Finite-rank complex perturbations of Hermitian random matrices

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Model

We are going to consider the random matrices of the form

$$\mathrm{H}_{\mathrm{eff}} = \mathrm{H} + i\Gamma,$$

where H is a random matrix ensemble with an appropriate symmetry (e.g., Hermitian or real symmetric), and Γ is a positive deformation of a constant rank M.

Most classical random matrix ensembles (such as Gaussian ensembles GUE/GOE, Wigner matrices, β-ensembles, etc.) are invariant under the unitary transformations, so one can consider

$$\Gamma = \begin{pmatrix} \gamma_1 & 0 & \dots & \dots & 0 & \dots & 0 \\ 0 & \gamma_2 & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \gamma_{\mathrm{M}} & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & 0 & \dots & 0 \end{pmatrix}.$$

For the most part of the talk we restrict ourself to the case of Hermitian matrices and rank-one perturbation, i.e. $\Gamma = \mathrm{diag}\{\gamma, 0, \dots, 0\}$.

Results for real perturbations

There are a lot of interesting works for the case of real deformation,

$$\mathrm{H}_{\text{eff}} = \mathrm{H} + \Gamma$$

In this situation the eigenvalues are real, the main part of the spectrum does not change but some “outlier" can be separated as γ's grows and we observe so called BBP transition.

- Baik, Ben Arous, and Peche (2005)
- Peche (2006)
- Capitaine, Donati-Martin, Feral (2009)
- Benaych-Georges, Guionnet, Maida (2011)
- Knowles, Yin (2014)
- ...

Anti-Hermitian perturbation

If we return to the anti-Hermitian deformation $H_{eff} = H + i\Gamma$, then the situation is much different since H_{eff} is not Hermitian anymore, and thus has complex eigenvalues.

However, in contrast to the classical non-Hermitian models such as Ginibre ensemble, if M is fixed and $N \to \infty$, matrices $\mathcal{H}_{eff}$ are weakly non-Hermitian.

It is straightforward to check that for $\gamma > 0$ (rank-one case) the eigenvalues of H_{eff} has the form

$$\lambda_j(\gamma) = \lambda_j(H) + \zeta_j(\gamma), \quad \operatorname{Im} \zeta_j > 0$$

Moreover, since and eigenvectors $\{\Psi_j\}$ of H (e.g. for GUE) are uniformly distributed over the sphere, it is naturally to expect that

$$\zeta_j(\gamma) \sim i\gamma(E_{11}\Psi_j, \Psi_j) \sim in^{-1}y_j$$

Hence it appears that the planar density of eigenvalues is concentrated in the strip $\operatorname{Im} z \sim n^{-1}$

This is indeed the case, and moreover one can show that for GUE (and, more generally, for Wigner matrices) the eigenvalues of H_{eff} are all in the upper half of the complex plane and for N large they all, except possibly one outlier, lie just above the interval $[-2, 2]$ of the real line. The presence of the outlier is determined by the value of γ ($\gamma < 1$ corresponds to no outliers; $\gamma > 1$ corresponds to one outlier lying much higher in the complex plane, its imaginary part is about $\gamma - 1/\gamma$).
Some results in this direction:

- O'Rourke, Renfrew (2014)
- O'Rourke, Wood (2017)
- Rochet (2017)
- Dubach, Erdös (2022)
- Fyodorov, Khoruzhenko, Poplavskyi (2023)
- ...

Apart from the mathematical curiosity, there is also motivation coming from physics. In the physics literature, the eigenvalues of H_{eff} are associated with the zeroes of a scattering matrix in the complex energy plane, and their complex conjugates with the poles of the same scattering matrix, known as "resonances". The latter are obviously the eigenvalues of matrices $H_{\text{eff}} = H + i\Gamma$ with γ's replaced by $-\gamma$'s. In this context the eigenvalues imaginary part is associated with the "resonance width" (see Verbaarschot, Weidenmüller, Zirnbauer '85; Sokolov, Zelevinsky '89; Fyodorov, Sommers '96,...)

In this context, one of the interesting questions about the spectral statistics of H_{eff} is the distribution of $\operatorname{Im} z_i$ (as was mentioned above, the planar density of the eigenvalues is concentrated in the strip $\operatorname{Im} z \sim N^{-1}$, so all but finitely many $\operatorname{Im} z_i \sim N^{-1}$.

