BIRS workshop proposal: Random Growth Models and KPZ Universality

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1 Overview of the Field

Irregular growth is a ubiquitous phenomenon in nature, from the growth of tumors, crystals, and bacterial colonies to the propagation of forest fires and the spread of water through a porous medium. Mathematical models of random growth have been a driving force in probability theory over the last sixty years and a rich source of important ideas [2].

The analysis of random growth models began in the early 1960s with the introduction of the *Eden model* by Eden [42] and *first-passage percolation* (FPP) by Hammersley and Welsh [51]. About two decades later, in 1979, early forms of a directed variant of FPP, *directed last-passage percolation* (LPP), appeared in a paper by Muth [66] in connection with *series of queues in tandem*. Soon after, Rost [71] introduced a random growth model, now known as the *corner growth model* (CGM), in connection with the *totally asymmetric simple exclusion process* (TASEP), a model of interacting particles first introduced by Spitzer [80]. In 1985 Huse and Henley [56] introduced the *directed polymer with bulk disorder* to model the domain wall in the *ferromagnetic Ising model* with random impurities.

In 1985 Kardar, Parisi, and Zhang [61] introduced the *KPZ equation*, an ill-defined stochastic PDE, as a proposed limit for the evolution of fluctuations of growing interfaces. Using non-rigorous renormalization group arguments, they predicted that scaling exponents of 1/3 and 2/3 should describe the fluctuations and correlations for a large class of systems, including the *asymmetric simple exclusion process* (ASEP) and the directed polymer model. These exponents were also predicted by van Beijeren, Kutner and Spohn in [86].

The next decade brought numerous mathematically rigorous results in the study of scaling limits of interacting particle systems [62, 63, 68, 70, 74]. The CGM arose naturally from LPP in queueing theory in the work of Szczotka and Kelly [81] and Glynn and Whitt [47]. Seppäläinen [74] connected the CGM and LPP to Hamilton-Jacobi equations and Hopf-Lax-Oleinik semigroups and Bertini and Giacomin [15] gave a rigorous derivation of the KPZ equation as the scaling limit of the *weakly asymmetric simple exclusion process*.

Also in the 1990s, Newman and coauthors [54, 55, 64, 67] pioneered the study of the geometric structure of the semi-infinite paths (called geodesics) that optimize the Hopf-Lax-Oleinik variational formula, in the context of FPP. In particular, borrowing ideas from classical metric geometry, [67] introduced the tool of *Busemann functions* into the field.

Later, Newman's program was implemented in LPP models [27–29, 44, 87] and certain stochastic Hamilton-Jacobi equations [4, 5, 7].

In 1999 the breakthrough of Baik, Deift, and Johansson [3] showed that the fluctuations of the Poissonian LPP have the same limit as the fluctuations of the largest eigenvalue of the *Gaussian unitary ensemble* derived by Tracy and Widom in [82]. This result was extended to the *exactly solvable* versions of the CGM by Johansson in [60]. The CGM and the related LPP and TASEP models were thus marked as members of the KPZ universality class.

These striking results caused a flurry of activity surrounding random growth models and new analytical tools from representation theory and combinatorics were developed. The subject of *integrable probability* was born. See the reviews [21, 23].

Over the last decade, ASEP, directed polymers, and the KPZ equation itself were shown to also belong to the KPZ universality class, along with many more newly discovered random growth models. See [1, 12, 16–20, 33, 40, 72, 84] (using integrable probability), [10, 30, 31, 75] (via coupling methods), and [50] (using Hairer's regularity structures [49]). The subject started moving at a high pace and the intensity of breakthroughs increased in other directions as well. We give some highlights of these recent developments in the next section.

2 Recap of objectives

The study of random growth models connects to a large number of areas in probability theory such as integrable probability, homogenization, percolation, disordered systems, interacting particle systems, random matrices, SPDEs, random polymer measures, random dynamical systems, and random walk in random environment.

