Temporal correlation in the inverse-gamma polymer

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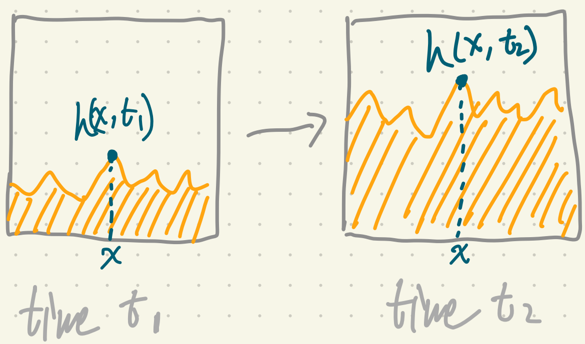
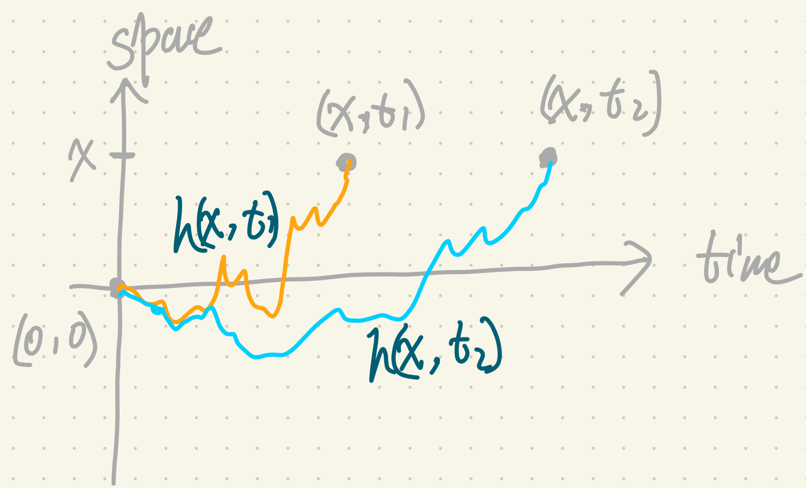
Joint works with Riddhipratim Basu & Timo Seppäläinen

Random Growth Models and KPZ Universality

The height function $h(x,t)$

x is spatial, $\mathbb{R}$
t is time, $\mathbb{R}_{\geq 0}$





Spatial statistics $x \mapsto h(x, t_0)$

Temporal process $t \mapsto h(x_0, t)$

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(Fix $x_0 = 0$)

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Baik, Liu, Johansson, Rahman

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What is $\mathrm{Corr}(h(0,t_1), h(0,t_2))$?

The study of $\mathrm{Corr}(h(0,t_1), h(0,t_2))$

- Physicis: Takeuchi, Sano, Singha, de Nardis, Le Doussal,
- Mathematics
  - Zero temperature (corner growth model)

    Basu, Ferrari, Ganguly, Occelli, Spohn, Zhang
  - Positive temperature

    (KPZ equation) Corwin, Ghosal, Hammond

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    (Inverse-gamma polymer) — Coupling methods &

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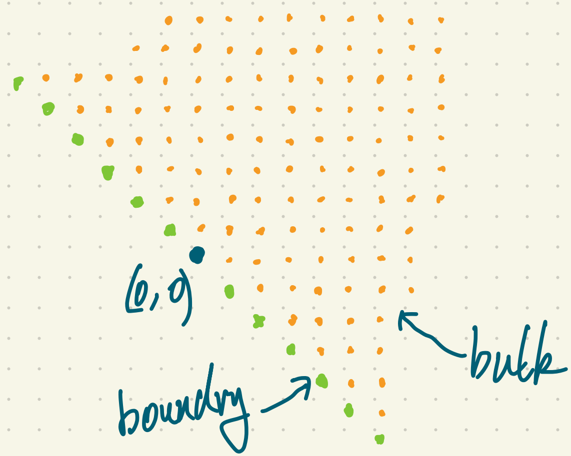
- Higher up in the hierarchy of KPZ models