Some results

GUE case (and some related models)

- Haake, Izrailev, Lehmann, Saher, Sommers '92
- Fyodorov, Sommers '96
- Fyodorov, Sommers '97
- Fyodorov, Khoruzhenko '99
- Fyodorov, Mehlig '02
- Fyodorov, Sommers '03

For the exact formulas for joint eigenvalue density for rank-one perturbation of β-ensembles see also

- Kozhan '17 (rank one perturbation of β-ensembles), Killip, Kozhan'17 (β-circular ensembles), Alpan, Kozhan '21 (same for chiral Gaussian β-ensembles)

General non-Hermitian random matrices: methods

Logarithmic potential approach (by Girko)

Based on the formula:

$$\nu(\zeta,\bar\zeta) = \frac{1}{2\pi}\Delta_\zeta \int \nu(z,\bar z)\log|\zeta - z| dz d\bar z,$$

Hence, introducing $X(z) = (H_{eff} - z)(H_{eff} - z)^*$, we have

$$\mathcal{N}_N[h] = \sum_i h(z_i,\bar z_i) = \sum_j \frac{1}{4\pi}\int h(z,\bar z)\Delta_z \log|z_j - z|^2 dz d\bar z$$
$$= \frac{1}{4\pi}\int \Delta h(z,\bar z)\cdot \log\det X(z) dz d\bar z$$

Since $X(z)$ is a hermitian matrix, one can find its limiting spectral distribution $\mu_n^{(z)}(\lambda)$. Then

$$\log\det X(z) = \int_0^\infty \log\lambda \, d\mu_N^{(z)}(\lambda)$$

In particular,

$$\mathbb{E}\{\mathcal{N}_N[h]\} = \frac{1}{4\pi}\int \Delta h(z,\bar{z})\cdot \mathbb{E}\{\log\det X(z)\}dzd\bar{z}$$
$$= \frac{1}{4\pi}\int h(z,\bar{z})\cdot \Delta\mathbb{E}\{\log\det X(z)\}dzd\bar{z}$$

and hence averaged density of the eigenvalues $z_j = X_j + iY_j$

$$\rho_N(X,Y) = \frac{1}{N}\mathbb{E}\{\sum_{j=1}^{n}\delta(X-X_j)\delta(Y-Y_j)\}$$

can be computed as

$$\rho_N(X,Y) = \frac{1}{\pi N}\frac{\partial^2}{\partial z\partial\bar{z}}\mathbb{E}\{\log\det X(z)\}$$

where $z = X + iY$.

As we discussed, we are interested in the scale $\mathrm{Im}\, z \sim N^{-1}$, so one need to define the rescaled version of $\rho_N(X, Y)$ for $y = N\rho_H(X)Y$:

$$\tilde{\rho}_N(X, y) = \frac{1}{N\rho_H(X)}\mathbb{E}\{\sum_{j=1}^{n}\delta(X - X_j)\delta(y - \rho_H(X)NY_j)\}, \quad X \in \mathrm{bulk}(\sigma(H)).$$

We are interested in the limit of this measure when the size of matrix N goes to infinity.

Averaging of logarithm

Averaging of logarithm (by Fyodorov and Sommers'96)

Technically, instead studying of $\mathrm{E}\{\log \det X(z)\}$ it is convenient to introduce the generating function

$$\mathcal{Z}_N(\kappa, z_1, z_2) = \mathrm{E}\Big\{\frac{\det(X(z_1) + \kappa^2/N^2)}{\det(X(z_2) + \kappa^2/N^2)}\Big\}$$

where z_1 and z_2 are auxiliary spectral parameters in the vicinity of $E + iy/N$:

$$z_l = E_l + \frac{iy_l}{N}, \quad E_l = E + \frac{x_l}{N}, \quad l = 1, 2.$$

Given $\mathcal{Z}_N(\kappa, z_1, z_2)$, the density can be obtained using the following identity:

$$\tilde{\rho}_N(E, y) = \frac{1}{4\pi} \lim_{\kappa \to 0} \Big(\frac{\partial}{\partial y_1} \Big(\lim_{y_2 \to y_1} \frac{\partial \mathcal{Z}_N}{\partial y_2} \Big) + \frac{\partial}{\partial x_1} \Big(\lim_{x_2 \to x_1} \frac{\partial \mathcal{Z}_N}{\partial x_2} \Big) \Big) \Big|_{\substack{y_1 = y, \\ x_1 = x_2 = 0}}$$