The last two decades have seen rapid advances in all of these directions, with a significant acceleration in progress in several of these subfields recently, including solutions of several long-open problems. This is an exciting time for the subject, with new possibilities in extending universality, new geometric approaches, and more. **The main objective** of this workshop was to bring together a number of top experts on these various subfields to disseminate these recent developments and exchange ideas that will fertilize the ground for yet another leap forward. We give below a partial list of key topics. We also used this opportunity to celebrate the work of Timo Seppäläinen in the field.

2.1 Busemann functions

Following Newman's introduction of Busemann functions [67] and the subsequent seminal work of Hoffman [53], these objects became a principal tool for studying semi-infinite geodesics and generating stationary distributions for growth models. The existence of these objects came originally as a consequence of geodesics coalescence and relied on strong hypotheses on the limit shape. More recently, generalized Busemann functions were constructed without assumptions on the limit shape [34, 35, 45, 46, 58]. One consequence of this construction is a connection between analytic properties of the Busemann functions and regularity properties of the limiting shape function. This provides an avenue for tackling the long-standing open questions of differentiability and strict concavity of the limiting shape.

The existence of Busemann functions in the case of the KPZ equation itself is also an open problem that now seems accessible using the recent techniques mentioned above. Busemann functions for the *KPZ fixed point* (a central object in the KPZ universality class, constructed recently by [65]) may also be accessible by working with the LPP-like description, made possible by the recent construction of the *directed landscape* [36, 37].

2.2 Integrable probability

The new field of *integrable probability* has been extremely effective in proving scaling limits for various random growth models and interacting systems. The basic idea is that for certain models due to some hidden algebraic or combinatorial structures one can compute the expected values of certain observables (often in the form of complex contour integrals) [1, 16, 21, 23, 24, 83]. By accessing a sufficient number of these expected observables this information can be used to derive and identify the scaling limit. This method has been successfully applied to a number of classical models and led to the discovery of new families of models in the KPZ universality class. It has also led to new results about the limiting objects; see the recent [65] on the KPZ fixed point and [36] on the directed landscape. This is a dynamically evolving field with a lot of activity and many more directions to explore.

2.3 Coupling techniques

The early use of coupling techniques in the study of random growth models was to prove hydrodynamic limits and large deviation results [43, 70, 71, 73]. The more recent uses of these techniques relied on various generalizations of Burke's property (a result from queuing theory) by finding generalized versions of systems of queues in certain random models. Although these techniques so far could not produce fluctuation limit theorems, they have successfully identified and bounded scaling exponents [9, 11, 26, 75, 76]. In the recent [8, 25] coupling techniques were used to prove the non-existence of bi-infinite geodesics in the exponential LPP and of bi-infinite Gibbs measures in the log-gamma polymer model.

2.4 Geometry of random geodesics

The last couple of years brought a deeper understanding of the geometry of random geodesics in random growth models. [13] and [8] proved (with two different approaches) that bi-infinite geodesics do not exist in the exponential LPP. Moreover, [59] gave the first complete description of the structure of semi-infinite geodesics in the LPP model with exponential weights. The analogous questions for models with general weight distributions are very much open, but the techniques developed for the recent results could provide a possible route to attack these problems.

2.5 Random matrices

Random matrix theory is closely connected to the study of random growth models. These connections can be observed both at the level of finite models and the limiting objects, and can be used to study these objects using the tools of random matrices. A successful example is the introduction and study of the *Airy line ensemble* using *Dyson's Brownian motion* [32], which was used in the recent work [36] in the study of the scaling limit of the Brownian LPP. [85] studied large deviation properties of the KPZ equation by connecting it to the edge point process limit of the Gaussian unitary ensembles via a distributional identity of [22]. The main tool is a description of the edge process as the spectrum of a random differential operator given in [69].

2.6 Random dynamical systems

The connection with the Hamilton-Jacobi equation allows one to view random growth models as random dynamical systems. This approach was initiated by Sinai [77], followed by the various extensions [39, 41, 48, 52, 57, 79]. These results are in direct relation with the more recent ones [4, 5, 7, 27, 28, 34, 35, 58] on existence of Busemann functions and the geometric properties of semi-infinite geodesics, mentioned above.

