"Stochastic homogenization of nonconvex viscous HJ equations in 1D"

Atilla Yilmaz, Temple University

Random Growth Models and KPZ universality

(Timo-Fest)

BIRS

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$$\frac{\partial u}{\partial t} = \underbrace{\mathrm{tr}\left(a(x) D^2 u\right)}_{\sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}} + H\Big(\underbrace{Du}_{\left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_d}\right)}, x\Big), \qquad (t,x) \in (0,+\infty) \times \mathbb{R}^d, \qquad d \geq 1.$$

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$$\left.\begin{array}{l}\text{Geometrical optics} - \text{Hamilton (1828)} \\ \text{Classical mechanics} - \text{Jacobi (1884)}\end{array}\right\}\ \text{Hamilton - Jacobi equation}$$

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Two-person differential games: $a \equiv 0$. H is general.

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Stochastic optimal control

Two-person stochastic differential games

$a \not\equiv 0$ (viscous).

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$$\left.\begin{array}{l}\text{Stochastic optimal control} \\ \text{Two-person stochastic differential games}\end{array}\right\} \quad a \not\equiv 0 \quad \text{(viscous).}$$

Suppose $a(x) = a(x, \omega)$ and $H(p, x) = H(p, x, \omega)$ are random.

$(\Omega, \mathcal{F}, \mathbb{P})$ probability space, $\omega \in \Omega$.

$x \longmapsto a(x, \omega)$ and $x \longmapsto H(p, x, \omega)$ are stationary & ergodic.

Let's zoom out:

$$(HJ_\varepsilon) \qquad \frac{\partial u^\varepsilon}{\partial t} = \varepsilon \operatorname{tr}\left(a\left(\frac{x}{\varepsilon}, \omega\right) D^2 u^\varepsilon \right) + H\left(Du^\varepsilon, \frac{x}{\varepsilon}, \omega \right), \qquad (t,x) \in (0,+\infty) \times \mathbb{R}^d.$$

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Questions:

① As $\varepsilon \to 0$, does this HJ equation effectively become

$$(\overline{HJ}) \qquad \frac{\partial \bar{u}}{\partial t} = \overline{H}(D\bar{u}), \qquad (t,x) \in (0,+\infty) \times \mathbb{R}^d,$$

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for some $\overline{H} : \mathbb{R}^d \to \mathbb{R}$? "Homogenization".

② If so, then $\underbrace{\overline{H}(p) = ?}_{\text{formula? properties?}}$ "Effective Hamiltonian" (deterministic).

Precisely: For every $g \in UC(\mathbb{R}^d)$, let u_g^ε and $\bar{u}_g$ be the unique viscosity solutions of these HJ equations with

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"Qualitative homogenization" (vs. "quantitative homogenization")

A strategy: Consider the static HJ equation

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Then, $u^\varepsilon(t,x,\omega) = \lambda(p)t + p\cdot x + \underbrace{\varepsilon F\left(\frac{x}{\varepsilon}, \omega\right)}_{\text{"correcting"}}$ solves $(\mathrm{HJ}_\varepsilon)$.

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with linear initial data, and $\overline{H}(p) = \lambda(p)$.

One then generalizes this to uniformly continuous initial data.

This strategy works when $(a(x), H(p,x))$ is periodic in each x_i.

compactness

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1. If $p \mapsto H(p,x,\omega)$ is convex, then (HJ_ε) homogenizes.

- $a \equiv 0$: Souganidis (1999)
  Rezakhanlou – Tarver (2000)
- $a \not\equiv 0$: Lions – Souganidis (2005)
  Kosygina – Rezakhanlou – Varadhan (2006)

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Subadditive ergodic theorem / large deviations techniques.

$\overline{H}(p)$ has a variational formula.

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Special case: $H(p,x,\omega) = \frac{1}{2}|p|^2 + b(x,\omega)\cdot p + V(x,\omega)$.

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- Quenched free energy of diffusions in random potential.

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Sublinear correctors correspond to Busemann functions.