Integral representation for $\mathcal{Z}(\kappa, z_1, z_2)$ (for GUE)

$$\mathcal{Z}(\kappa, z_1, z_2) = n^4 \int_{|u_1|=1} \int_{|u_2|=1} du_1 du_2 \int_{-\infty}^{\infty} da_1 da_2$$
$$\exp\{n(\phi(u_1, z_{\kappa,1}) + \phi(u_2, z_{\kappa,1}) - \phi(a_1, z_{\kappa,2}) - \phi(a_2, z_{\kappa,2}))\}$$
$$\times F(u_1, u_2, a_1, a_2, U, S) dU dS$$

where U is a unitary 2×2 matrix ($U \in U_j \in \mathring{U}(2)$) and S is a hyperbolic 2×2 matrix ($S \in \mathring{U}(1,1)$)

$$z_{\kappa,l} = E + in^{-1}\sqrt{y_l^2 + \kappa^2}, \quad \phi(u, z) = \frac{u^2}{2} - izu - \log u$$

F is a rather complicated function of u_1, u_2, a_1, a_2, U, S which does not contain n in the main order.

The analysis of $\mathcal{Z}(z, z_b)$ is a standard but rather involved problem of the saddle point method, since there are 4 saddle points and the factor n^4 before the integral makes it necessary to take into account all terms of the forth order in the expansion near the saddle points.

Density for GUE plus rank 1 complex perturbation

Fyodorov and Sommers '96

Recall that $z = E + iy/n\rho_H(E)$.

$$\tilde{\rho}(E, y) = \lim_{N\to\infty} \tilde{\rho}_N(E, y) = -\frac{d}{dy}\left(e^{-y\tau}\frac{\sinh y}{y}\right) \quad (1)$$

where $\tau = (2\pi\rho_H(E))^{-1}\left(\gamma + \gamma^{-1}\right)$ and

$$\rho_H(E) = \rho_{sc}(E) = \frac{1}{2\pi}\sqrt{4 - E^2}, \quad E \in (-2, 2)$$

Given $\tilde{\rho}(E, y)$, expected fraction of the eigenvalues of H_{eff} which lie above the level $\operatorname{Im} z = Y$ can be computed as (here $y = \rho_{sc}(E)NY$)

$$\int_{-2}^{2} \rho_{sc}(E)dE \int_{y}^{\infty} \tilde{\rho}(E, y')dy' \sim \frac{e^{-y(\gamma+\gamma^{-1})}}{y} I_1(2y)$$

where I_1 is the modified Bessel function.

Random band matrices

$$
H = \begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}
$$

Density is still semicircle if the width of the band $W \to \infty$ together with N. However, varying W, one can observe the transition in the local statistics

Conjecture (in the bulk of the spectrum):

$W \gg \sqrt{N}$	Delocalization, GUE statistics
$W \ll \sqrt{N}$	Localization, Poisson statistics

Block random band matrices (Wegner model)

One of the possible realization of RBM is

$$H = \begin{pmatrix} A_1 & B_1 & 0 & 0 & 0 & \dots & 0 \\ B_1^* & A_2 & B_2 & 0 & 0 & \dots & 0 \\ 0 & B_2^* & A_3 & B_3 & 0 & \dots & 0 \\ . & . & B_3^* & . & . & . & . \\ . & . & . & . & . & A_{n-1} & B_{n-1} \\ 0 & . & . & . & 0 & B_{n-1}^* & A_n \end{pmatrix}$$

A_j – GUE $W \times W$ matrices with variance $(1-2\beta)/W$; B_j - Ginibre $W \times W$ matrices with variance β/W, so the variance of entries in each (i,j)-block $(i,j=1,..,n)$ is J_{jk} with $J = I_n/W + \beta\Delta_n/W$, $\beta < \frac{1}{4}$.

Since the size of the matrix is $N = Wn$, the transition should happen at $W \sim n$.

