Expanding Blaschke Products for the Lee-Yang zeros on the Diamond Hierarchical Lattice.

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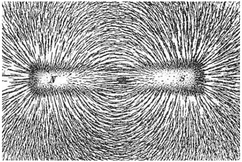

https://arxiv.org/pdf/1009.4691.pdf

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- Sketch of the proofs

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$$H_n(\sigma) = -J \cdot I(\sigma) - h \cdot M_n(\sigma),$$

where $J > 0$.

Gibbs Distribution and the Partition Function

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$$W_n(\sigma) = e^{-H_n(\sigma)/T},$$

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An obvious danger occurs at those values of h, T for which $Z_n(h, T) = 0$. Luckily, this never happens for $h, T \in \mathbb{R}$.

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Fundamental symmetry of the Ising model!

Thermodynamic quantities in terms of zeros of $Z_n(z, t)$.

For each $t \in \mathbb{C}^*$ $Z_n(z, t) = 0$ has $2|E_n|$ roots $z_i(t) \in \mathbb{C}$.

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$$F_n(z, t) := -T \log Z_n(z, t) = -T \sum \log |z - z_i(t)| + |E_n| T (\log |z| + \frac{1}{2} \log |t|)$$

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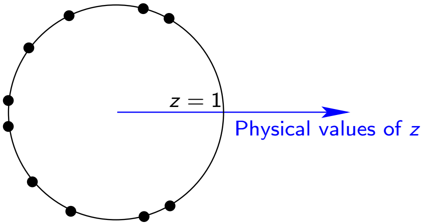

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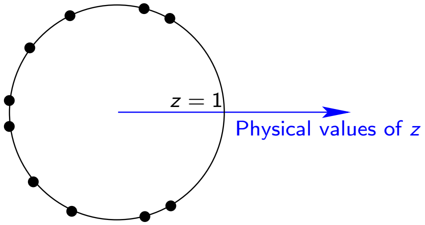

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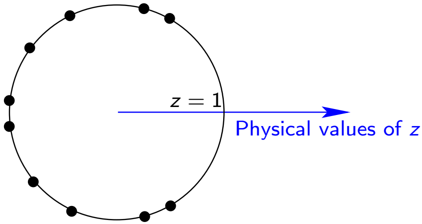

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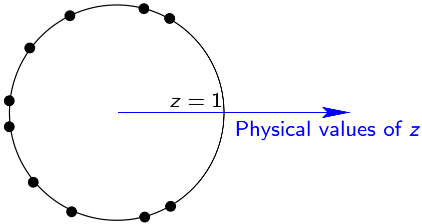

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J. Borcea and P. Brändén The Lee-Yang and Pólya-Schur programs. I. Linear operators preserving stability. *Invent. Math.* (2009).

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For each $t \in [0,1]$ there is a measure μ_t on $\mathbb{T}$ describing the asymptotic distribution of Lee-Yang zeros.

Phase transitions in terms of Lee-Yang distribution

$$F(z,t) = -2T\int_{\mathbb{T}} \log|z-\zeta|\,d\mu_t(\zeta) + T\log|z| + \frac{1}{2}\log|t|$$

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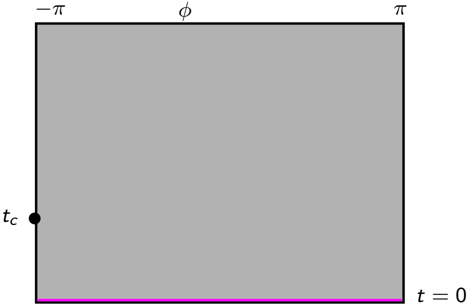
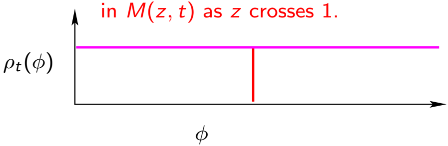
Understanding how the Lee-Yang distributions $\mu_t(\phi)$ vary with t and ϕ is essential to understanding phase transitions of the model.

Expected limiting distributions of Lee-Yang zeros for $\mathbb{Z}^2$



$\rho_t(0) > 0$ gives jump discontinuity in $M(z, t)$ as z crosses 1.