Inverse-gamma polymer

CGM   Brownian LPP   OC-Y polymer   KPZ equation

The model

- $\{W_{(k,-k)}\}_{k\in\mathbb{Z}}$, 2-sided (multiplicative) walk, $W_{(0,0)}=1$
- $\{Y_z\}_{z \text{ above } x-y=0}$ i.i.d. $Ga^{-1}(\mu)$



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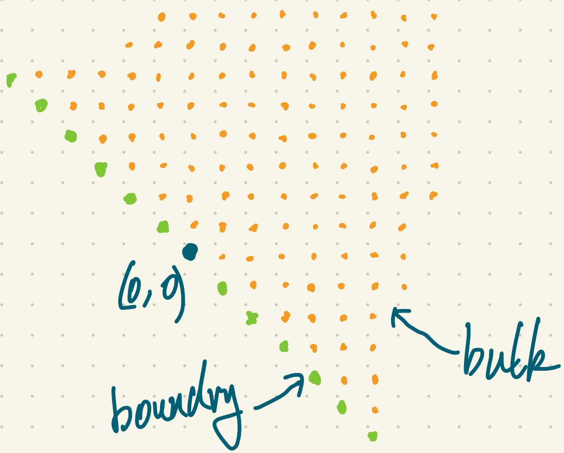

- Bulk partition function

$$\tilde{Z}_{a,b}=\sum_{\gamma\in\mathbb{X}_{a,b}}\prod_{z\in\gamma_{>0}}Y_z\,,\quad \tilde{Z}_{a,a}=1$$

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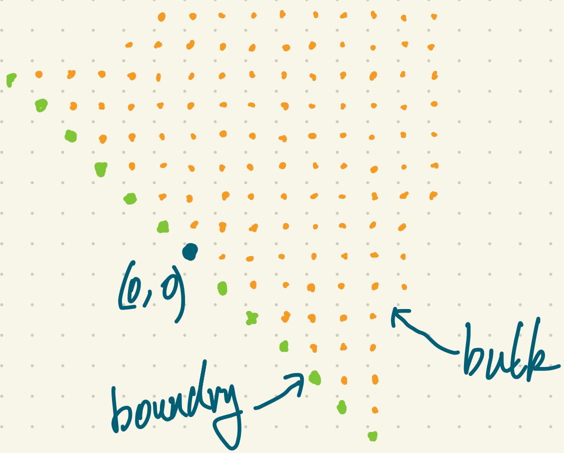

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- Partition function

$$Z^W_b = \sum_{k\in\mathbb{Z}} W_{(k,-k)}\cdot \tilde{Z}_{(k,-k),b}$$

$\boxed{W_{(k,-k)} = W_k}$

Initial (boundary) conditions

- Droplet: $W_k = \begin{cases} 1 & \text{for } k=0 \\ 0 & \text{for } k\neq 0 \end{cases}$

