Instability in the KPZ equation

Evan Sorensen

Joint work with Sean Groathouse, Firas Rassoul-Agha, and Timo Seppäläinen

University of Wisconsin - Madison

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Happy Birthday Timo!

The KPZ equation is

$$\partial_t h(t,x) = \frac{1}{2}\partial_{xx} h(t,x) + \frac{1}{2}(\partial_x h(t,x))^2 + W(t,x), \quad h(s,x) = h_s(x).$$

We consider the Cole-Hopf solution $h = \log Z$, where Z solves the SHE

$$\partial_t Z(t,x) = \frac{1}{2}\partial_{xx} Z(t,x) + Z(t,x) W(t,x), \quad Z(s,x) = e^{h_s(x)}.$$

We will often work with the four-parameter field $\{Z(t,y|x,s) : t > s, x,y \in \mathbb{R}\}$ of narrow wedge solutions studied by Alberts, Khanin, Quastel (2012) and Alberts, Janjigian, Rassoul-Agha, Seppäläinen (2022).

The Busemann process

Janjigian, Rassoul-Agha, and Seppäläinen (2022) constructed the Busemann process $\{b(s,x;t,y) : s,x,t,y, \in \mathbb{R}\}$ defined for all directions simultaneously.

They define

$$\Lambda^{\omega} = \{ \in \mathbb{R} : b(s,x;t,y) \neq b^{+}(s,x;t,y) \text{ for some } (s,x;t,y) \in \mathbb{R}^4\}.$$

Busemann functions for the SHE

Theorem (Janjigian, Rassoul-Agha, Seppäläinen, 2022)

The Busemann functions are space-time stationary global solutions to the KPZ equation: for $s \in \mathbb{R}$, and $t > r$,

$$e^{b(s,x;t,y)} = \int_{\mathbb{R}} e^{b(s,x;r,z)} Z(t,y\,|\,r,z)\,dz.$$

The One Force–One Solution principle

Theorem (Janjigian, Rassoul-Agha, Seppäläinen, 2022)

For fixed $\in \Lambda$, whenever $f : \mathbb{R} \quad \mathbb{R} \to \mathbb{R}$ satisfies appropriate growth conditions, the following limit holds uniformly on compact sets of $(s,x,t,y) \in \mathbb{R}^4$:

$$\lim_{r \to \infty} \frac{\int_{\mathbb{R}} e^{f(r,z)} Z \,(t,y|r,z)\,dz}{\int_{\mathbb{R}} e^{f(r,z)} Z \,(s,x|r,z)\,dz} = e^{b \;(s,x,t,y)}.$$

The One Force–One Solution principle

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For fixed $\in \Lambda$, whenever $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies appropriate growth conditions, the following limit holds uniformly on compact sets of $(s, x, t, y) \in \mathbb{R}^4$:

$$\lim_{r \to \infty} \frac{\int_{\mathbb{R}} e^{f(r,z)} Z(t, y | r, z)\, dz}{\int_{\mathbb{R}} e^{f(r,z)} Z(s, x | r, z)\, dz} = e^{b(s,x,t,y)}.$$

One way for f to satisy these conditions is if $f(r, z) = g(z)$ with $\lim_{|x| \to +\infty} \frac{g(z)}{z} =$.

We call this the One Force–One Solution principle (1F1S)

The 1F1S principle fails exactly when $\in \Lambda$.

History of the 1F1S principle

1F1S principles have been established in a fixed direction for the stochastic Burgers equation and the Burgers equation with random forcing.

Sinai (1997), Kifer (1997), Khanin-Mazel-Sinai (2000), Hoang-Khanin (2003), Dirr-Souganidis (2005), Gomes, et. al (2005), Bakhtin-Khanin (2010), Bakhtin (2013), Bakhtin-Cator-Khanin (2014), Bakhtin (2016), Bakhtin-Li (2016), Drivats et. al (2022).