Gaussian case results (without deformation):

- M. Shcherbina, TS'21: GUE local statistics $W \gg N^{1/2}$ $(W \gg n)$
- Goldstein '22: Localization and Poisson statistics $W \ll N^{1/2}$ $(W \ll n)$

Now we consider $\mathcal{H} = \mathrm{H} + i\Gamma$, where H is a Gaussian block band matrix above, and $\Gamma = \mathrm{diag}\{\gamma_1, \dots, \gamma_M, 0, \dots, 0\}$. Recall we need to study

$$Z_{\beta nW}(\kappa, z_1, z_2) = \mathbb{E}\left[\frac{\det\Big\{(\mathcal{H} - z_1)(\mathcal{H} - z_1)^* + \dfrac{\kappa^2}{N^2}\Big\}}{\det\Big\{(\mathcal{H} - z_2)(\mathcal{H} - z_2)^* + \dfrac{\kappa^2}{N^2}\Big\}}\right],$$

where $z_l = E + \dfrac{x_l}{N} + \dfrac{iy_l}{N}, \quad l = 1, 2.$

Generally, similarly to GUE case, one can write an integral representation for $Z_{\beta nW}(\kappa, z_1, z_2)$ and consider the limit $N, W \to \infty$, $W \gg \sqrt{N}$. This representation will give a complicated statistical mechanic system on the lattice $\mathbb{Z} \cap [1, n]$ whose “spins" are 4×4 supermatrices (i.e., matrices containing both usual complex and Grassmann (anticommuting) variables). However, it is much easier to consider first so-called sigma-model approximation, which is often used by physicists to study complicated statistical mechanics systems.

Mathematically, we first rescale $\beta \to \beta/W$ (so the covariance become $J = I_n/W + \beta\Delta_n/W^2$), and then first consider the limit $W \to \infty$ (β and n are fixed), and then in the obtained model consider the limit $\beta, n \to \infty$ (delocalized regime will correspond to $\beta \gg n$).

Theorem (M. Shcherbina, TS '23)

- if $J = I_n/W + \beta\Delta_n/W^2$, then, as $W \to \infty$,

$$Z_{\beta nW}(\kappa, z_1, z_2) \to Z_{\beta n}(\kappa, z_1, z_2)$$

  where $Z_{\beta n}(\kappa, z_1, z_2)$ is a sigma-model approximation (defined below).
- the asymptotic behavior of the sigma-model approximation $Z_{\beta n}(\kappa, z_1, z_2)$ in the delocalized regime $\beta \gg n$ coincides with those for GUE.

Corollary

The density of the imaginary parts of complex eigenvalues of $H_{\text{eff}} = H + i\Gamma$ for the Gaussian block band matrices H and $\Gamma = \text{diag}\{\gamma_1, \dots, \gamma_M, 0, \dots, 0\}$ in the regime $W \gg n$ coincides with density (1) obtained for GUE in a sigma-model approximation.

On a physical level of rigour, the counterpart is also known:

- Fyodorov, Skvortsov, Tikhonov '22: in the regime $W \ll n$ ($W \ll N^{1/2}$) the density is different!

How the sigma-model approximation looks like?

$$Z_{\beta n}(\kappa, z_1, z_2) = e^{E(x_1-x_2)} \int \exp\Big\{ -\frac{\tilde\beta}{4} \sum \mathrm{Str}\, Q_j Q_{j-1} + \frac{c_0}{2n} \sum \mathrm{Str}\, Q_j \Lambda_{\kappa, y_1, y_2} \Big\}$$
$$\times \prod_{a=1}^{M} \mathrm{Sdet}^{-1}\Big(Q_1 - \frac{iE}{2\pi\rho(E)} + \frac{i\gamma_a}{\pi\rho(E)} \mathcal{L}\Sigma \Big) dQ,$$

where $\tilde\beta = (2\pi\rho(E))^2\beta$, Q_j are 4×4 super-matrices depending on 4 Grassmann parameters, and 2×2 unitary matrix U_j and hyperbolic matrix S_j, $Q_j^2 = I$, and

$$\Lambda_{\kappa, y_1, y_2} = \begin{pmatrix} \kappa & -iy_1 & 0 & 0 \\ iy_1 & -\kappa & 0 & 0 \\ 0 & 0 & \kappa & -iy_2 \\ 0 & 0 & iy_2 & -\kappa \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}.$$

Analysis is based on the supersymmetric transfer matrix approach (proposed by Efetov'82, Fyodorov, Mirlin '91-94), so we write

$$\mathrm{Z}_{\beta\mathrm{n}}(\kappa, \mathrm{z}_1, \mathrm{z}_2) = (\mathcal{K}_{\beta\mathrm{n}}^{\mathrm{n}-1}\mathrm{f}, \mathrm{g})$$

where $\mathcal{K}_{\beta\mathrm{n}}$ is an integral operator with the kernel

$$\mathrm{K}_{\beta\mathrm{n}}(\mathrm{Q}_1, \mathrm{Q}_2) = \exp\{\mathrm{F}(\mathrm{Q}_1)/2\} \exp\Big\{ -\frac{\tilde{\beta}}{4}\mathrm{Str}\,\mathrm{Q}_1\mathrm{Q}_2\Big\} \exp\{\mathrm{F}(\mathrm{Q}_2)/2\}$$

with $\mathrm{F}(\mathrm{Q}) = \dfrac{\mathrm{c}_0}{2\mathrm{n}}\mathrm{Str}\,\mathrm{Q}\Lambda_{\kappa,\mathrm{y}_1,\mathrm{y}_2}$.
The main task is to perform the spectral analysis of $\mathcal{K}_{\beta\mathrm{n}}$.