It is believed [6] that the statistics and structure of the shocks formed by the solutions of the associated stochastic Burgers equation (studied in [4, 14, 41, 59, 78]) are closely related to the KPZ universality phenomenon. This is yet another widely open direction in the field. A promising starting point is offered by the following idea: The shocks in the KPZ equation should be in duality with the geodesics coming from the aforementioned, recently constructed, directed landscape [36–38].

3 Presentation Highlights

The workshop centered around 29 30-minute talks that covered all the different approaches that have been successful in the subject. The broad spectrum of topics occurred organically; participants were invited to voluntarily give a talk on a topic of their choice, and we naturally found cohesion within the scientific program. Here we provide an overview of each, arranged loosely according to topic.

3.1 Interacting particle systems

Márton Balázs explained how the two-parameter stationary last passage percolation picture and some planarity tricks can be used to establish stabilisation of the point-to-point geodesic tree to the semi-infinite one.

Eric Cator introduced a periodic version of the polynuclear growth model (PNG) and showed that it is a solvable model. He described the stationary measures at a fixed time and the distribution of the space-time paths.

Ivan Corwin introduced a new method to construct the stationary measure for open boundary KPZ models, focusing on geometric LPP, the log-gamma polymer, and the KPZ equation. These stationary measures are realized as marginals of two-layer Gibbs measures by constructing local Markov dynamics that preserve this class of measures and project on the top layer to the KPZ models.

Pablo Ferrari spoke about the KMP model where each vertex of a finite graph is assigned a random nonnegative energy value and then when a Poisson clock rings at an edge, the energies of the two vertices are randomly redistributed. The main result described the invariant measure of the process. Applications included hydrodynamic limits.

In his talk, Nicos Georgiou introduced a new totally asymmetric exclusion process on a rooted Galton-Watson tree without leaves and then focus on the aggregated current of particles across generations. He showed how one can use a coupling with last passage percolation to obtain upper and lower bounds for the aggregated current of particles across generations, depending on the tree structure and jump rates on each node. As a consequence, one can construct time intervals where the current across a fixed generation jumps from 0 to linear order.

Vadim Gorin described his work on the six-vertex model in a large square with Domain Wall Boundary Conditions where he found that the local description of the model near a straight segment of the boundary depends on the value of a single parameter Δ, with a single limiting object for all and a richer class of limits at $\Delta > 1$.

Sunder Sethuraman considered the one dimensional asymmetric simple exclusion process (ASEP), starting from a stationary state, and studied the typical behavior of a tagged particle, conditioned to deviate to an atypical position at a fixed time. Among their results is an upper tail large deviations bound for the position of the tagged particle.

3.2 Random geometry

Duncan Dauvergne talked about the directed landscape model which is the scaling limit for models in the KPZ universality class. He gave a full description of the exceptional pairs of points on the plane between which there are multiple paths that maximize the model's action functional.

Bálint Virág talked about several new results for models in the KPZ universality class. He explained how the Wick-ordered stochastic heat equation with planar white noise can be defined as the free energy of a certain undirected polymer in a random environment, and how one can recover one-dimensional KPZ fluctuations from this model in an appropriate limit. He also presented results about the Brownian web distance, an integer-valued scale invariant random metric with scaling exponents 0:1:2 – how this metric can be obtained from coalescing simple random walks on the plane and how in the shear limit one can recover the Airy process.

Joseph Yukich showed that the rescaled maximal radial and longitudinal fluctuations of the boundary of the convex hull of n i.i.d. random points uniformly distributed on a smooth convex set asymptotically follow the Gumbel law as n tends to infinity and, in $d = 2$, they asymptotically exhibit $(1/3, 2/3)$ scaling, with precise logarithmic corrections. When the convex set is the unit disc, the radial fluctuations satisfy process level convergence. He also introduced a dual space-time two parameter growth process, which for $t = 1$ coincides with the support function of the convex hull, displays 1:2:3 scaling, and converges to a two-parameter limit process given by the Hopf-Lax formula.

3.3 Limit shapes, Busemann functions, and directed polymers

Yuri Bakhtin considered a class of continuous-space polymer models in positive and zero temperature for which he showed that the shape function is differentiable.