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Busemann functions in FPP/LPP/polymer models: Hoffman (2005), Hoffman (2008), Damron-Hanson (2014), Hanson (2018), Georgiou-Rassoul-Agha-Seppäläinen (2017), Janjigian-Rassoul-Agha (2020), Seppäläinen (2020), Seppääinen-S. (2021), Seppäläinen-S. (2023), Rahman-Virǵ (2021), Busani-Seppäläinen-S. (2022), Ganguly-Zhang (2022).

Discontinuities of the Busemann process

Theorem (Janjigian, Rassoul-Agha, Seppäläinen, 2022)

For each $\quad 2\ \mathrm{R}$, $\mathrm{P}(\quad 2\ \Lambda^{!}) = 0$. Furthermore, exactly one of the following is true:

- $\mathrm{P}(\Lambda^{!} = ?) = 1$
- $\mathrm{P}(\Lambda^{!}$ is countably infinite and dense in $\mathrm{R}) = 1$.

In exponential LPP (Janjigian, Rassoul-Agha, Seppäläinen), Brownian LPP (Seppäläinen, S.), and the directed landscape (Busani, Seppäläinen, S.), discontinuities exist.

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Theorem (Groathouse, Rassoul-Agha, Seppäläinen, S., 2023+)

P($\Lambda^{!}$ is countably infinite and dense in R) $= 1$.

Describing the Busemann process

It is enough to describe the distribution of the Busemann process for a fixed time level:

$$\{b \quad (s,0;s,x) : x, \quad \in \mathbb{R}\}.$$

In this case, each $x \mapsto b \quad (s,0;s,x)$ is a two-sided Brownian motion with diffusivity and drift .

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Coupling of Brownian motions with drift

We construct a process $\{F\}_{\in \mathbb{R}}$ whose finite-dimensional marginals are described as follows: for $_1 < {}_2 < \quad < {}_k$,

$$(F^{1}, F^{2}, \dots, F^{k}) \overset{d}{=} (Y^1, D^{(2)}(Y^2, Y^1), \dots, D^{(k)}(Y^k, \dots, Y^1)),$$

where $Y^1, \dots, Y^k$ are independent Brownian motions with drifts $_1, \dots, {}_k$,

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where $Y^1, \dots, Y^k$ are independent Brownian motions with drifts $_1, \dots, \, _k$, and

$$D^{(2)}(Y^2, Y^1)(y) = Y^1(y) + \frac{1}{} \log \int_{\infty}^{y} e^{(Y^2(x) \quad Y^1(x))}\,dx \quad \frac{1}{} \log \int_{\infty}^{0} e^{(Y^2(x) \quad Y^1(x))}\,dx,$$

Coupling of Brownian motions with drift

We construct a process $fF \ g_{\ 2\mathbb{R}}$ whose finite-dimensional marginals are described as follows: for $\ _1 < \ _2 < \quad < \ _k$,

$$(F^{\ 1}; F^{\ 2}; :::; F^{\ k}) \overset{d}{=} (Y^1; D^{(2)}(Y^2; Y^1); :::; D^{(k)}(Y^k; :::; Y^1));$$

where $Y^1; :::; Y^k$ are independent Brownian motions with drifts $\ _1; :::; \ _k$, and

$$D^{(2)}(Y^2; Y^1)(y) = Y^1(y) + \frac{1}{\ } \log \ Z_{1}^{y} e^{\ (Y^2(x) \ \ Y^1(x))} dx$$
$$\frac{1}{\ } \log \ Z_{1}^{0} e^{\ (Y^2(x) \ \ Y^1(x))} dx;$$

$$D^{(k)} = D^{(2)}(D^{(k \ \ 1)}(Y^k; :::; Y^2); Y^1):$$

The KPZ horizon

We call the process $fF \;\; g \;\; _{2\mathrm{R}}$ the KPZ horizon.

The fact that $D^{(2)}(Y^2; Y^1)$ is a BM with the same drift as Y^2 is a result of O'Connell and Yor (2001).

