delta, \xi\rangle = M_n |\Delta, \xi\rangle = 0\ \forall\, n > 0$.
- $|\psi_R\rangle = |h, \bar{h}\rangle \xrightarrow[\text{limit}]{\text{NR}} |\Delta, \xi\rangle$ in the NR limit $(L_p = \mathscr{L}_p + \bar{\mathscr{L}}_p,\ \ M_p = \epsilon \left(\mathscr{L}_p - \bar{\mathscr{L}}_p \right),\ \ \epsilon \to 0)$
- Due to the structure of the cost function, only L_0 modes contribute while taking the trace.

BMS Complexity : From the limit

- Inverse diffeomorphism F : $F(t, f(t,\sigma)) = \sigma$, Group velocity: $\epsilon(\tau,\sigma) = -\dfrac{\dot{F}(\tau,\sigma)}{F'(\tau,\sigma)}$
- Generators of BMS_3 in terms of conserved currents : $L_n = -\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} d\sigma e^{in\sigma} j(\sigma),\ M_n = -\dfrac{1}{2\pi}\int_0^{2\pi} d\sigma e^{in\sigma} p(\sigma)$
- Under the finite BMS_3 transformations $\tilde{\sigma} = F(\sigma),\ \ \tilde{t} = tF'(\sigma) + \alpha(\sigma),$ the currents transform as,

$$U_f^\dagger(t) j(\sigma) U_f(t) = \tilde{j}(f) = F'^2(\sigma)\Big(\partial_F p(F)\alpha(F) + 2\partial_F\alpha(F)p(F) - \frac{c_2}{24\pi}\partial_F^3\alpha(F)\Big) + F'^2(\sigma) j(F) - \frac{c_1}{24\pi}\{F,\sigma\}$$ [Only, this contributes!]

$$U_f^\dagger(t) p(\sigma) U_f(t) = \tilde{p}(f) = F'(\sigma)^2 p(F) - \frac{c_2}{24\pi}\{F,\sigma\},$$ where, $c_1 = -2\pi c_L,\ c_2 = -2\pi c_M.$

- Contribution due to central term: (Maurer-cartan extension for BMS_3): $C_2 = \dfrac{1}{2\pi}\displaystyle\int dt \int d\sigma \left(-\frac{\dot{F}}{F'}\right)\left[\left(\frac{c_1}{48\pi}\right)\left(\frac{F''}{F'}\right)\right]'.$

✷ $$C = C_1 + C_2 = -\frac{1}{2\pi}\int dt\ d\sigma \left[\dot{F}F' j_0(F) + \frac{c_1}{48\pi}\frac{\dot{F}''}{F'} + \dot{F}F'\Big(\partial_F p_0(F)\alpha(F) + 2\partial_F\alpha(F)p_0(F) - \frac{c_2}{24\pi}\partial_F^3\alpha(F)\Big)\right]$$

✷ Circuit complexity functional for BMS_3= geometric (co-adjoint orbit) action of BMS_3 [from the limiting analysis]

✷ No contributions from the super-transaltions in the complexity calculation!!

Complexity in 2d BMSFT (Refined Intrinsic Analysis)