Sample covariance case

One can apply the same techniques to the deformed sample covariance matrices, i.e. to $H_{\mathrm{eff}} = H + i\Gamma$ with $H = n^{-1}X^*X$ where X is a rectangular $m \times n$ matrix with iid mean zero variance 1 entries (in our case Gaussian), and we assume $m/n \to c \in [1, +\infty)$.

Marchenko-Pastur law:

$\rho_{\mathrm{mp}}(E) = (2\pi E)^{-1}\sqrt{(\lambda_+ - E)(E - \lambda_-)}, \quad E \in (\lambda_+, \lambda_-), \quad \lambda_\pm = (1 \pm \sqrt{c})^2.$

Theorem (TS'23)

The density of the imaginary parts of complex eigenvalues of $H_{\mathrm{eff}} = H + i\Gamma$ for Gaussian sample covariance matrices H and $\Gamma = \mathrm{diag}\{\gamma_1, \dots, \gamma_M, 0, \dots, 0\}$ is similar to the GUE case. If $M = 1$, it is

$$\rho(E, y) = -\frac{d}{dy}\left(e^{-y\tau}\frac{\sinh y}{y}\right)$$

where $z = E + iy/(n\rho_{\mathrm{mp}}(E))$, $\tau = \frac{1}{2\pi\rho_{\mathrm{mp}}(E)}\left(\frac{\gamma}{E} + \frac{1}{\gamma}\right)$

(Recall: for GUE $\tau = \frac{1}{2\pi\rho(E)}\left(\gamma + \gamma^{-1}\right)$)

Corrected conjecture (Fyodorov)

For all Hermitian matrices with a local behaviour of GUE type the density $\rho(\mathrm{z})$ with $\mathrm{z} = \mathrm{E} + \mathrm{i}y/\mathrm{n}\rho_{\mathrm{H}}(\mathrm{E})$ is defined by formula above with $\tau = \frac{\mathrm{R(E)}}{2\pi\rho_{\mathrm{H}}(\mathrm{E})}\left(\tilde{\gamma} + \tilde{\gamma}^{-1}\right)$ and $\tilde{\gamma} = \gamma \cdot \mathrm{R(E)}$ where $\mathrm{R(E)} = \lim_{\eta\to+0} |\mathbb{E}(\mathrm{G_{ii}(z)})|$ and $\mathrm{G} = (\mathrm{H} - \mathrm{z})^{-1}$, $\mathrm{z} = \mathrm{E} + \mathrm{i}\eta$ (this limit is actually the Stieltjes transform of ρ_{H} at point E).

For the GUE case the Stieltjes transform is

$$\mathrm{m_{sc}(E)} = \frac{-\mathrm{E} + \mathrm{i}\sqrt{4 - \mathrm{E}^2}}{2} \Longrightarrow |\mathrm{m_{sc}(E)}| = 1 \Longrightarrow \tau = \frac{1}{2\pi\rho_{\mathrm{sc}}(\mathrm{E})}\left(\gamma + \gamma^{-1}\right)$$

For the Marchenko-Pastur law the Stieltjes transform is

$$\begin{aligned}\mathrm{m_{mp}(E)} &= \frac{-(\mathrm{E} + 1 - \mathrm{c}) + \mathrm{i}\sqrt{2\mathrm{Ec} + 2\mathrm{E} + 2\mathrm{c} - \mathrm{E}^2 - \mathrm{c}^2 - 1}}{2\mathrm{E}} \\ &\Longrightarrow |\mathrm{m_{mp}(E)}| = \mathrm{E}^{-1/2} \Longrightarrow \tau = \frac{1}{2\pi\rho_{\mathrm{mp}}(\mathrm{E})}\left(\frac{\gamma}{\mathrm{E}} + \frac{1}{\gamma}\right).\end{aligned}$$