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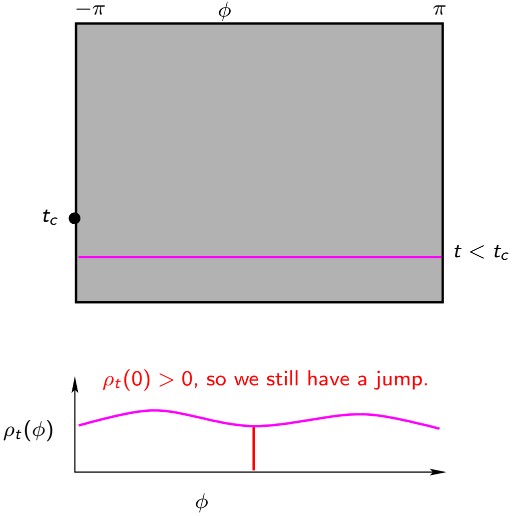

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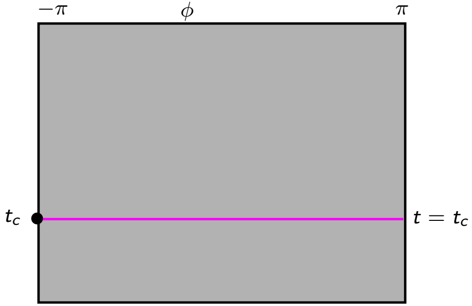
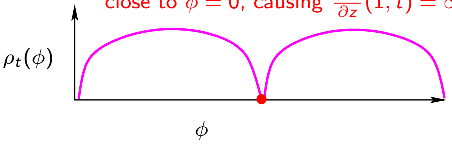
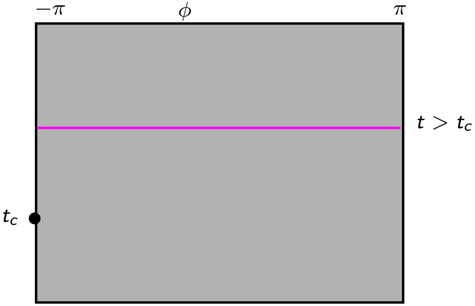


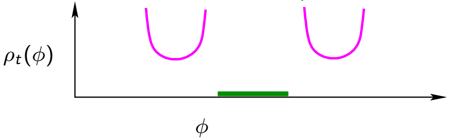
Now, $\rho_t(0) = 0$, so no jump.
However, they accumulate arbitrarily close to $\phi = 0$, causing $\frac{\partial M}{\partial z}(1, t) = \infty$



Expected limiting distributions of Lee-Yang zeros for $\mathbb{Z}^2$



Now, we have a nice interval around $\phi = 0$ with $\rho_t(\phi) \equiv 0$. Causes $M(z, t)$ to be differentiable at $z = 1$ (and hence everywhere).



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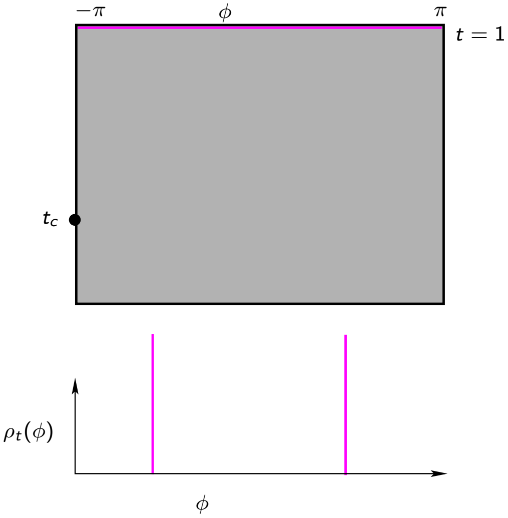