Erik Bates delivered a presentation on the Busemann process for directed polymers. He revealed certain parallels with the zero-temperature last-passage percolation model and, at the same time, intriguing distinctions. For instance, within the solvable realm of the inverse-gamma model, the Busemann process's countably dense set of discontinuities remains the same across all edges of the square lattice. This stands in contrast to the exponential last-passage percolation model, where an edge's set of discontinuities is countable but nowhere dense, while the union of all discontinuities across edges is dense.

Ofer Busani presented recent progress in the understanding of the multi-type stationary measures of the KPZ fixed-point as well as the scaling limit of multi-type stationary measures of two families of models in the KPZ class: metric-like models (e.g. last passage percolation) and particle systems (e.g. exclusion process).

Ofer Zeitouni described the evolution of high moments of the partition function of two-dimensional directed polymers in the weak disorder regime.

3.4 Fluctuation bounds and scaling limits

Elnur Emrah talked about the limiting boundary fluctuations of the eigenvalue minor process of a unitarily invariant random Hermitian matrix with eigenvalue sequence a_n. He presented a classification theorem that identifies five fluctuation regimes in terms of the parameters a_n and describes the corresponding limit processes.

Milind Hegde gave a proof of a recent conjecture stating that the geodesic in the directed landscape between $(0, 0)$ and $(0, 1)$, conditioned on the event that its weight is at least some large value L, and suitably rescaled, converges as $L \to \infty$ to a Brownian bridge.

Xiao Shen gave upper and lower bounds for the correlation between the free energies of two inverse-gamma polymers with endpoints in close proximity or far apart.

Philippe Sosoe presented a general methodology, based on a formula of Rains and Emrah-Janjigian-Seppalainen, to obtain scaling and tail bounds in several KPZ models, including some that are not known or expected to be integrable.

Xuan Wu proved the convergence of the KPZ equation to the directed landscape.

3.5 Algebraic methods and integrable probability

In the heart of the construction of the KPZ fixed point (by Matetski-Quastel-Remenik) lies the solution of TASEP with general initial conditions in terms of a random walk hitting representation. Nikos Zygouras presented a step-by-step derivation of this via principles of the RSK correspondence.

3.6 Homogenization and Hamilton-Jacobi equations

Jessica Lin presented quantitative estimates on the the parabolic Green function and the stationary invariant measure in the context of stochastic homogenization of elliptic equations in nondivergence form. Then, she discussed implications of these homogenization results, such as a quenched, local CLT for the corresponding diffusion process and a quantitative ergodicity estimate for the environmental process.

Hamilton-Jacobi PDEs with random stationary Hamiltonian functions are popular toy models for studying the dynamics of interfaces in various phenomena in physics and biology. There is a one-to-one correspondence between piecewise smooth solutions and marked tessellations. Fraydoun Rezakhanlou explained how a kinetic theory can be developed to describe the dynamics of such tessellations. As an application, he derived kinetically describable Gibbsian solutions for such PDEs.

Atilla Yilmaz considered viscous Hamilton-Jacobi equations with a general (nonconvex) continuous superlinear momentum and a bounded Lipschitz continuous time-independent potential. He showed that under a certain "hill and valley" condition on the diffusivity-potential pair, the equation homogenizes and there are sublinear correctors outside of the linear segments of the effective Hamiltonian.

3.7 Random matrices

The complex eigenvalues of non-Hermitian random matrices have attracted much research interest due to their relevance to several branches of theoretical physics, and in particular to the study of scattering chaotic systems. In her talk, Tatyana Shcherbina discussed a certain form of universality appearing in the distribution of the "resonance widths" in the case of a few classical ensembles of random matrices.

3.8 SPDE, KPZ, SHE

Chirs Janjigian discussed some recent results on the ergodicity properties of the Brownian motion stationary distributions of the KPZ equation through the lens of the synchronization by noise phenomenon. He showed that the solution to the equation started in the distant past from an initial condition with a given slope will converge almost surely to a Brownian motion with that same drift for all but a random countable set of slopes.