The KPZ horizon and the Busemann process

Lemma (GRASS 2023+)

The KPZH describes the Busemann process for the SHE. That is,

$$\{b^{(\ \ )+}(0,0;0;\ )\}_{\ \in \mathbb{R}} \overset{d}{=} \{\ F\ \}_{\ \in \mathbb{R}}.$$

For i.i.d Brownian motions $\{B_r\}_{r\in\mathbb{Z}}$, $m \le n$, and $x \le y$, define

$$Z^{\text{sd}}(n,y|m,x) = \int_{x=x_{m-1}<x_m<\cdots<x_{n-1}<x_n=y} \exp\Big(\sum_{r=m}^{n} B_r(x_{r-1},x_r)\Big)\, dx_m \cdots dx_{n-1}$$

For an appropriate random or deterministic initial function $f:\mathbb{R}\to\mathbb{R}$, define, for $n \ge 0$ and $y \in \mathbb{R}$,

$$Z^{\text{sd}}(n,y|f) = \int_{-\infty}^{y} f(x)\, Z^{\text{sd}}(n,y|0,x)\, dx$$

$$\overline{Z^{\text{sd}}}(n,y|f) = \frac{Z^{\text{sd}}(n,y|f)}{Z^{\text{sd}}(n,0|f)}.$$

Lemma (GRASS 2023+)

Let $fF\ g\ _{2\mathrm{R}}$ be the KPZH. Then, for $0 < \ _1 < \quad < \ _k$ and each $n \quad 0$,

$$\overline{Z^{\mathrm{sd}}}(n;\ je\ ^{F\ 1});:::;\overline{Z^{\mathrm{sd}}}(n;\ je\ ^{F\ k}) \ \stackrel{d}{=}\ e\ ^{F\ 1};:::;e\ ^{F\ k}$$

Lemma (GRASS 2023+)

Let $\{F_\theta\}_{\theta\in\mathbb{R}}$ be the KPZH. Then, for $0 < \theta_1 < \cdots < \theta_k$ and each $n \ge 0$,

$$\left(\overline{Z^{\mathrm{sd}}}(n,\cdot\,|\,e^{F_{\theta_1}}),\ldots,\overline{Z^{\mathrm{sd}}}(n,\cdot\,|\,e^{F_{\theta_k}})\right) \overset{d}{=} \left(e^{F_{\theta_1}},\ldots,e^{F_{\theta_k}}\right)$$

The recentered versions of Z^{sd} satisfy

$$\log \overline{Z^{\mathrm{sd}}}(n,\cdot\,|\,f) = D^{(2)}(\log \overline{Z^{\mathrm{sd}}}(n-1,\cdot\,|\,f), B_n).$$

The proof of the theorem follows by an intertwining argument originating from the work of Ferrari and Martin (2007) and adapted in Fan and Seppäläinen (2020), and Seppäläinen and S. (2021).

Scaling invariance of the KPZH

For $\;> 0$, the KPZH satisfies the following scaling invariance:

$$\exp \; N^{\;1/4} \; F_{N^{\;1/4}}^{(\;+1/2)N^{\;1/4}+N^{1/4}}(y\sqrt{N}) \quad (\sqrt{N}+1/2)y \;:\; y \in \mathbb{R} \quad _{\in \mathbb{R}}$$

$$\overset{d}{=} \; \exp \; F \; (y) \;:\; y \in \mathbb{R} \quad _{\in \mathbb{R}}.$$

Scaling invariance of the KPZH

For > 0, the KPZH satisfies the following scaling invariance:

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$$\stackrel{d}{=} \exp \quad F \; (y) \; : y \in \mathbb{R} \quad {}_{\in\mathbb{R}}.$$

Combined with the convergence of the O'Connell-Yor polymer to the SHE (Nica, 2016) and some additional details, we obtain

Lemma (GRASS 2023+)