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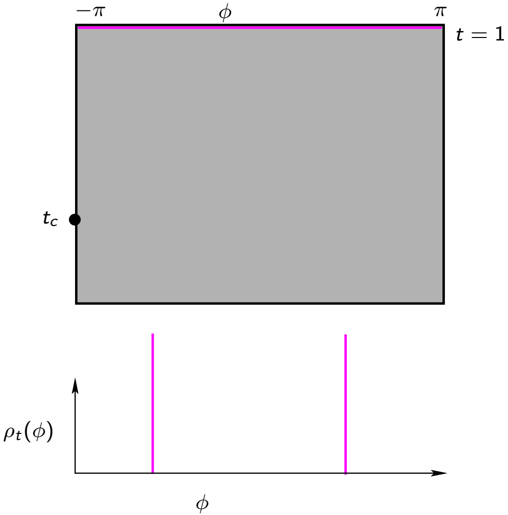
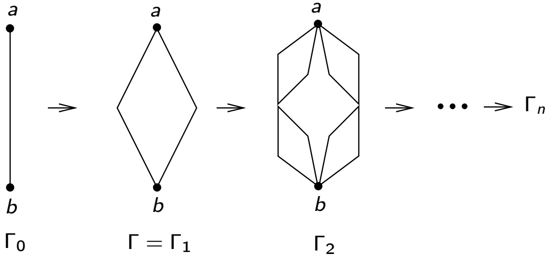


Zakhar Kabluchko: Lee-Yang zeros for the Curie Weiss model match this conjectureal description. `http://arxiv.org/pdf/2203.05533.pdf`

Hierarchical Lattices

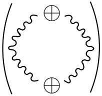
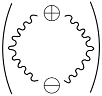
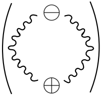
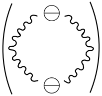


The Diamond Hierarchical Lattice (DHL).

Γ_n is obtained by replacing each edge of generating graph Γ (a diamond) with a copy of Γ_{n-1}, considering the marked vertices a and b as the "endpoints" of Γ_{n-1}.

Migdal-Kadanoff Renormalization[1,2,3]

Consider the conditional partition functions:

$U_n := Z_n\Big($  $\Big)$, $V_n := Z_n\Big($  $\Big) = Z_n\Big($  $\Big)$, $W_n := Z_n\Big($  $\Big)$

The total partition function is equal to $Z_n = U_n + 2V_n + W_n$.

[1] A.A. Migdal. Recurrence equations in gauge field theory. *JETP*, (1975).
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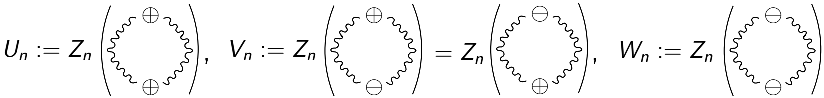

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Migdal-Kadanoff RG Equations:

$$U_{n+1} = (U_n^2 + V_n^2)^2, \quad V_{n+1} = V_n^2(U_n + W_n)^2, \quad W_{n+1} = (V_n^2 + W_n^2)^2.$$

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MK renormalization in the (z, t) coordinates:

We can lift R from the $[U : V : W]$ coordinates (downstairs) to the $[z : t : 1]$ coordiantes upstairs:

$$\begin{array}{ccc} \mathbb{CP}^2 & \xrightarrow{\mathcal{R}} & \mathbb{CP}^2 \\ \downarrow \Psi & & \downarrow \Psi \\ \mathbb{CP}^2 & \xrightarrow{R} & \mathbb{CP}^2 \end{array} \tag{1}$$

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and Ψ is some degree 2 rational map.

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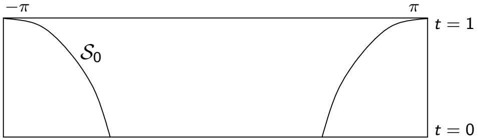
It is this recursive relationship between $\mathcal{S}_{n+1}$ and $\mathcal{S}_n$ that makes a study of the Lee-Yang zeros tractable for hierarchical lattices.