Hao Shen consider the Langevin dynamics of a large class of lattice gauge theories on the 2D torus and proved that these discrete dynamics all converge to the Markov process induced by the local solution to the stochastic Yang-Mills equation on the torus. Using this universality result for the dynamics, he showed that the Yang-Mills measure on the 2D torus is the universal limit for these lattice gauge theories. He also proved that the Yang-Mills measure is invariant under the dynamics. thod for invariant measures.

Evan Sorensen showed the existence of a random, countably infinite dense set of directions for which the Busemann process for the KPZ equation is discontinuous. This demonstrated the failure of the One Force–One Solution principle in those exceptional directions and resolved a recent conjecture of Janjigian, Rassoul-Agha, and Seppäläinen.

Li-Cheng Tsai considered the n-point, fixed-time large deviations of the KPZ equation with the narrow wedge initial condition. He explained how to analyze the multipoint moments of the Stochastic Heat Equation via a system of attractive Brownian particles and how to use the moments to obtain the n-point Large Deviation Principle and spacetime limit shape.

3.9 Random turn games

Many combinatorial games, such as chess, Go and Hex, are zero-sum games in which two players alternate in making moves. In a random turn variant, each player wins the right to move at any given turn according to the flip of a fair coin. In 2007, Peres, Schramm, Sheffield and Wilson [PSSW] found explicit optimal strategies for a broad class of random-turn games, including Hex. The gameplay in random turn Hex is a novel random growth process related to planar critical percolation; other random turn games offer growth processes related to other universality classes.

In his talk, Alan Hammond discussed stake-governed random-turn games and showed how to evaluate the relative strategic importance of intermediate positions in multi-turn games.

4 Open problems session

Yuri Bakhtin posed the problem of pushing the work he presented in his talk to try and get strict convexity of the limiting free energy.

Ivan Corwin asked about stationary distributions for the KPZ equation on $[0, \infty)$ with initial condition $h(0, x) = vx$ and Neumann boundary condition $\partial_x h(t, 0) = u$ (for given parameters $u, v \in \mathbb{R}$).

Alan Hammond considered the planar stochastic process that is rotating at angular speed n and whose radius is a standard Brownian motion, conditioned on staying positive. He asked about the scaling limit (as $n \to \infty$) of this process, when conditioned on not intersecting itself.

In the standard first-passage percolation model, Chris Hoffman asked about the shape of the boundary of the (random) ball of radius t near its right-most point.

Xuan Wu noted that in the exponential last passage percolation model, the eigenvalues of the Laguerre unitary ensemble (LUE) and the last passage times are equal in distribution. On the LUE side, one may introduce its dynamical version by replacing the Gaussian entries with Brownian motions, resulting in an evolving system with rich integrable structures. The open problem is then to figure out how to introduce a dynamical last passage percolation model to lift the identification to a dynamical version.

5 Mentoring Program

The conference had a mentoring program associated to it. All participants were invited to participate as both mentors and mentees within the program. Small mentor groups of 4 members (2 mentors and 2 mentees) were formed. The matchings were made based on (i) requests of the participants (ii) pairing disjoint groups of people together (i.e. no mentor/mentees were supervisors/trainees, nor collaborators). In total, 4 groups of 4 were formed (constituting 16 participants total). Participants were encouraged to sit together for at least one meal and exchange contact info in order to find out about social activities. Moreover, mentees were encouraged to solicit feedback on their talks from mentors; this involved some preparatory feedback on slides and feedback after the talk took place.

6 Demographics

21% of the invitees self-identified as being from an underrepresented group. 16% of the invitees were females. 16% of the invitees were postdoctoral fellows and 14% were tenure-track assistant professors.

18% of the participants self-identified as being from an underrepresented group. 13% of the participants were females. 18% of the participants were postdoctoral fellows and 16% were tenure-track assistant professors.

7 Conclusion

The workshop was a resounding success in many ways. The research presentations shared cutting-edge developments in the study of random growth models and adjacent, related topics. The infrastructure provided by BIRS created a welcoming environment to promote interaction and discussions. The mentoring program encouraged engagement between participants of all levels, and it was greatly appreciated by junior participants who are at early stages in their careers.

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