For ${}_1 < \quad < {}_k$,

$$\left(\frac{\int_{\mathbb{R}} e^{F^{\ i}(x)} Z\,(t; \,|0; x)}{\int_{\mathbb{R}} e^{F^{\ i}(x)} Z\,(t; 0|0; x)} \right)_{1 \; i \; k} \stackrel{d}{=} \{e^{F^{\ i}}\}_{1 \; i \; k}$$

Consequence for Busemann process

Corollary (GRASS 2023+)

The KPZH describes the Busemann process for the SHE. That is,

$$\{b^{(\ )+}(0,0,0,\ )\}_{\ \in \mathbb{R}} \overset{d}{=} \{F\ \}_{\ \in \mathbb{R}}.$$

Consequences for the Busemann process

Fix $y > 0$. Then, the process $\mapsto F(y)$ is strictly increasing and has stationary increments.

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Lemma (GRASS 2023+)

Let $\mapsto X(\ )$ be an increment-stationary, nondecreasing, almost surely continuous process with $E[X(1) - X(0)] < \infty$. Then, for every $\varepsilon > 0$,

$$\lim_{n \to \infty} n P(X(n^{-1}) - X(0) > \varepsilon) = 0.$$

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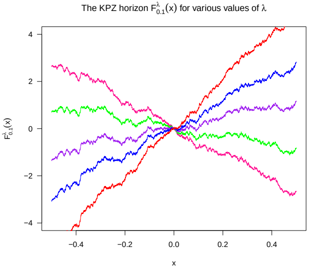
Let $\mapsto X(\ )$ be an increment-stationary, nondecreasing, almost surely continuous process with $E[X(1) \quad X(0)] < \infty$. Then, for every $\varepsilon > 0$,

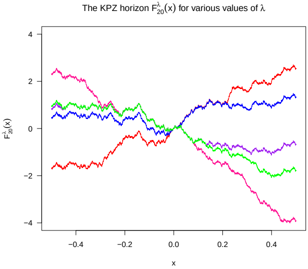
$$\lim_{n \to \infty} n P(X(n^{\ 1}) \quad X(0) > \varepsilon) = 0.$$

After a computation, the KPZH satisfies, for $y, \varepsilon > 0$

$$\liminf_{\searrow 0} \ ^{1} P(F \ (y) \quad F^{0}(y) > \varepsilon) > 0.$$

Limits of the KPZH in the parameter





Theorem (GRASS, 2023+)

As $\to \infty$, we have the distributional convergence

$$(F^{1}, \dots, F^{k}) \Longrightarrow (G_{1}, \dots, G_{k}),$$

where $\{G\}_{\in \mathbb{R}}$ is the stationary horizon (coupled invariant measures for the KPZ fixed point).

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$$(F^{1}, \dots, F^{k}) \Longrightarrow (B( ) + {}_{1}, \dots, B( ) + {}_{k}),$$

where B is a standard Brownian motion.

By the scaling relations of the KPZH, for any $\quad > 0$,

$$\{F^{\,i}_{2\;\;1/3\,T^{1/3}}\}_{1\;\;i\;\;k} \overset{d}{=} \Big\{ 2^{1/3}T\;\;{}^{1/3}F + 2^{1/3}T\;\;{}^{1/3}\;{}_{i}\;\; \frac{2^{1/3}T^{2/3}}{2} \qquad \frac{2^{2/3}T^{1/3}}{} \Big\}_{1\;\;i\;\;k}.$$

By the scaling relations of the KPZH, for any $\quad > 0$,

$$\{F_{2}{}^{i}{}_{1/3\,T^{1/3}}\}_{1\quad i\quad k}$$
$$\stackrel{d}{=}\Big\{\,2^{1/3}T\quad{}^{1/3}F\;+\;2^{1/3}T\quad{}^{1/3}\,{}_{i}\;\;\frac{2^{1/3}T^{2/3}}{2}\qquad\frac{2^{2/3}T^{1/3}}{}\Big\}_{1\quad i\quad k}.$$

Combined with recent results of Wu (2023), the 1:2:3 scaling of the coupled stationary KPZ equation converges to the coupled stationary KPZ fixed point (as a process in space-time).