- Flat: $W_k = 1$ for each $k \in \mathbb{Z}$

- Random: $\{\log W_k\}_{k\in\mathbb{Z}}$ is a random walk, $\log W_0 = 0$

  Ratio-stationary initial (boundary) condition

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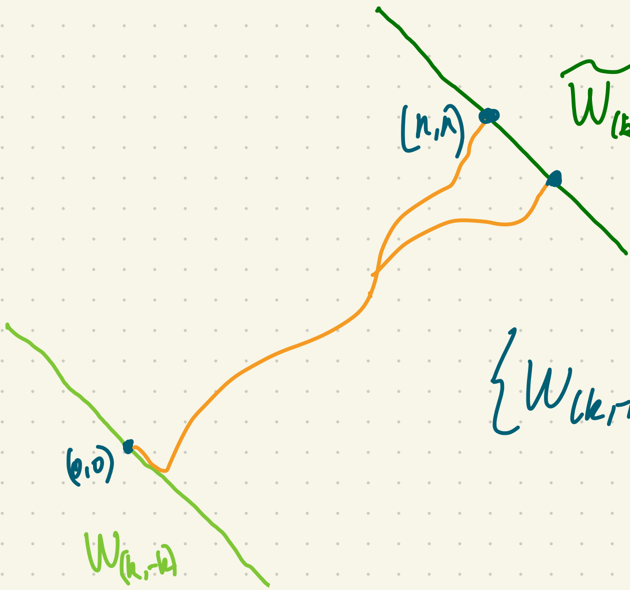
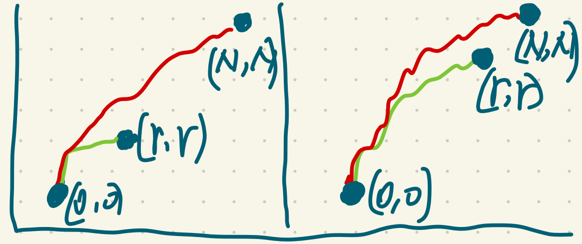
$$\frac{W_{k+1}}{W_k} \stackrel{d}{=} \frac{Ga^{-1}(\mu/2)}{Ga^{-1}(\mu/2)} \quad \text{i.i.d.}$$



$$\widetilde{W}_{(k,-k)} = \frac{Z^w_{(n+k,\,n-k)}}{Z^w_{(n,n)}}$$

$$\{W_{(k,-k)}\} \stackrel{d}{=} \{\widetilde{W}_{(k,-k)}\}$$

Main results



For droplet (Basu-Seppäläinen-S) and random* (Basu-S) initial conditions. $\exists\, C_1, C_2, C_3, C_4, r_0\ N_0$

- for $r_0 \le r \le N/2$

$$C_1\left(\frac{r}{N}\right)^{\frac{1}{3}} \le \mathrm{Corr}\left(\log Z^{w}_{(r,r)},\ \log Z^{w}_{(N,N)}\right) \le C_2\left(\frac{r}{N}\right)^{\frac{1}{3}}$$

- for $N \ge N_0$, $N/2 \le r \le N - r_0$

$$1 - C_3\left(\frac{N-r}{N}\right)^{\frac{2}{3}} \le \mathrm{Corr}\left(\log Z^{w}_{(r,r)},\ \log Z^{w}_{(N,N)}\right) \le 1 - C_4\left(\frac{N-r}{N}\right)^{\frac{2}{3}}$$

Droplet IC

Same results:

- Basu, Ganguly in the CGM
- Corwin, Ghosal, Hammond for the KPZ equation

Asymptotic result: $r = \varepsilon N$ or $r = (1-\varepsilon)N$, and $N \to \infty$

- Ferrari, Occelli in the CGM (explicit constants)

Random IC

Ferrari, Occelli (CGM) $r = (1-\varepsilon)N$, and $N \to \infty$

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1. Small r regime
2. Much broader class of random IC.

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(iv) $c_1 N \le \mathrm{Var}(\log W_N) \le C_2 N$

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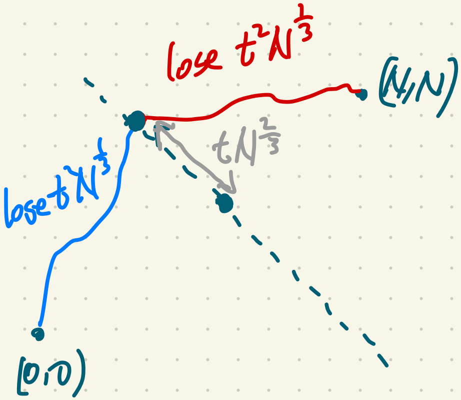
③ Estimate for finite N and r, where r or $N-r \ll N$.

