Point vortex for the lake equations

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Fluid Equations, A Paradigm for Complexity: Regularity vs Blow-up, Deterministic vs Stochastic
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Outline

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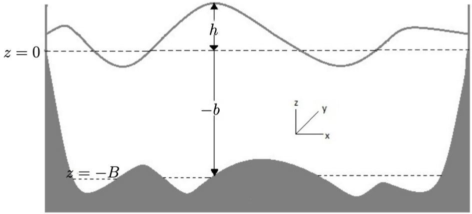
The lake equations

Fluid domain $\{(\tilde{x}, z),\ \tilde{x} \in \Omega,\ -b(\tilde{x}) < z < h(\tilde{x})\}$



We consider $v(t, \tilde{x})$ the vertically averaged horizontal component of the velocity of an incompressible fluid satisfies:

$$\begin{cases} \partial_t(bv) + \mathrm{div}\,(bv \otimes v) + b\nabla p = 0, \\ \mathrm{div}\,(bv) = 0, \quad (bv)\cdot n = 0. \end{cases}$$

Derivation: Greenspan 68 (p. 235), Bresch-Métivier 10 (from the shallow water wave equations in the low Froude number limit), Mésognon.

Potential vorticity

Like for the 2D Euler equations, a crucial quantity for this problem is the potential vorticity:

$$\omega = \frac{1}{b}\mathrm{curl}\,(v) = \frac{\partial_1 v_2 - \partial_2 v_1}{b}$$

which satisfies the continuity equation

$$\partial_t(b\omega) + \mathrm{div}\,(bv\omega) = 0, \qquad \mathrm{div}\,(bv) = 0.$$

Remark

$$b^{1/p}\omega_0 \in L^p \Longrightarrow b^{1/p}\omega(t,\cdot) \in L^p \ \forall t > 0.$$

$bv = \nabla^\perp \psi$ where $\mathrm{div}\left(\frac{1}{b}\nabla\psi\right) = b\omega$.

Well posedness results

Theorem (Levermore - Oliver - Titi 96)

If b and $\partial\Omega \in C^3$, $\frac{1}{C} \leq b(x) \leq C$, $b^{1/p}\omega_0 \in L^p \Longrightarrow u \in W^{1,p} \Longrightarrow$ existence and uniqueness in the Yudovich class.

Theorem (Bresch - Métivier 06)

If b and $\partial\Omega \in C^3$, $b(x) = c(x)d(x, \partial\Omega)^\alpha$ $(\alpha > 0)$, $b^{1/p}\omega_0 \in L^p \Longrightarrow u \in W^{1,p} \Longrightarrow$ existence and uniqueness in the Yudovich class.

Theorem (Al Taki - C.L. 23)

If b and $\partial\Omega \in C^3$, $b(x) = c(x)d(x, \partial\Omega)^\alpha$ $(\alpha > 0)$, and $\omega_0 \in L^\infty(\Omega)$ then $u \in$LogLip and the solution is lagrangian.

Other reduced model

- 2D Euler equations: if $u(t,x) = (u_1, u_2, 0)(t, x_1, x_2)$, then $\operatorname{curl} u = \omega e_3$ verifies the lake equation with $b \equiv 1$.
- Axisymmetric 3D Euler equations without swirl: if $u(t,r,\theta,z) = (u_r, 0, u_z)(t,r,z)$, then $\operatorname{curl} u = (0, \omega_\theta, 0)(r,z)$ verifies the lake equation with $b(r,z) \equiv r$.
- Helicoidal 3D Euler equations without swirl.

Commun property: the vorticity is scalar and transported.

Concentrated vortex?

Texoma lake (US, Oklahoma) 2015 [DR / US Corps of Engineers]

Concentrated vortex?

Assumption

$\omega_{0,\varepsilon} = \sum_{i=1}^{N_v} \omega_{i,\varepsilon}$ *where* $\operatorname{supp} \omega_{i,\varepsilon}(0) \subset B(z_{0,i}, M\varepsilon)$,

$$\int b\omega_{i,\varepsilon}(0) = \gamma_i, \quad 0 \leq \delta_i \omega_{i,\varepsilon}(0) \leq \frac{M}{\varepsilon^2} \quad (\delta_i = \pm 1).$$

Question:

1. does the vorticity remain concentrated? $\omega_{i,\varepsilon}$ " $\approx$ "$\gamma_i \delta_{z_i(t)}$
2. equation verified by $z_i(t)$?

Two notions of “being concentrated”

Definition (weakly concentrated (the mass))

$$\frac{1}{\gamma_i}\int_{B(z_\varepsilon(t),r_\varepsilon)} b\omega_{i,\varepsilon} \geq 1-\eta_\varepsilon \text{ where } r_\varepsilon,\eta_\varepsilon \to 0 \text{ as } \varepsilon \to 0.$$

Definition (strongly concentrated (the support))

$\mathrm{supp}\, \omega_{i,\varepsilon}(t) \subset B(z_\varepsilon(t), r_\varepsilon)$ where $r_\varepsilon \to 0$ as $\varepsilon \to 0$.