Lee-Yang zeros as pull-backs under $\mathcal{R}$



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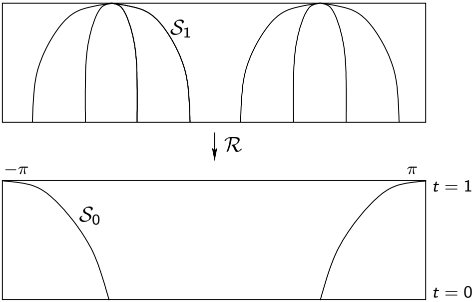

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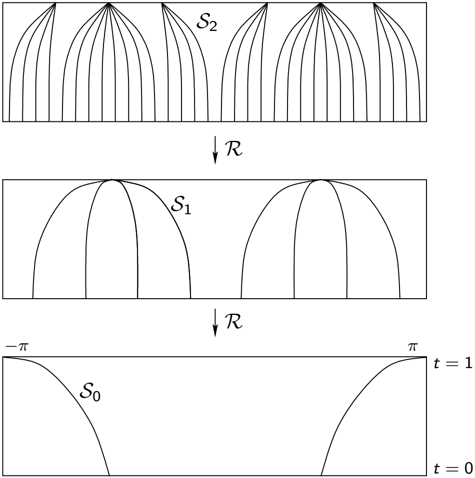

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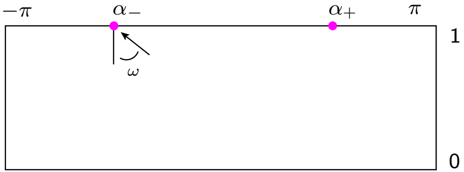

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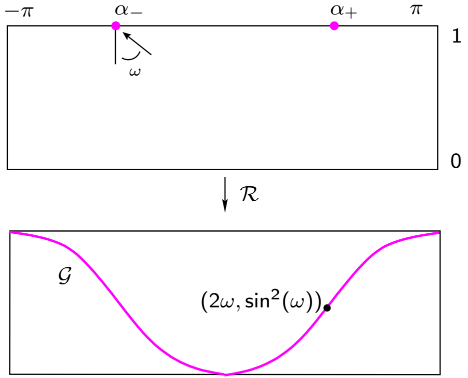

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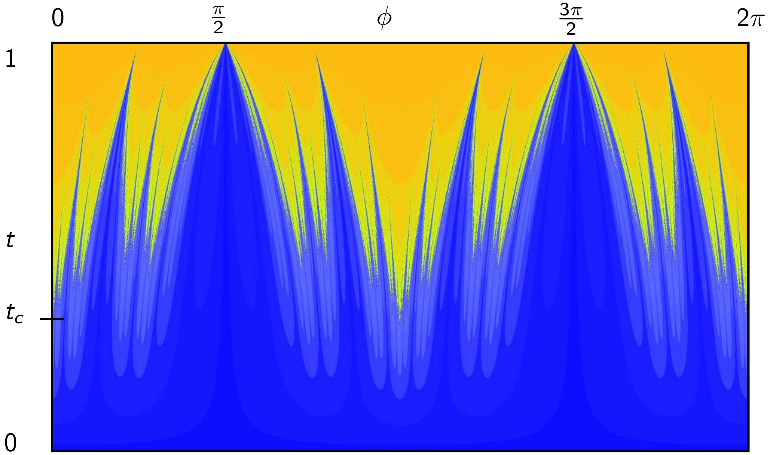
Numerical Experiment



$\mathcal{W}^s(\mathcal{B})$ is colored blue and $\mathcal{W}^s(\mathcal{T})$ is colored orange.

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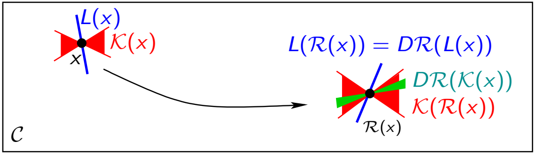

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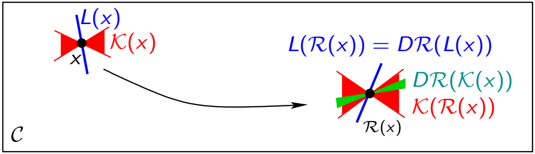

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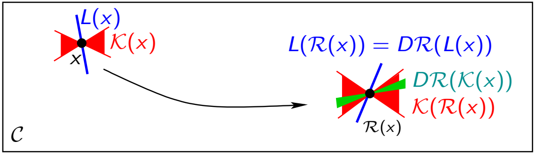

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The idea of this proof that this conefield is invariant seems to play a role in the recent work of Dang-Grigorchuk-Lyubich about the Basilica IMG.

