

Upper tail scaling limit of continuum path measures in KPZ

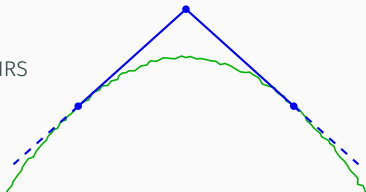
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(based on joint works with Shirshendu Ganguly and Lingfu Zhang)

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The directed landscape and continuum directed random polymer

- Directed landscape $\mathcal{L} : \mathbb{R}^4 \rightarrow \mathbb{R}$ is a random continuous function expected to be a universal KPZ scaling limit.
- It is a **last passage percolation** problem: continuous paths $\gamma : [s, t] \rightarrow \mathbb{R}$ are given a random **weight** $w(\gamma)$, and

$$\mathcal{L}(x, s; y, t) = \sup_{\substack{\gamma : [s, t] \rightarrow \mathbb{R} \\ \gamma(s) = x, \gamma(t) = y}} w(\gamma).$$

The **geodesic** is the path achieving the maximum

- $\mathcal{L}(0, 0; \cdot, 1)$ is the **weight profile** of the geodesic.



The directed landscape and continuum directed random polymer

- Positive temperature analogue: continuum directed random polymer (CDRP).
- White noise environment ξ on $\mathbb{R} \times \mathbb{R}$, continuous paths $\gamma : [s, t] \rightarrow \mathbb{R}$ have a weight $w(\gamma)$,

$$w(\gamma) = \int_s^t \xi(z, \gamma(z)) dz.$$

- Polymer measure defined via partition function Z^β :

$$Z(x, s; y, t) = \mathbf{E}^{x, s, y, t}[\exp(w(\gamma))].$$

Z is a function of ξ ; $\mathbf{E}^{x, s, y, t}$ is over γ only and distributes it as a Brownian bridge from (x, s) to (y, t) .

- $\mathfrak{h}(0, 0; \cdot, 1) = \log Z(0, 0; \cdot, 1)$ is the free energy profile.



- For both the DL and the CDRP, the location of the geodesic Γ or marginal of the polymer measure μ at a height s are given by convolution formulas:

$$\Gamma(s) = \operatorname{argmax}_{x \in \mathbb{R}} \mathcal{L}(0, 0; x, s) + \mathcal{L}(x, s; 0, 1)$$
$$\mu(\Gamma(s) = x) = \frac{\exp(\mathfrak{h}(0, 0; x, s) + \mathfrak{h}(x, s; 0, 1))}{\int_{\mathbb{R}} \exp(\mathfrak{h}(0, 0; y, s) + \mathfrak{h}(y, s; 0, 1)) dy}.$$

- Much structure of the path measures can be understood via the profile processes \mathcal{L} and \mathfrak{h} .

- The free energy profile \mathfrak{h} solves the KPZ equation, given by

$$\partial_t h = \frac{1}{4}(\partial_x h)^2 + \frac{1}{4}\partial_x^2 h + \xi,$$

where ξ is space-time white noise on $\mathbb{R} \times (0, \infty)$ and $h : \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}$.

- We will use the Cole-Hopf notion of solution to the KPZ equation, i.e., \mathfrak{h} is defined via $\log Z$ where Z solves the multiplicative SHE:

$$\begin{cases} \partial_t Z(y, t \mid x, s) = \frac{1}{4}\partial_y^2 Z(y, t \mid x, s) + \xi(y, t)Z(y, t \mid x, s) \\ Z(y, s \mid x, s) = \delta_0(x - y) \end{cases} \quad \text{for all } s > 0$$

- Introduced by Alberts-Khanin-Quastel, regularity recently studied by Alberts-Janjigian-Rassoul-Agha-Seppäläinen.

The upper tails and upper large deviations of these two processes have been studied for quite some time, eg.

- One-point large deviations/upper tails for \mathcal{L} were known from work of Tracy-Widom, see also Rider-Ramirez-Virág.
- Seppäläinen and Johansson studied one-point large deviations of prelimiting zero temp. models (TASEP and geometric LPP resp.)
- Quastel-Tsai studied profile large deviations of TASEP.
- Corwin-Ghosal, Ganguly-H., Tsai-Lin studied upper tails of \mathfrak{h} .
- Prelimiting models for \mathfrak{h} : ASEP by Das-Zhu and Damron-Petrov-Sivakoff.
- and more...

The random path under the upper tail conditioning

Interested in the behaviour of the geodesic or polymer measure when $\mathcal{L}(0, 0; 0, 1)$ or $\mathfrak{h}(0, 0; 0, 1)$ is large, say $> \theta$.

An **energy-entropy** tradeoff occurs: larger fluctuations give the geodesic more choice of paths, but the cost grows with θ .

So the path measure will become more **rigid**, i.e., have much smaller transversal fluctuations. (It also becomes a “highway” for geodesics to nearby points.)