Outline

Expansion of the Green-kernel

We have $bv_\varepsilon = \nabla^\perp \psi_\varepsilon^0 +$ harmonic part related to islands, where $\psi_\varepsilon^0(x) = \int G_{\Omega,b}(x,y)(b\omega_\varepsilon)(y)\,dy$ solves $\mathrm{div}\,(\frac{1}{b}\nabla\psi_\varepsilon^0) = b\omega_\varepsilon$ in Ω and $\psi_\varepsilon^0|_{\partial\Omega} = 0$.

Lemma

$$G_{\Omega,b}(x,y) = \sqrt{b(x)}\sqrt{b(y)}G_\Omega(x,y) + S_{\Omega,b}(x,y)$$

where G_Ω is the usual Green kernel (Δ_D^{-1}) and

$$\mathrm{div}\,_x\Big(\frac{1}{b(x)}\nabla_x S_{\Omega,b}(x,y)\Big) = G_\Omega(x,y)\sqrt{b(y)}\Delta_x\Big(\frac{1}{\sqrt{b(x)}}\Big) \;\; in \;\; \Omega,$$

$$S_{\Omega,b}(\cdot,y)|_{\partial\Omega} = 0$$

Corollary: $G_{\Omega,b}(x,y) = \frac{1}{2\pi}\sqrt{b(x)}\sqrt{b(y)}\ln|x-y| + R_{\Omega,b}(x,y)$ with $R_{\Omega,b} \in W^{1,\infty}_{loc} \cap W^{2,p}_{loc}$.

Biot-Savart type formula

The solution of $\operatorname{div} \frac{1}{b}\nabla\psi_\varepsilon = b\omega_\varepsilon$ can be written as

$$\psi_\varepsilon(x) = \frac{1}{2\pi}\int \ln|x-y|\sqrt{b(x)b(y)}(b\omega_\varepsilon)(y)dy + R_\varepsilon$$

hence $v_\varepsilon = \frac{\nabla^\perp\psi_\varepsilon}{b} = v_K + v_L + v_R$ where

- $v_K(x) = \dfrac{1}{2\pi}\displaystyle\int \frac{(x-y)^\perp}{|x-y|^2}\sqrt{\frac{b(y)}{b(x)}}(b\omega_\varepsilon)(y)dy = \mathcal{O}(\varepsilon^{-1})$ the spinning around a straight vortex filament;
- $v_L(x) = \dfrac{1}{4\pi}\dfrac{\nabla^\perp b(x)}{b(x)}\displaystyle\int \ln|x-y|\sqrt{\frac{b(y)}{b(x)}}(b\omega_\varepsilon)(y)dy = \mathcal{O}(\ln\varepsilon)$;
- $v_R = \mathcal{O}(1)$.

Outline

The 2D Euler equations

Limit trajectories: point vortex system

$$\dot{z}_i(t) = \sum_{j \neq i} \gamma_j \frac{(z_i(t) - z_j(t))^\perp}{2\pi |z_i(t) - z_j(t)|^2}.$$

- Marchioro-Pulvirenti 93: weakly concentrated.
- Marchioro-Pulvirenti 94: strongly concentrated, $r_\varepsilon \leq \varepsilon^\beta$ with $\beta < 1/300$.
- Marchioro 98, $r_\varepsilon \leq \varepsilon^\beta$ with $\beta < 1/3$.
- Buttà-Marchioro 18: time where some vorticity meet $\partial B(z_i, \varepsilon^\beta)$ with $\beta < 1/2$.

Marchioro's strategy

a) Consider that there is only one vortex patch and an exterior **lipchitz** force F_ε. Ok if strongly concentrated.

$$u_\varepsilon(t,x) = \frac{1}{2\pi}\int \frac{(x-y)^\perp}{|x-y|^2}\omega_\varepsilon(t,y)dy + F_\varepsilon(t,x)$$

b) Vortex center: $z_\varepsilon(t) := \int x\omega_\varepsilon(t,x)dx$: then

$$\begin{aligned}\dot{z}_\varepsilon(t) &= -\int x\,\mathrm{div}\,(u_\varepsilon(x)\omega_\varepsilon(x))dx = \int u_\varepsilon(x)\omega_\varepsilon(x)dx\\ &= \frac{1}{2\pi}\iint \frac{(x-y)^\perp}{|x-y|^2}\omega_\varepsilon(x)\omega_\varepsilon(y)dxdy + \int F_\varepsilon\omega_\varepsilon\\ &= \frac{1}{4\pi}\iint \Big[\frac{(x-y)^\perp}{|x-y|^2} - \frac{(x-y)^\perp}{|x-y|^2}\Big]\omega_\varepsilon(x)\omega_\varepsilon(y)dxdy + \mathcal{O}(1) = \mathcal{O}(1\end{aligned}$$