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- there are some positive results in $d \geq 2$, e.g.
  - + Armstrong - Souganidis (2013): $a \equiv 0$, H is quasiconvex;
  - + Armstrong - Tran - Yu (2015): $a \equiv 0$, H is rotation invariant;
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- there are counterexamples to homogenization in $d \geq 2$.
  - Ziliotto (2017); Feldman – Souganidis (2017): $a \equiv 0$;
  - Feldman – Fermanian – Ziliotto (2020): $a \not\equiv 0$.

  Two-person (stochastic) differential game, H has a saddle point.

From now on: $d=1$, no periodicity or convexity assumption.

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$\sqrt{a(\cdot,\omega)}$ and $V(\cdot,\omega)$ are Lipschitz.

$\beta > 0$.

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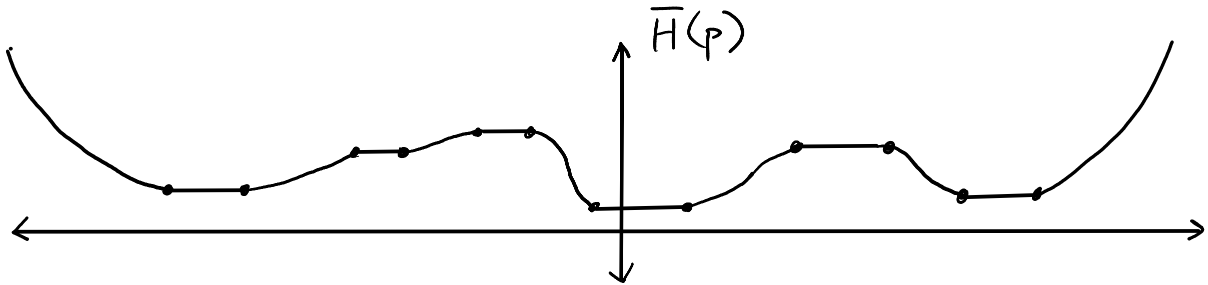

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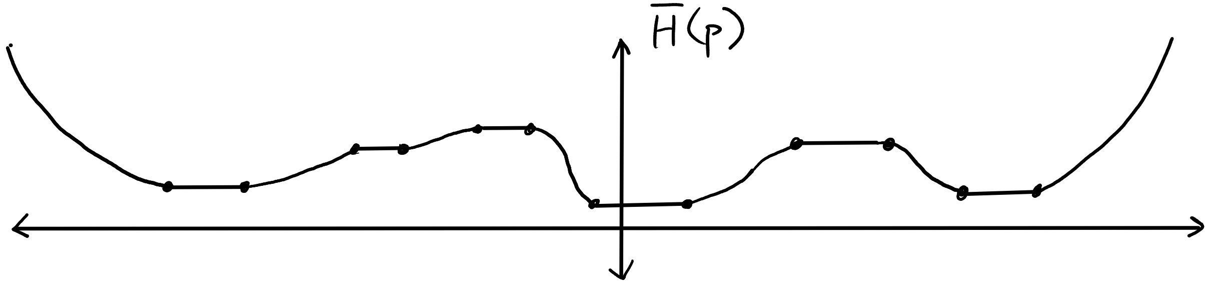

Generalized to nonseparated $H(p, x, \omega)$ by Gao (2016).

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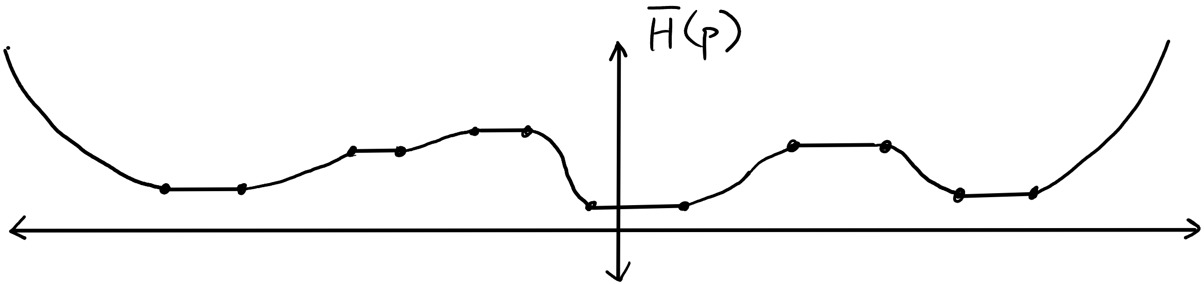