Local Fluctuations



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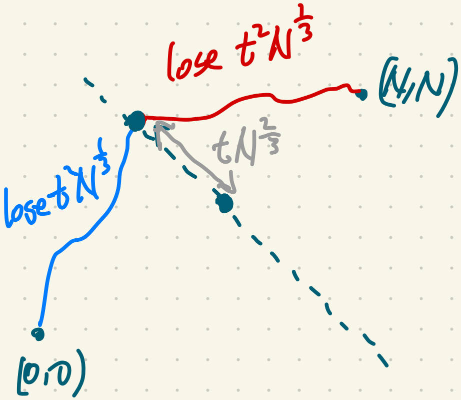

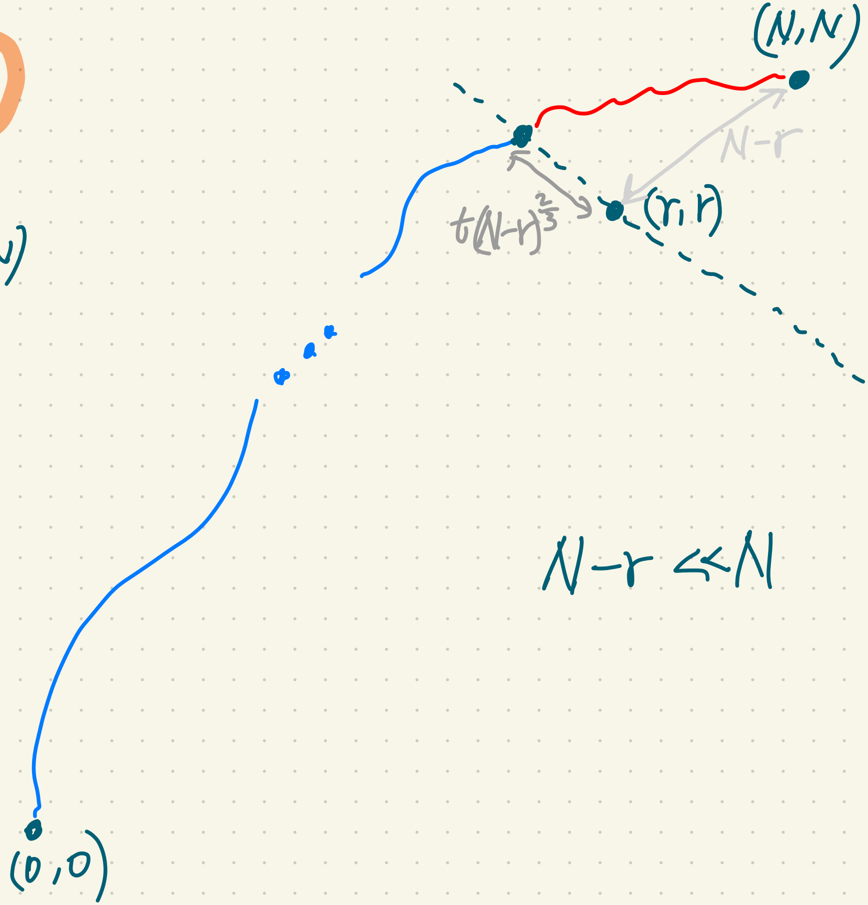

$N - r \ll N$

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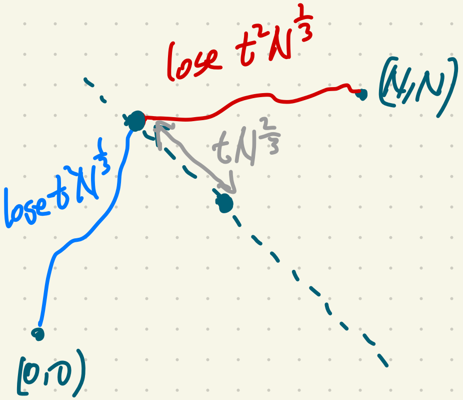