Marchioro's strategy

c) Moment of inertia: $I_\varepsilon(t) := \int |x - z_\varepsilon(t)|^2 \omega_\varepsilon(t,x)dx$: then

$$\begin{aligned}
\dot{I}_\varepsilon(t) =& \int 2(x - z_\varepsilon) \cdot (u_\varepsilon(x) - \dot{z}_\varepsilon)\omega_\varepsilon(x)dx \\
=& \frac{1}{\pi} \iint (x - z_\varepsilon) \cdot \frac{(x-y)^\perp}{|x-y|^2} \omega_\varepsilon(x)\omega_\varepsilon(y)dxdy \\
&+ \iint 2(x - z_\varepsilon) \cdot (F_\varepsilon(x) - F_\varepsilon(y))\omega_\varepsilon(x)\omega_\varepsilon(y)dxdy \\
=& \frac{1}{2\pi} \iint (x - y) \cdot \frac{(x-y)^\perp}{|x-y|^2} \omega_\varepsilon(x)\omega_\varepsilon(y)dxdy + \dots \\
\leq& C I_\varepsilon \quad \Longrightarrow I_\varepsilon(t) \leq C\varepsilon^2.
\end{aligned}$$

Corollary: weakly concentrated $\int_{B(z_\varepsilon, r_\varepsilon)^c} \omega_\varepsilon \leq \frac{I_\varepsilon}{r_\varepsilon^2}$.

Marchioro's strategy

d) $R_t := \max_{x \in \mathrm{supp}\,\omega_\varepsilon(t)} |x - z_\varepsilon(t)| = |X_{x_0}(t) - z_\varepsilon(t)|$, then

$$\frac{d}{dt}|X_{x_0}(t) - z_\varepsilon(t)| \leq f\Big(t, m_t\Big(\frac{R_t}{2}\Big)\Big) \text{ where } m_t(R) := \int_{B(z_\varepsilon, R)^c} \omega_\varepsilon.$$

e) $\lim_{\varepsilon \to 0} \varepsilon^{-\ell} m_t(\varepsilon^\beta) = 0$ for any $\ell \in \mathbb{N}$.
VERY technical: $\mu_t(h) := 1 - \int W_h(x - z_\varepsilon)\omega_\varepsilon(x)dx$ (so that $\mu_t(h) \leq m_t(h) \leq \mu_t(\frac{h}{2})$) and prove that $\frac{d}{dt}\mu_t(h) =$ "$u_{\text{radial}} \cdot \nabla W_h$" $\leq A_h m_t(h)$ + iteration.

f) Conclusion by a continuity argument and bootstrap argument.

Other references

- Iftimie-Sideris and Gamblin
- Smets-Van Schaftingen
- D. Cao et al.
- Davila-Del Pino-Musso-Wei
- Gallay

Outline

Local Induction Approximation

We expect a fast motion, self induced in the direction of $\nabla^\perp b$ (on the level set $\mathcal{C}_i$).

Main Theorem (Hientzsch - Miot - C.L. 23

After a time change of var., if $z_i(0) \in \mathcal{C}_i$ ($\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$), the limit trajectories is $\dot{z}_i = -\frac{\gamma_i}{4\pi}\frac{\nabla^\perp b(z_i)}{b(z_i)}$, and the vorticity is weakly concentrated and strongly concentrated **around the level set**:

$$\operatorname{supp} \omega_{\varepsilon,i}(t,\cdot) \subset \left\{ x \,:\, |b(x) - b(z_i^0)| \le \frac{C_{k,T}}{|\ln \varepsilon|^k} \right\}.$$

Strategy (Buttà-Cavallaro-Marchioro 22): moment of inertia in the transverse direction $J_\varepsilon(t) = \int_\Omega |b(x) - b(z^0)|^2 (b\omega_\varepsilon)(x) dx$.

References

Main references:

- Benedetto-Caglioti-Marchioro 00: one vortex ring.
- Buttà-Marchioro 20: several vortex rings (short time).
- Buttà-Cavallaro-Marchioro 22: several vortex ring ($r_i \neq r_j$).

Further references

- J. J. Thomson 1883.
- Da Rios 1906: formal derivation for vortex filament.
- Jerrard-Smets 15, Jerrard-Seis 17.
- Richardson 00: formal derivation for the lake equation.
- Dekeyser-Van Schaftingen 20: one point and $b(x) \geq b_0 > 0$ (weakly concentrated).

Existence of a particular solution: Frankel 70 and....
Slightly viscous fluid: Gallay-Sverak 16,
Bedrossian-Germain-Harrop Griffiths 18.

Main open questions

Study of the filamentation.

Leapfrogging phenomenon.

Vortex ring emanated from Etna.

Source: Siciliafan.

Thank you for your attention!!

Crater lake (OregonUS) 2015 pierre leclerc photography