- Take contributions from both super-translation and super-rotation generators.
- $C = \int dt \mathcal{F} = \frac{1}{2\pi}\int dt \int d\sigma \left[\epsilon_L(t,\sigma)\langle\psi_R|U^\dagger(t)j(\sigma)U(t)|\psi_R\rangle + \epsilon_M(t,\sigma)\langle\psi_R|U^\dagger(t)p(\sigma)U(t)|\psi_R\rangle\right]$, $(\epsilon_L, \epsilon_M) =$ two different instantaneous velocities corresponding $f(t,\sigma)$ (diffeomorphism) and $\alpha(t,\sigma)$ (supertranslations).
- Find the group velocities from BMS_3 transformations: $\sigma \to f(\sigma), t \to tf'(\sigma) + \alpha(\sigma)$, So that: $(\sigma, t) \xrightarrow{(f_1,\alpha_1)} (\sigma_1, t_1) \xrightarrow{(f_2,\alpha_2)} (\sigma_2, t_2)$.
- For two infinitesimally close points in the path of the circuit: $\big(f(t+dt,\sigma), \alpha(t+dt,\sigma)\big) = e^{(\epsilon_L(t,\sigma),\epsilon_M(t,\sigma))dt}\ (f(t,\alpha), \alpha(t,\sigma))$
- Expanding in the first order, $\epsilon_L(t,\sigma) = \frac{\partial f(t, F(t,\sigma))}{\partial t} = -\frac{\dot{F}(t,\sigma)}{F'(t,\sigma)}, \epsilon_M(t,\sigma) = \dot{\alpha}(t,F) + \alpha(t,F)\left(\frac{\dot{F}'}{F'} - \frac{\dot{F}F''}{F'^2}\right)$.
- Finally, add the contribution due to Maurer-Cartan extension for BMS_3 to write the complexity functional

$$C[F] = \frac{1}{2\pi}\int dt\, d\sigma \left[-\dot{F}F'\Big[\partial_F p_0(F)\alpha(F) + 2\partial_F\alpha(F)p_0(F) - \frac{c_2}{24\pi}\partial_F^3\alpha(F) + j_0(F)\Big] - \frac{c_1}{48\pi}\frac{\dot{F}''}{F'} + \left(\dot{\alpha}(F) + \alpha(F)\left(\frac{\dot{F}'}{F'} - \frac{\dot{F}F''}{F'^2}\right)\right)\left(F'(\sigma)^2 p_0(F) - \frac{c_2}{24\pi}\{F,\sigma\}\right)\right]$$

- The BMS complexity functional $\neq$ Geometric Action for BMS_3. [Bhattacharya, Nandi 2023]
- Extremizing $C[F]$ and solving with the periodicity conditions: simple optimal path: $f(\sigma,t) = \sigma + a_1 t, \alpha = a_3(t)$.
- And, $C[T] = a_1\, j_0 T + p_0(a_3(T) - a_3(0)), j_0 = |\Delta - \frac{c_1}{24\pi}|,\ p_0 = |\xi - \frac{c_2}{24\pi}|$.

Summary and Conclusions

- For Direct product groups (Virasoro,Kac Moody) "Complexity functional= Geometric Action" holds [Caputa, Megan 2018, Erdmenger et al 2020].
- Starting from two copies of the Virasoro algebra, it is possible to reach BMS_3 algebra, the asymptotic symmetry algebra for 3d asymptotically flat spacetimes using the Carrollian limit.
- From the limiting perspective Complexity functional resembles the geometric action for BMS_3 group.
- However, the limit fails to capture contributions from super-translations while deriving BMS complexity from its relativistic Virasoro counterpart.
- From the intrinsic analysis (using symmetry transformations), we found that the proposal "Complexity functional=Geometric Action" is not true for semi-direct product groups, e.g BMS_3. [And also Warped Conformal Symmetry group [Bhattacharyya, Katoch, Roy 2022]]
- Our analysis for BMS circuit complexity functional matches with the deformed geometric action (by addition of Hamiltonian) of BMS_3 [Merbis, Reigler 2019] only if the group velocity ϵ_M is set to zero.
- The simplest solution for the optimal path in BMS complexity matches with solutions to gravitational saddle points with constant orbit representatives j_0, p_0 in [Merbis, Reigler 2019] and Flat Space Cosmologies [FSC]s.

Future Directions

Loads of things to explore in Carrollian/BMS Field Theories

- Fubini-Study construction (Work in process with Bhattacharyya)
- Other notions of complexity (Krylov, Spread…) for BMSFT
- Higher dimensional generalisation

Thank you for your attention!