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This is the "intertwined basins" phenomenon studied by Kan-Yorke, Bonifant-Milnor, Ilyashenko-Kleptsyn-Saltykov....

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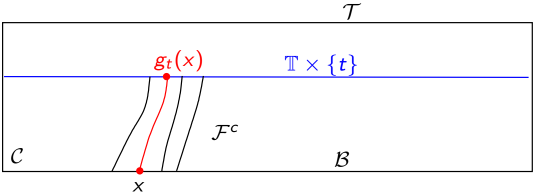

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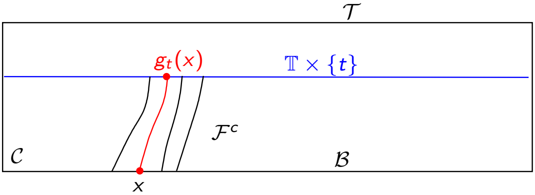
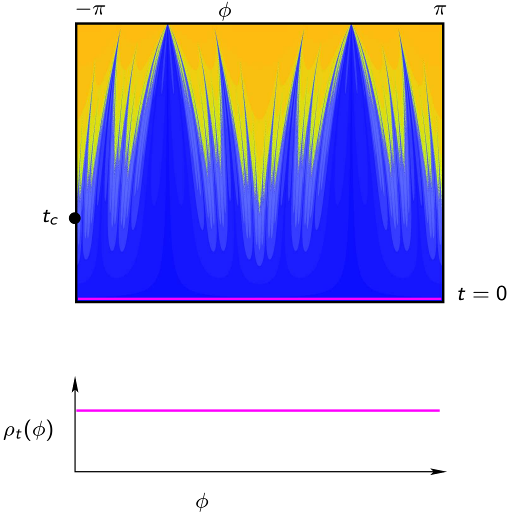


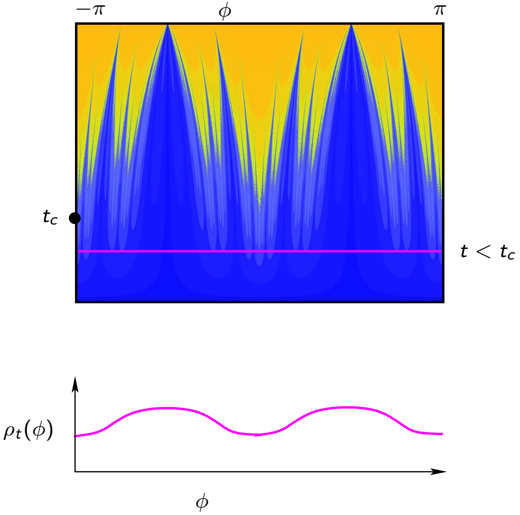
Theorem (BLR)

The asymptotic distribution of Lee-Yang zeros at a temperature $t_0 \in [0,1)$ is given by under holonomy by $\mu_t = (g_t)_*(\mu_0)$ where μ_0 be the Lebesgue measure on $\mathcal{B}$.

Geometric view of Lee-Yang distributions for the DHL



Geometric view of Lee-Yang distributions for the DHL



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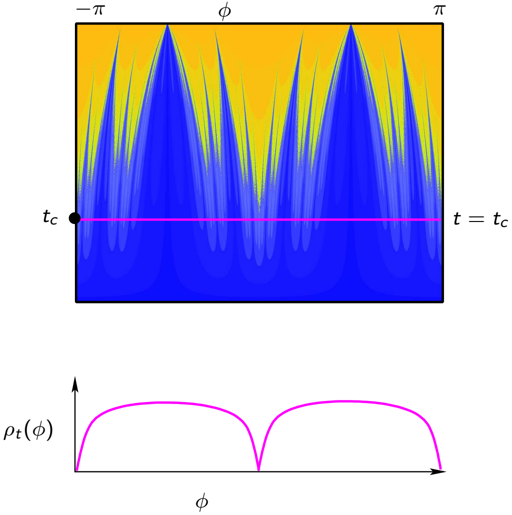