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The scaling limit of the geodesic under upper tail conditioning

Let $\Gamma_\theta : [0, 1] \rightarrow \mathbb{R}$ be the **geodesic** in the directed landscape from $(0, 0)$ to $(0, 1)$, conditioned on $\mathcal{L}(0, 0; 0, 1) > \theta$.

Theorem (Ganguly-H.-Zhang)

$\theta^{-1/4} \Gamma_\theta \xrightarrow{d} \frac{1}{2} B$ in the uniform topology with $B =$ standard *Brownian bridge*.

Note that we identify the fluctuation scale to be $\theta^{-1/4}$ as well as the scaling limit.

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This result had been conjectured by Zhipeng Liu, who proved the one-point scale and one-point convergence using exact formulas.

A similar result had earlier been conjectured by Basu-Ganguly for the geodesic in exponential LPP under a **large deviation** conditioning.

Let $\Gamma_\theta^{\text{ann}} : [0, 1] \rightarrow \mathbb{R}$ be a sample from the **annealed polymer measure** from $(0, 0)$ to $(0, 1)$ in the CDRP, under the conditioning that $\mathfrak{h}(0, 0; 0, 1) > \theta$.

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What about the **quenched** situation? The polymer measure concentrates in a $O(\theta^{-1/2})$ window around a random "backbone" $\Gamma_\theta^{\text{back}}$, and $\theta^{-1/4} \Gamma_\theta^{\text{back}} \xrightarrow{d} \frac{1}{2} B$.

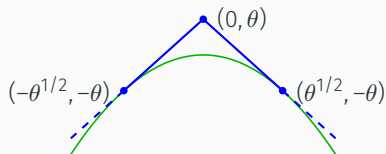
A crucial ingredient: an upper tail limit shape

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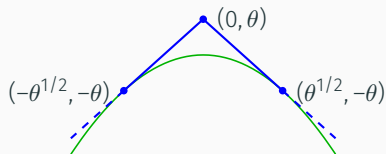


The linear portions of Triangle_θ are *tangent* to $-x^2$ at $\pm\theta^{1/2}$.

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Theorem (Ganguly-H.)

There exist θ_0 and $c > 0$ such that, for all $t \geq 1$, $\theta > \theta_0$, and $M > 0$,

$$\mathbb{P} \left(\sup_{x \in [-\theta^{1/2}, \theta^{1/2}]} |\mathfrak{h}(x) - \text{Triangle}_\theta(x)| > M\theta^{1/4} \mid \mathfrak{h}(0) = \theta \right) \leq \exp(-cM^2).$$

A second crucial ingredient: an upper tail comparison

From the limit shape, one can obtain sharp asymptotics for the upper tail:

Theorem (Ganguly-H.)

There exist $C < \infty$ and θ_0 such that, for all $\theta > \theta_0$,

$$\exp\left(-\frac{4}{3}\theta^{3/2} - C\theta^{3/4}\right) \leq \frac{1}{d\theta} \mathbb{P}(\mathfrak{h}(0) \in d\theta) \leq \exp\left(-\frac{4}{3}\theta^{3/2} + C\theta^{3/4}\right).$$

As an immediate consequence, the same bounds also hold for $\mathbb{P}(\mathfrak{h}(0) > \theta)$.

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By more refined coupling arguments, we also get a comparison statement:

Theorem (Ganguly-H.-Zhang)

There exist $C < \infty$ and θ_0 such that, for all $\delta > 0$ and $\theta > \theta_0$,

$$\frac{\mathbb{P}(\mathfrak{h}(0) \geq \theta + \delta)}{\mathbb{P}(\mathfrak{h}(0) \geq \theta)} = \exp\left(-2\delta\theta^{1/2} + O(\delta)\right).$$

Heuristics and proof ideas

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- $x \mapsto \mathfrak{h}(0, 0; x, 1) + x^2$ and $x \mapsto \mathcal{L}(0, 0; x, 1) + x^2$ are stationary.
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- So the geodesic/polymer fluctuating by ε means it suffers a loss of $O(\varepsilon^2)$.
- Under the conditioning of being $> \theta$, this loss has to be made up; akin to $\mathfrak{h}(0, 0; 0, 1) > \theta + O(\varepsilon^2)$ (by stationarity).
- But $\frac{\mathbb{P}(\mathfrak{h}(0, 0; 0, 1) > \theta + O(\varepsilon^2))}{\mathbb{P}(\mathfrak{h}(0, 0; 0, 1) > \theta)} \approx \exp(-C\varepsilon^2\theta^{1/2})$.
- This is $O(1)$ exactly when $\varepsilon = O(\theta^{-1/4})$.