Regular hill and valley condition: For every $h \in (0,1)$ and $y > 0$,

$$\mathbb{P}\left(V(\cdot,\omega) \underset{\leq}{\geq} h \text{ on } [0,y] \right) > 0 \quad \begin{array}{c}(\text{hill}). \\ (\text{valley})\end{array}$$

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Holds for a large class of potentials,
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By ergodicity, for every $h \in (0,1)$, $y > 0$ and $\mathbb{P}$-a.e. ω,

$$\exists\, \ell \in \mathbb{R} \text{ s.t. } \quad V(\cdot,\omega) \geq h \text{ on } [\ell, \ell+y] \qquad (\text{hill}).$$
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Scaled hill and valley condition: For every $h \in (0,1)$, $y > 0$ and $\mathbb{P}$-a.e. ω,

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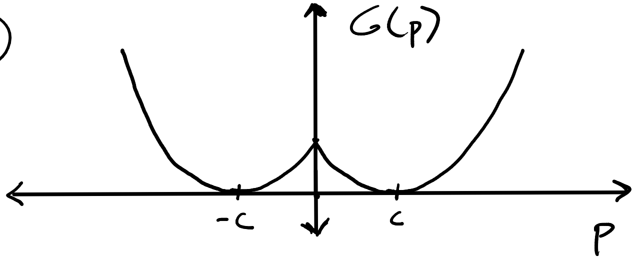
When $a \equiv 0$, the scaled hill and valley condition is trivially satisfied.

Previous results with special choices for G :

①



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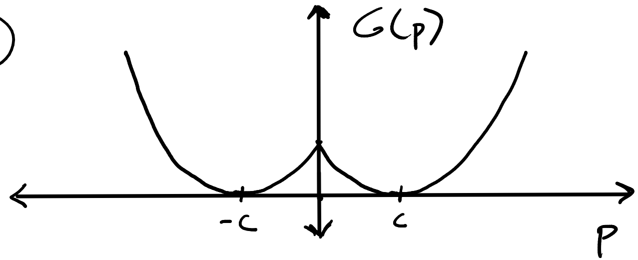
① 

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Risk-sensitive stochastic optimal control in a random potential.

- Y. – Zeitouni (2019): random walk
- Kosygina – Y. – Zeitouni (2020): Brownian motion

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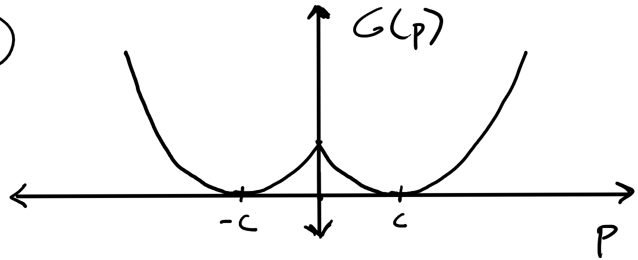

$$G(p) = \min\{(p-c)^2, (p+c)^2\}.$$

Risk-sensitive stochastic optimal control in a random potential.

- Y.-Zeitouni (2019): random walk
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Regular hill and valley condition is introduced.

Previous results with special choices for G :

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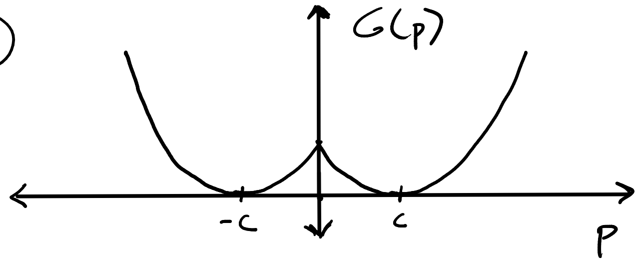