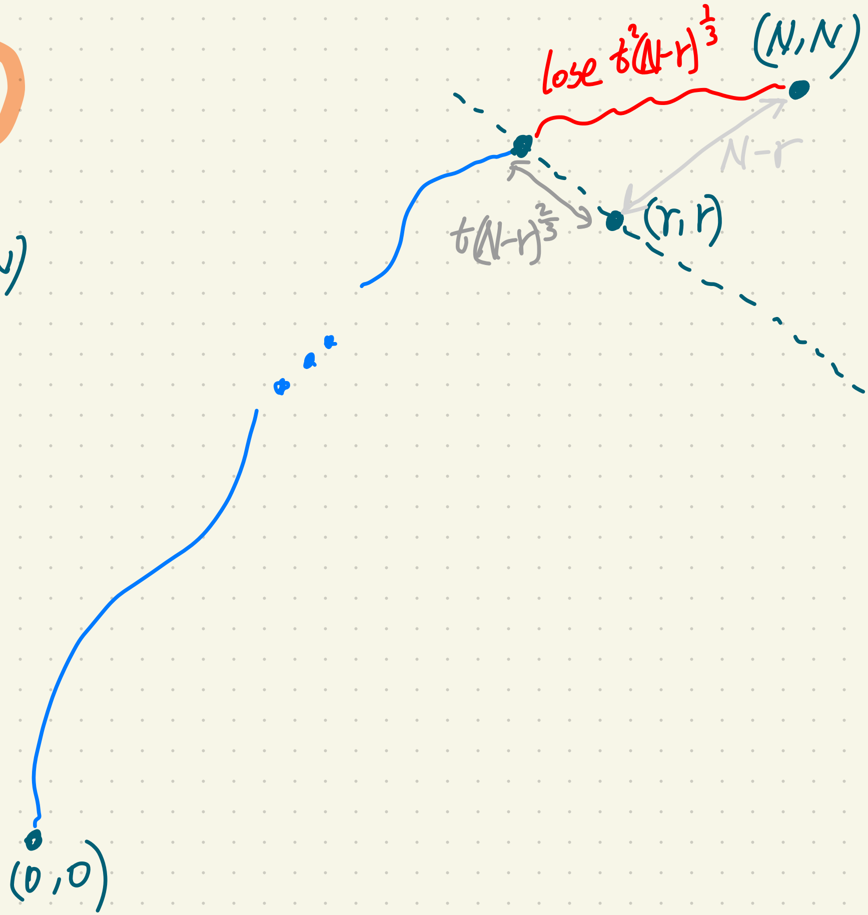

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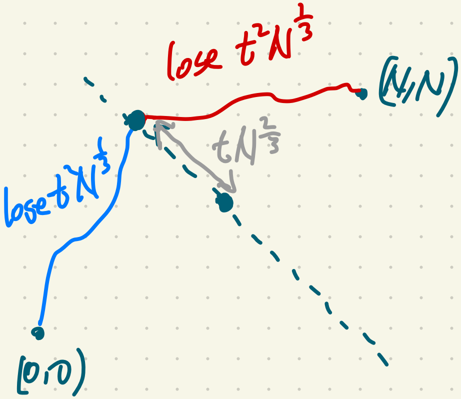
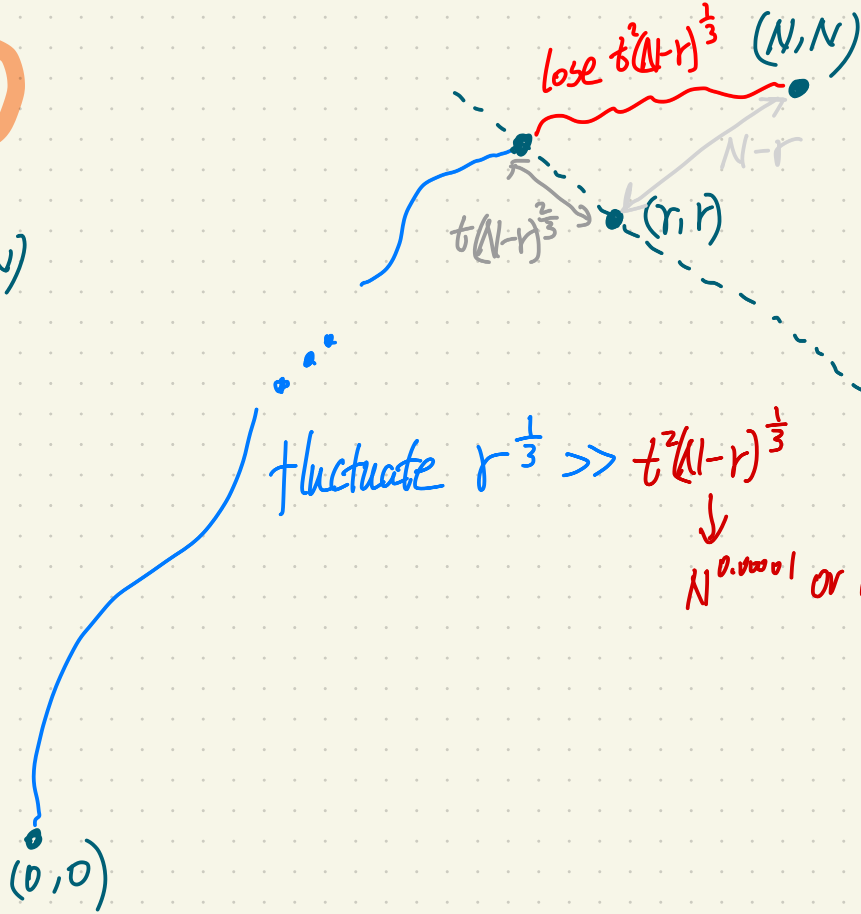