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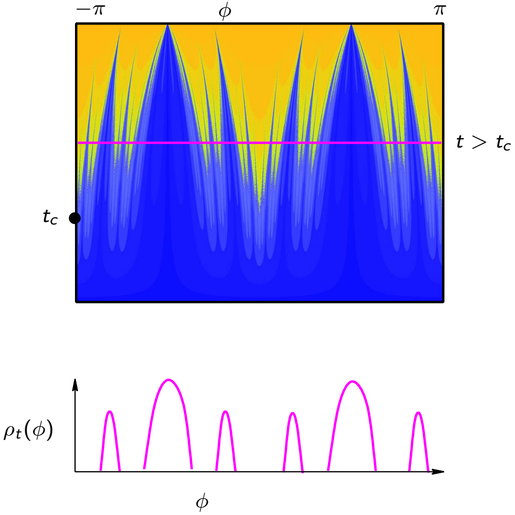

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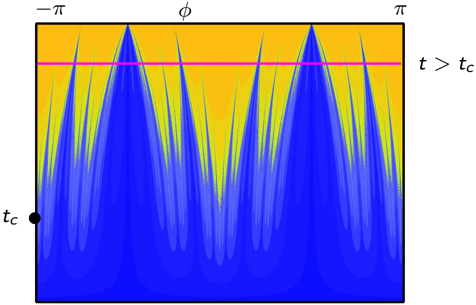

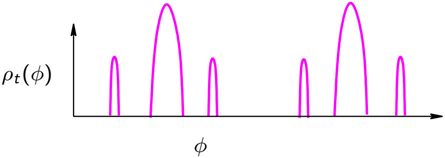

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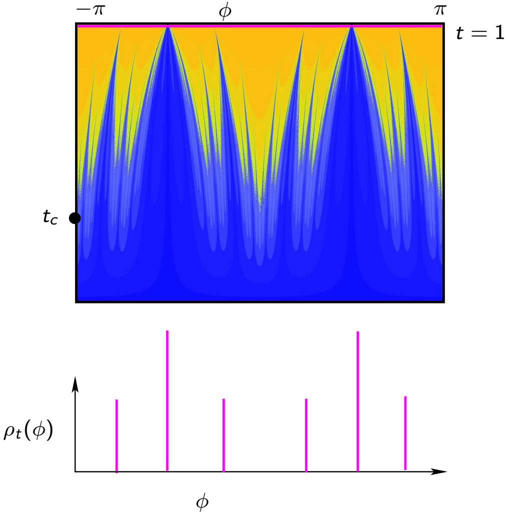

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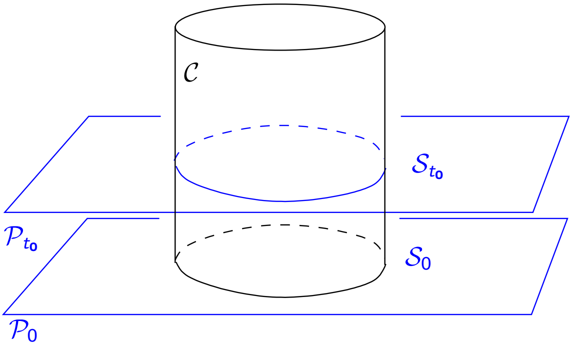

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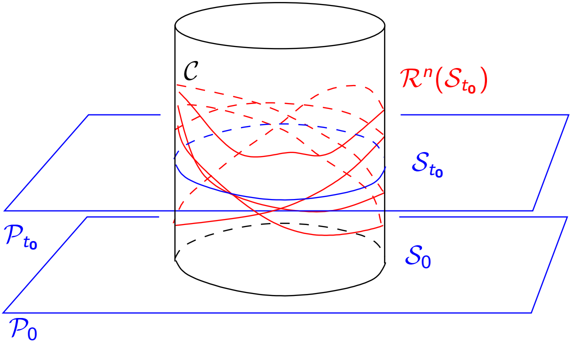