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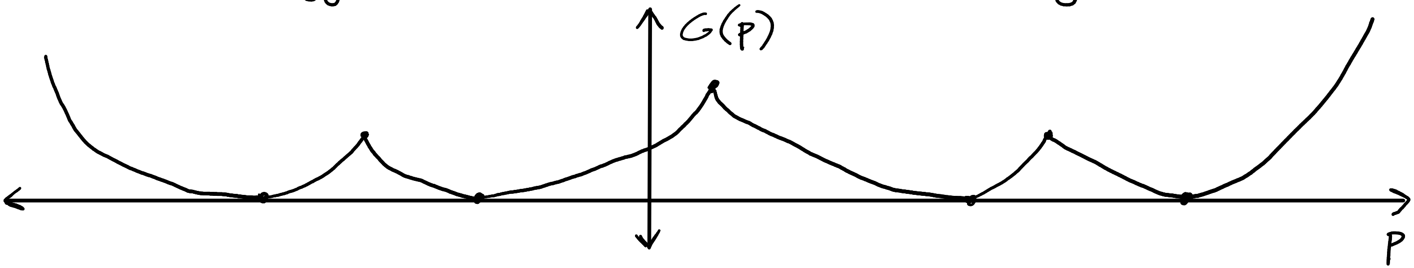
Sublinear correctors have explicit control representations.

(2) Davini - Kosygina (2022): scaled hill and valley, $a \in [0,1]$.



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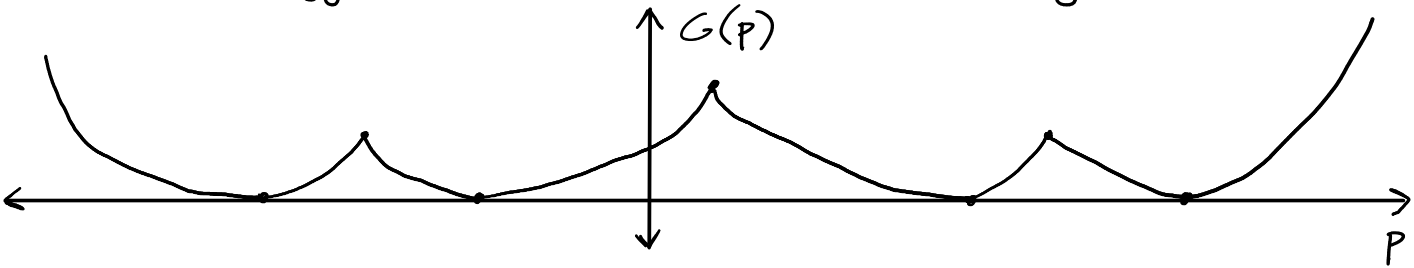

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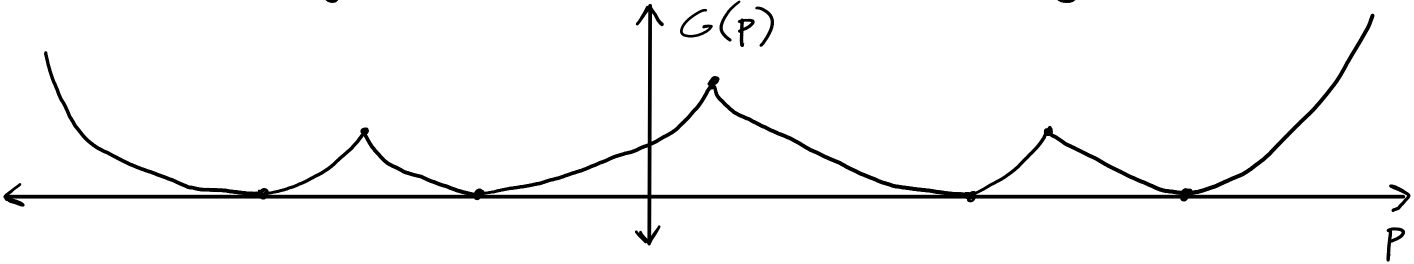

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Adaptation of the proof of the existence of Busemann functions in first passage percolation by Damron – Hanson (2014).