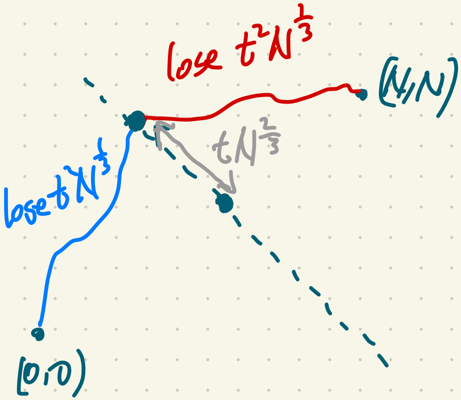


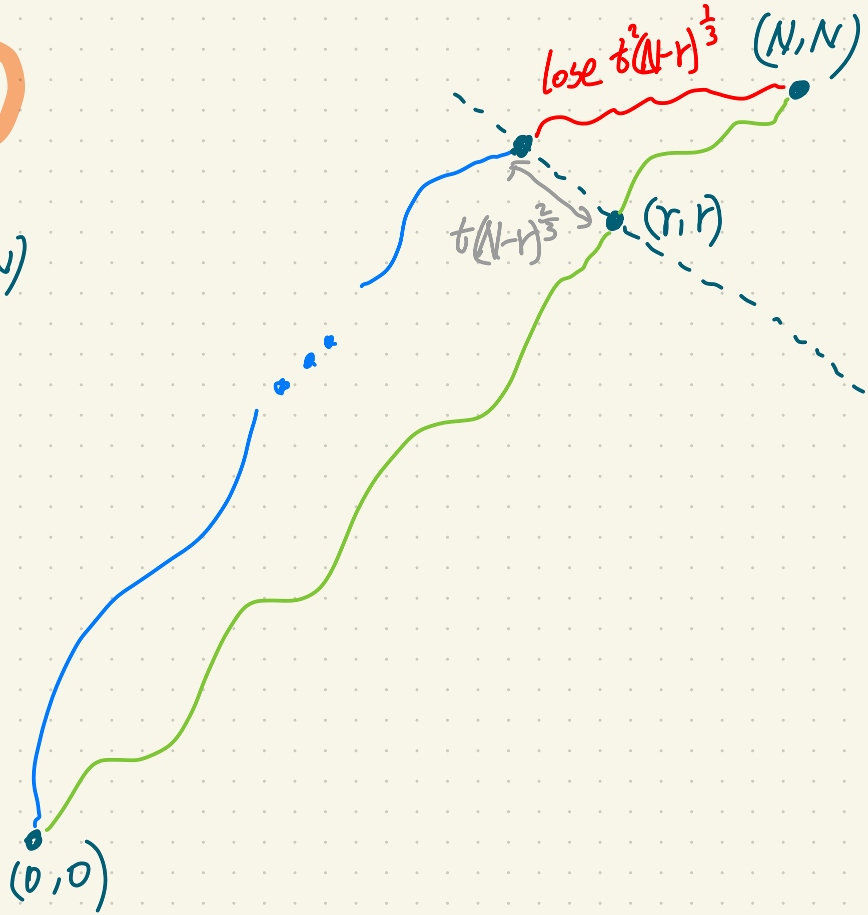
fluctuate $r^{\frac{1}{3}} >> t^2(N-r)^{\frac{1}{3}}$

