WOMAP 2023

Hyperkähler manifolds and Lagrangian fibrations

<u>Riemannian manifold</u> A C^∞ manifold is called Riemannian if it has a Riemannian metric, i.e., a $(2,0)$-tensor $g \in C^\infty((T_M^*)^2)$ which is symmetric:

$$g \in C^\infty(\mathrm{Sym}^2 T_M^*),$$

and defines a positive definite quadratic form on $T_{M,x}$ for all $x \in M$.

Fundamental result of Riemannian Geometry:

every C^∞ manifold can be endowed with a Riemannian metric

Riemannian manifolds have canonical connections on their tangent bundles: Levi-Civita connections

The holonomy group of a connection ∇ at a point x, is the group generated by parallel transport along loops based at x. $Hol_x(\nabla) \hookrightarrow GL(T_x)$

If M is connected, then $Hol_x(\nabla)$ is independent of x, up to conjugation.

The holonomy group leaves g invariant:

$$\mathrm{Hol}(\nabla) \hookrightarrow SO(g)$$

<u>De Rham's theorem</u>: (M,g) Riemannian, complete, simply connected. Then (M,g) is isometric to a product $(M_0, g_0) \times \dots \times (M_k, g_k)$ where M_0 is a Euclidean space and $(M_1, g_1), \dots, (M_k, g_k)$ are irreducible. The decomposition is unique up to reordering $M_1, \dots, M_k$.

<u>Berger's theorem:</u> (M,g) Riemannian, complete, connected, non-symmetric, irreducible. Then $\mathrm{Hol}^0(g)$

$:=$ connected comp. of id. of $\mathrm{Hol}(\nabla)$ $= \mathrm{Hol}(g)$

is one of the following:

1. $\mathrm{Hol}^0 \cong SO(n)$ (generic metric)

2. $n = 2m \geq 4$ $\mathrm{Hol}^0 \cong U(m) \hookrightarrow SO(n)$ (Kähler)

3. $n = 2m \geq 4$ $\mathrm{Hol}^0 \cong SU(m) \hookrightarrow SO(n)$ (Calabi-Yau, Kähler)

4. $n = 4r \geq 4$ $\mathrm{Hol}^0 \cong Sp(r) \hookrightarrow SO(n)$ (Hyperkähler)

5. $n = 4r \geq 8$ $\quad \mathrm{Hol}^0 \cong Sp(r)Sp(1) \hookrightarrow SO(n)$

(quaternionic-Kähler)

6. $n = 7$ $\quad \mathrm{Hol}^0 \cong G_2$ (exceptional)

7. $n = 8$ $\quad \mathrm{Hol}^0 = \mathrm{Spin}\,7 \hookrightarrow SO(8)$ (exceptional)

Hyperkähler (irreducible): $H^2(M,\mathbb{C}) = H^{2,0}(M) \oplus H^{1,1}(M) \oplus H^{0,2}(M)$

$$H^{2,0}(M) = \mathbb{C}\,\omega$$

$\uparrow$ symplectic form

$$\omega : \Lambda^2 T_M \longrightarrow \mathcal{O}_M$$

$$T_M \xrightarrow{\simeq} T_M^* \quad \text{(nondegenerate)}$$

Beauville-Bogomolov: $\exists$ nondegenerate $q: H^2(M,\mathbb{Z}) \to \mathbb{Z}$

with signature $(3, b_2 - 3)$ s.t. $\exists\, \lambda_M \in \mathbb{Q}$ with

$$\int_M \alpha^{2n} = \lambda_M\, q(\alpha)^n \qquad \forall\, \alpha \in H^2(M,\mathbb{C})$$

$$(2n = \dim_{\mathbb{C}} M)$$

also $q(\omega) = 0 \qquad q(\omega + \bar{\omega}) > 0$

Period domain: **Lattice** : (Γ, q_Γ)

$\Gamma \cong \mathbb{Z}^d$ as groups q_Γ nondeg. quadratic

$$Q_\Gamma := \{\alpha \mid q_\Gamma(\alpha) = 0,\ q_\Gamma(\alpha + \bar{\alpha}) > 0\} \subset \overline{Q}_\Gamma \subset \mathbb{P}(\Gamma \otimes \mathbb{C})$$

Marking: $\varphi : (H^2(M, \mathbb{Z}), q_M) \xrightarrow{\cong} (\Gamma, q_\Gamma)$

abstract Lattice

$\mathcal{M}_\Gamma :=$ moduli space of marked hyperkähler manifolds (for Γ)

$\mathcal{P} : \mathcal{M}_\Gamma \longrightarrow Q_\Gamma$ period map

Maps from hyperkähler manifolds:

Matsushita: If $f: M \longrightarrow B$ with $0 < \dim B < \dim M$,

then every smooth fiber of f is an abelian variety.

Lagrangian: $\omega|_{T_{f^{-1}b}} = 0$ as a symplectic form.

Hyperkähler manifolds with (Lagrangian) fibrations form codimension 1 subsets of $\mathcal{M}_\Gamma$ and Q_Γ.

Conversely: every Lagrangian torus is a fiber of a Lagrangian fibration. (Hwang-Weiss and others)

Matsushita-Hwang: If $f: M \longrightarrow B$ $\quad 0 < \dim B < \dim B$

and B is smooth, then $B \cong \mathbb{P}^n$. $\supset$ hyperplane.

A projective K3 surface is elliptic iff it admits a nonzero divisor whose square is 0.

On M: $\quad D^{n+1} = 0 \iff q_M(D) = 0$ (Verbitsky)

SYZ (Strominger-Yau-Zaslow) conjecture.

Conj 1: M has a Lagrangian fibration iff

$\exists\, D \neq 0$, D nef. s.t. $q_M(D) = 0$

Conj. 2: If $M \supset D$ s.t. $D \not\equiv 0$ and $q_M(D) = 0$, then $\exists\ M' \overset{\text{birat.}}{\dashrightarrow} M$ s.t. M' has a Lagrangian fibration.

$$
\begin{array}{ccccc}
 & \mathcal{M}_T & \longrightarrow & \mathcal{Q}_T & \\
 & \cup & & \cup & \\
\alpha_{\text{isot.}} \subset & (\mathcal{M}_T)_{\text{isot.}} & & (\mathcal{Q}_T)_{\text{isot.}} & \supset \alpha^{\perp} \\
 & \cup & & \cup & \\
\alpha_{\text{rat.}} & (\mathcal{M}_\Gamma)_{\text{rat.}} & & (\mathcal{Q}_\Gamma)_{\text{rat.}} & \alpha_{\text{rat.}} \\
 & \cup & & \cup & \\
\alpha_{\text{reg.}} & (\mathcal{M}_\Gamma)_{\text{reg.}} & & (\mathcal{Q}_\Gamma)_{\text{reg.}} & \alpha_{\text{reg.}}
\end{array}
$$

Kamenova-Verbitsky-Matsushita:

$(\mathcal{M}_\Gamma)_{reg.}$ and $(\mathcal{Q}_\Gamma)_{reg}$ are dense.

Our result: $(\mathcal{M}_\Gamma)_{isot} \cap V$ and $(\mathcal{Q}_\Gamma)_{isot} \cap W$

are dense in V and W

for all $V, W \subset \mathcal{M}_\Gamma, \mathcal{Q}_\Gamma$ of $\dim \geq 1$

Further goal: prove the same for $(\mathcal{M}_\Gamma)_{reg} \cap V$

and $(\mathcal{Q}_\Gamma)_{reg} \cap V$.