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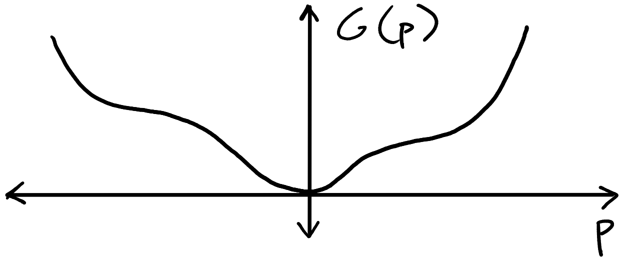

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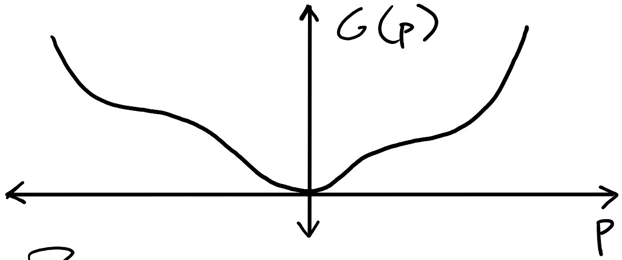

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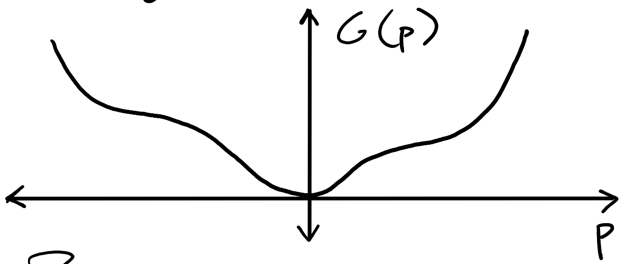

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(SHJ) $$\lambda = a(x,\omega)F'' + G(p+F') + \beta V(x,\omega), \qquad x \in \mathbb{R}.$$

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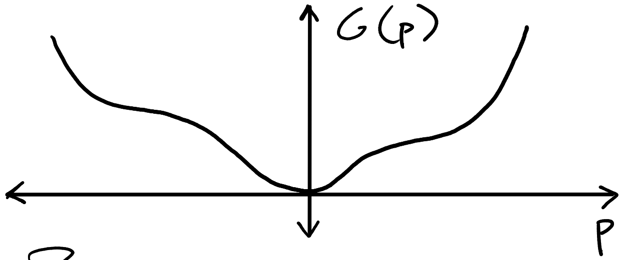

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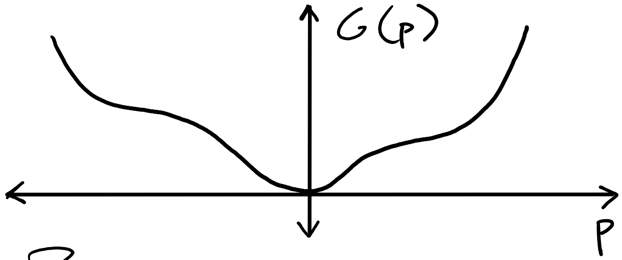

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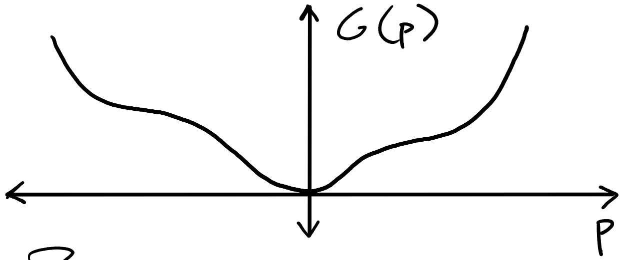

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Restate the classical strategy: Suppose that, for every $p \in \mathbb{R}$, there exist $\lambda = \lambda(p)$ and an $f(\cdot, \omega)$ s.t.

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Relaxed strategy: Suppose that, for every $p \in \mathbb{R}$, there exist $\lambda = \lambda(p)$ and stationary & ergodic $\underline{f}(\cdot,\omega)$, $\bar{f}(\cdot,\omega)$ (not necessarily distinct) that solve (ODE),

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distinct if $\underline{f}(\cdot,\omega) < \overline{f}(\cdot,\omega)$

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This is where the scaled hill and valley condition is used.

Cheers,
Timo!