$\downarrow$

$N^{0.00001}$ or constant

Local fluctuations





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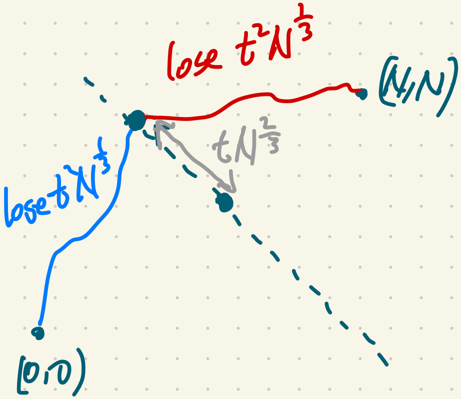

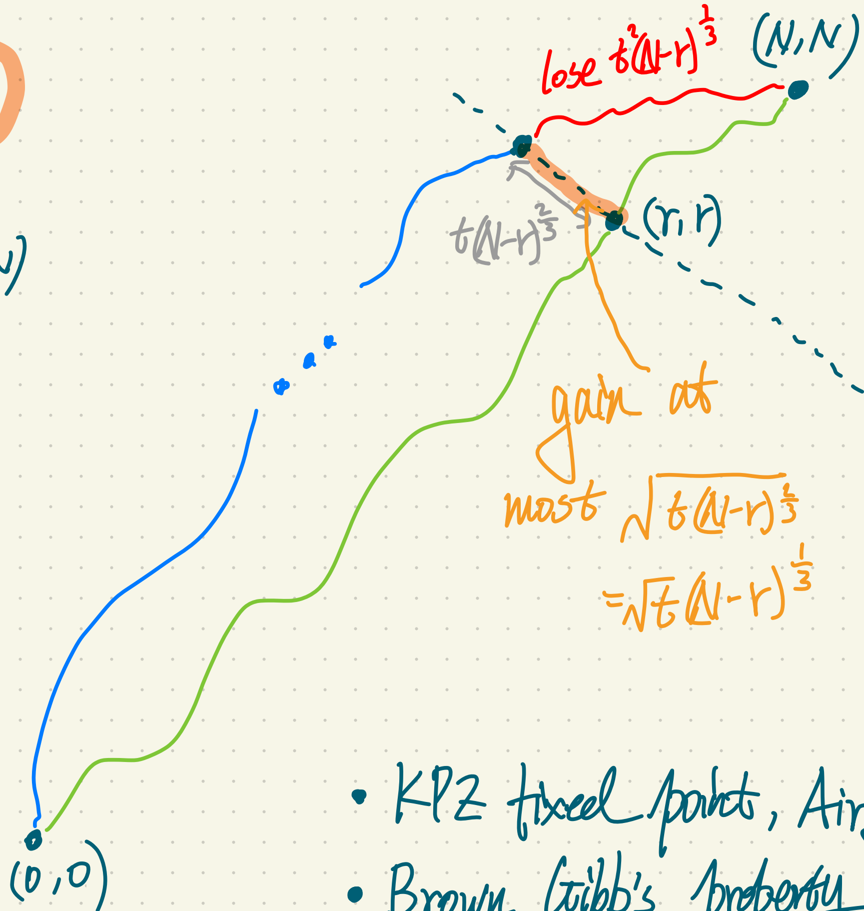

- KPZ fixed point, Airy
- Brown Gibb's property
- Coupling method

Large r regime

Replace $\log Z^{w}_{0,N}$ by $\log Z^{w}_{0,r} + \log \tilde{Z}_{r,N}$

independent

(notation $Z_{a,b} = Z_{(a,a),(b,b)}$)

$$\boxed{\inf_{\lambda\in\mathbb{R}} Var(U-\lambda V) = (1-Corr(U,V)^2)\,Var(U)}$$