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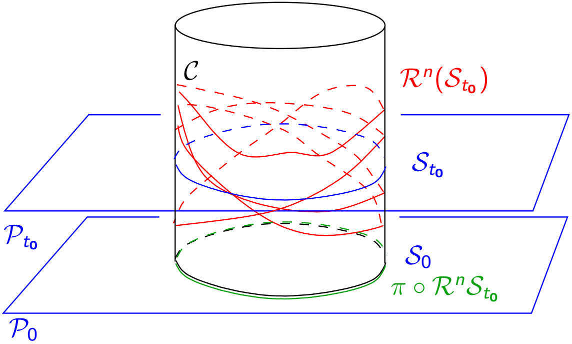

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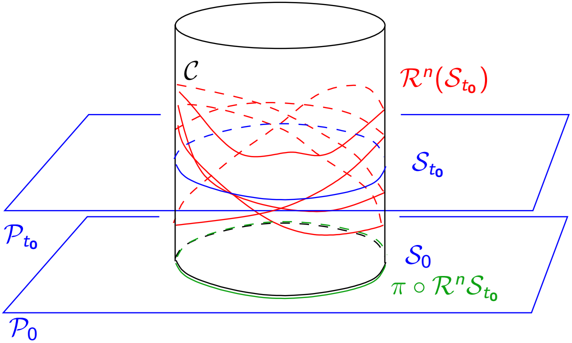

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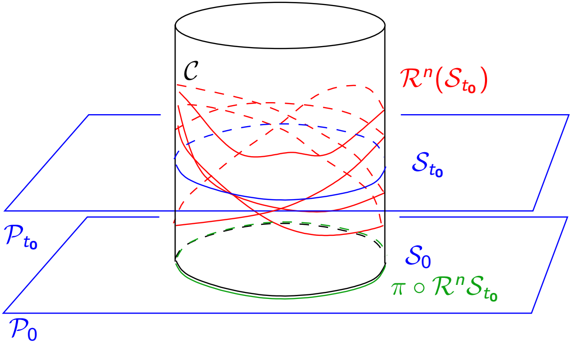

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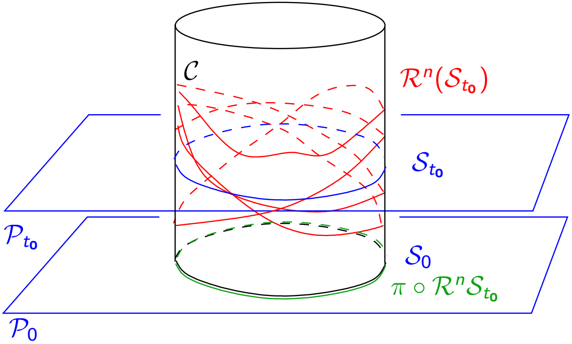

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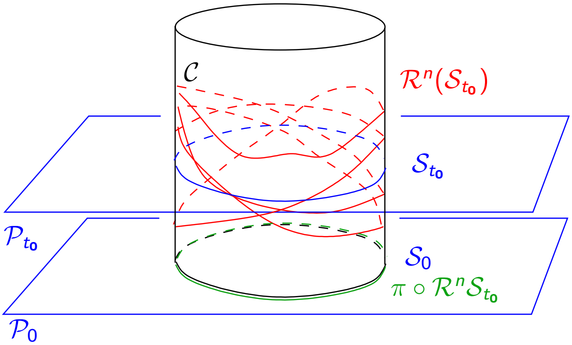

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Algebraic instability: $4^n < \deg(\mathcal{R}^n) < (\deg(\mathcal{R}))^n = 6^n$.

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Original idea actually works in these coordinates!

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Claim: $\psi_n : \mathbb{C} \to \mathbb{C}$ is an Blaschke product preserving the unit disc $\mathbb{D}$, expanding the circle $\mathbb{T} = \partial\mathbb{D}$ by a factor of 2^{n+1}.