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- Scale $x \mapsto x\theta^{-1/4}$. By the comparison theorem, this ratio is

$$\exp\left(-2\theta^{1/2}(x\theta^{-1/4})^2[s^{-1} + (1-s)^{-1}]\right) = \exp\left(-\frac{x^2}{2 \times \frac{1}{4}s(1-s)}\right),$$

i.e., the exponent of the density of $\frac{1}{2}B(s) = N(0, \frac{1}{4}s(1-s))$.

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- These are provided by **coalescence**.
- $(\Gamma(s), \Gamma(t)) = \operatorname{argmax}_{z_1, z_2} \mathcal{L}(0, 0; z_1, s) + \mathcal{L}(z_1, s; z_2, t) + \mathcal{L}(z_2, t; 0, 1)$
- Coalescence gives quadrangle equality:
 $\mathcal{L}(x_1 \rightarrow y_1) + \mathcal{L}(x_2 \rightarrow y_2) = \mathcal{L}(x_1 \rightarrow y_2) + \mathcal{L}(x_2 \rightarrow y_1)$
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- So the double argmax **decouples**.
- Coalescence also guarantees the two argmaxes occur in **disjoint** time strips, so **independent**.



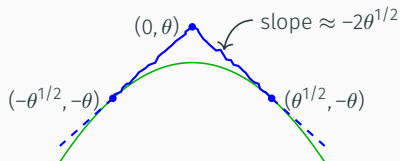
The source of $\theta^{-1/2}$ window around backbone

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Why does the polymer measure concentrate in a $\theta^{-1/2}$ window around $\Gamma_\theta^{\text{back}}$?

- Recall that when $\mathfrak{h}(0, 0; 0, 1) > \theta$, the profile has slope approximately $-2\theta^{1/2}$.
- So at distance $O(\theta^{-1/2})$, the loss in free energy is $O(1)$; all such locations are therefore competitive for the polymer measure.
- Different scale in zero temp: the argmax location will be on scale θ^{-1} , as then the slope loss and Brownian fluctuations are of the same order, $\theta^{-1/2}$.



Summary

- Using geometric methods + Brownian Gibbs properties, we can obtain the **shape** of the weight and free energy profiles under **upper tail** events.
- These also give sharp upper tail **asymptotics** and probability **comparison** statements.
- With these + “tent” picture, can prove that geodesic/polymer measure rescaled by $\theta^{-1/4}$ converges to a **Brownian bridge**, under upper tail.
- Further, the polymer measure fluctuates on scale $\theta^{-1/2}$ around a random “**backbone**” curve.

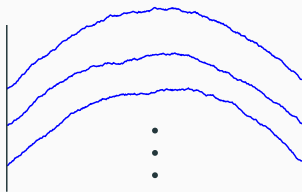
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Happy birthday Timo!

The Brownian Gibbs property

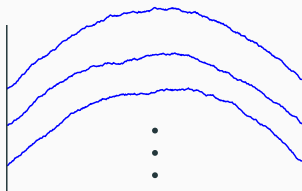
The resampling property

Both $\mathfrak{h}(0, 0; \cdot, 1)$ and $\mathcal{L}(0, 0; \cdot, 1)$ can be embedded as the top/lowest-indexed curve in a \mathbb{N} -indexed ensemble of random continuous curves.



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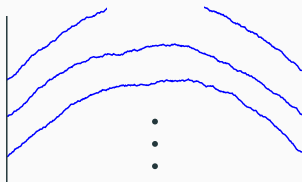


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For the \mathcal{L} ensemble, it says the conditional distribution of \mathcal{L} on an interval is a **non-intersecting Brownian bridge** (of rate 2).

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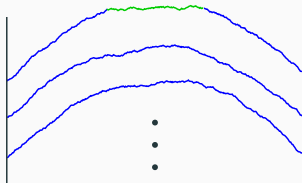


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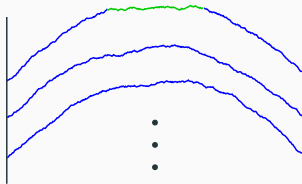


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A useful heuristic to keep in mind:

\mathfrak{h} and \mathcal{L} are like Brownian bridges conditioned to stay above a parabola $-x^2$ with which they share endpoints.

A convex consequence

A key idea is that the resampling in terms of Brownian bridges implies that the limit shapes should be **convex**.

Indeed, suppose the limit shape of the top curve is not convex in some neighbourhood. This pushes the **second curve** down on the interval.

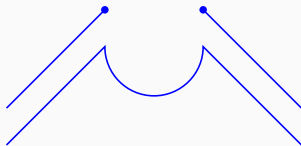


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Then **resample** the **top curve** on that interval. *Since the non-convexity means the second curve is far away, Brownian bridge naturally avoids it.*



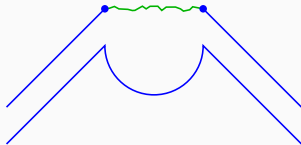
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