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$$C\,\frac{(N-r)^{\frac{2}{3}}}{N^{\frac{2}{3}}} \le \cdots \le C\,\frac{(N-r)^{\frac{2}{3}}}{N^{\frac{2}{3}}}$$

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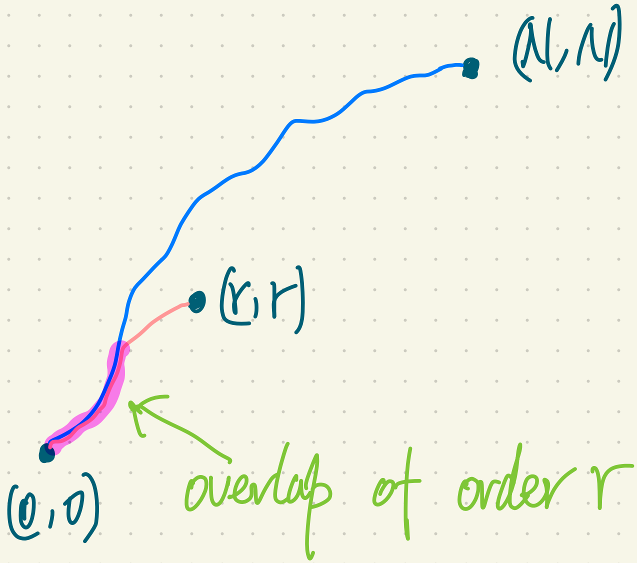
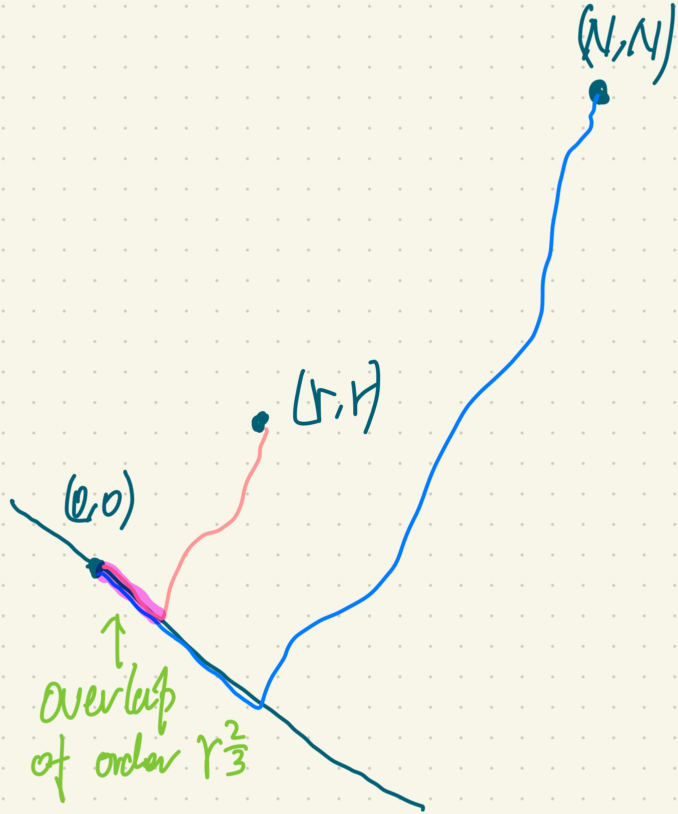


$$\mathrm{Cov}(\log Z_{0,r}, \log Z_{0,N})$$
$$= \mathrm{Var}(\text{overlap in the bulk})$$
$$= r^{\frac{2}{3}}$$



$$\mathrm{Cov}(\log Z^{w}_{0,r}, \log Z^{w}_{0,N})$$
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Thank you!

Happy birthday, Timo