Conditional partition functions and their symmetries

$$\begin{aligned} U_n(z,t) &= \sum_{\sigma(a)=\sigma(b)=+1} W(\sigma) = \sum_{\sigma(a)=\sigma(b)=+1} t^{-I(\sigma)} z^{-M(\sigma)} \\ &= a_d^+(t) z^d + \cdots + a_{-d}^+(t) z^{-d}, \end{aligned}$$

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Reason for 2: With -1 spins at the marked vertices a, b, we can't get more than $4^n - 2^{n+1}$ edges with $++$, so $M(\sigma) \leq 4^n - 2^{n+1}$.

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Are the other zeros b_i within the unit disc $\mathbb{D}$?

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$$\psi_n(z) = \frac{W_n(z)}{U_n(z)} = z^{2^{n+1}} \prod \frac{z - b_i}{1 - \overline{b_i} z}$$

is a Blaschke product with 2^{n+1} zeros at $z = 0$.
Are the other zeros b_i within the unit disc $\mathbb{D}$?
If yes, then $\psi_n(z)$ is a Blaschke product that expands the circle $\mathbb{T}$ by at least 2^{n+1}

Proof of horizontal expansion, part IV

Factor $U_n(z) \equiv U_n(z, t_0)$ and $W_n(z) \equiv W_n(z, t_0)$ as

$$\begin{aligned} W_n(z) &= z^{-4^n+2^{n+1}} \prod (z - b_i) \\ U_n(z) &= z^{-4^n} \prod (1 - b_i z) = z^{-4^n} \prod (1 - \overline{b_i} z) \end{aligned}$$

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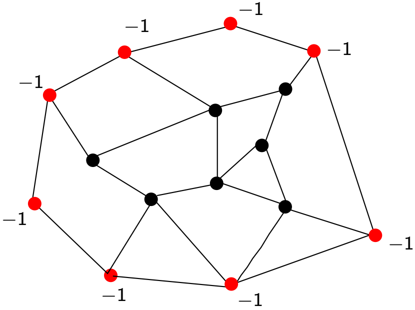
Are the other zeros b_i within the unit disc $\mathbb{D}$?
If yes, then $\psi_n(z)$ is a Blaschke product that expands the circle $\mathbb{T}$ by at least 2^{n+1}
so we'd be done!

Lee-Yang Theorem with Boundary conditions

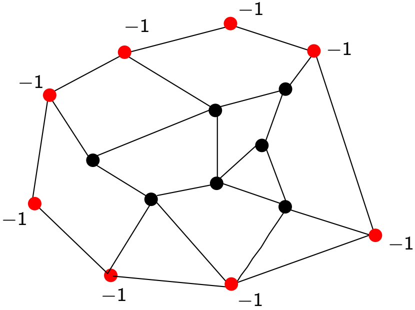


S is the vertices in red.

Theorem (Bleher, Lyubich, R)

Consider a ferromagnetic Ising model on a connected graph Γ *and let* $\sigma_S \equiv -1$ *on a nonempty subset* S *of the vertex set* V.

Lee-Yang Theorem with Boundary conditions



S is the vertices in red.

Theorem (Bleher, Lyubich, R)

Consider a ferromagnetic Ising model on a connected graph Γ *and let* $\sigma_S \equiv -1$ *on a nonempty subset* S *of the vertex set* V*.*
Then, for any temperature $t \in (0,1)$ *the Lee-Yang zeros* $z_i^-(t)$ *of the conditional partition function* $Z_{\Gamma|\sigma_S}$ *lie inside the open disc* $\mathbb{D}$*.*

Thank you for listening!

Pavel Bleher, Mikhail Lyubich, and Roland Roeder. *Lee-Yang Zeros for the DHL and 2D Rational Dynamics, I. Foliation of the Physical Cylinder.* Journal de Mathématiques Pures et Appliquées, 107(5): 491-590, 2017.

For those who like joint spectra:

Pavel Bleher, Mikhail Lyubich, and Roland Roeder. *Lee-Yang-Fisher zeros for DHL and 2D rational dynamics, II. Global Pluripotential Interpretation.* Journal of Geometric Analysis, 30(1): 777-833, 2020.