Energy transfer for solutions to the Nonlinear Schrödinger Equation

Gigliola Staffilani

MIT

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

SIMONS FOUNDATION

NSF

Fluid and Dispersive Equations

Merle - Raphael - Rodnianski - Szeftel

Smooth Implosion of Compressible Euler and Navier-Stokes

Madelung transformation

Smooth blow-up of certain energy super critical defocusing NLS in $\mathbb{R}^n$

growth of Sobolev norms of periodic SQG eq

??

Energy spectrum of periodic subcritical NLS

Global well-posedness and properties

Consider the Initial Value Problem

$$\begin{cases} i\partial_t u + \Delta u = \lambda |u|^2 u \\ u|_{t=0} = u_0(x) \quad x \in \mathbb{T}^2 \end{cases} \qquad \lambda = \pm 1$$

Using Strichartz estimates and a fixed point argument one can claim that this Initial Value Problem is locally well-posed in $H^s(\mathbb{T}^2)$, $s>0$. If $\lambda = 1$ (defocusing) then energy conservation $\Longrightarrow$ global well-posedness for $s \geq 1$.

<u>Question</u>: Can we learn more about the behaviour of the solution $u(t,x)$ as $t \to \infty$?

Energy Spectrum

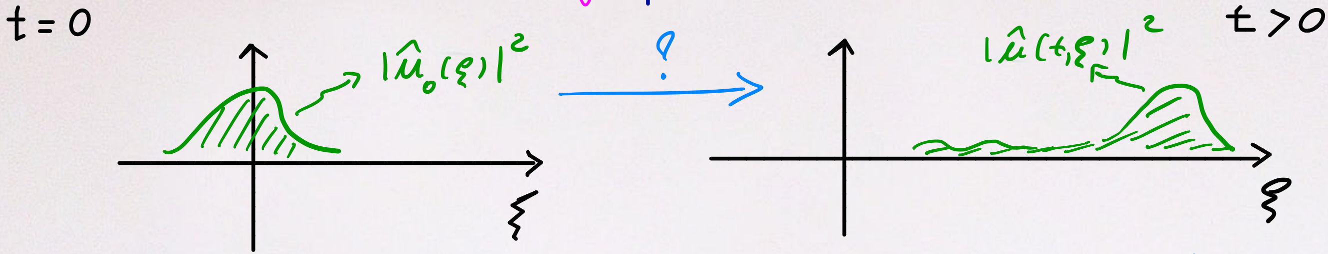
Given a periodic solution $u(t,x)$ of a (dispersive) PDE, we call Energy Spectrum the set

$$\Sigma(t) = \{ |\hat{u}(t,n)|^2, \ n \in \mathbb{Z}^d \}$$

Questions:

- What is the dynamics of $\Sigma(t)$ as $t \to \pm\infty$?
- Is there an evolution equation for $\Sigma(t)$?

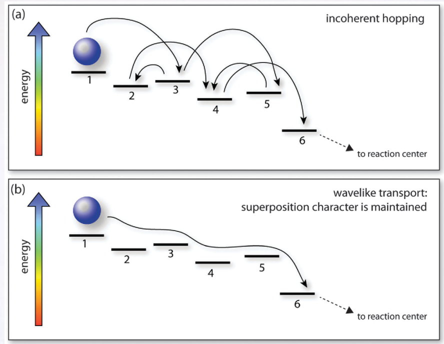
Transfer of energy



Question: 1) Does the support of $|\hat{u}(t,\xi)|^2$ move to higher frequencies?

(Weak turbulence, forward cascade)

2) If such a "migration" happens, is it done in a incoherent hopping way or more like a wavelike transport?



Two different approaches

Approach # 1 : We look at $\sum_k |\hat{u}(t,k)|^2 \langle k \rangle^{2s} =: \|u(t)\|_{H^s}^2$

and we study $\lim_{t \to \infty} \|u(t)\|_{H^s}^2$.

PDE Approach : Bourgain, Kuksin, S., Sohinger, Deng-Germain, Colliander-Keel-S.-Takaoka-Tao, Carles-Faou, S.-Wilson, Hani-Pausader-Tzvetkov-Visciglia ...

Computational Approach : Colliander-Sulem, Faou, Y. Pan ...

Dynamical System Approach : Haus-Procesi, Kaloshin-Guardia, Berti-Maspero, Giuliani-Guardia ...

Approach # 2 : This is based on finding an effective dynamics.
One approximate the equation, where the nonlinearity is
weak $(\lambda \to 0)$, (this is done in various ways)

and then "take limits" to get to the wave kinetic equation

$\Longrightarrow$ $\boxed{\text{Wave Turbulence Theory.}}$

Fundamental original work on this by:

Peierls, Hasselman, Zakharov, Newell, L'vov,
Pomeau, Nazarenko, ...

Approach #1: Growth of Sobolev Norms

Fact 1: Complete integrability may prevent the growth of Sobolev norms (1D cubic NLS, KdV)

Fact 2: Scattering prevents the growth of Sobolev norms:

(Defocusing cubic NLS in $\mathbb{R}^2$. If $u(t,x)$ is solution in $H^s(\mathbb{R}^2)$ then $\exists\, u^+ \in H^s(\mathbb{R}^2)$ s.t.

$$s \geq 0 \qquad \boxed{\| S(t)u^+ - u \|_{H^s} \xrightarrow[t \to +\infty]{} 0} \qquad \text{(Dodson '16)}$$

As a consequence for $t \gg 1$

$$\|u(t)\|_{H^s} \leq \|S(t)u^+ - u\|_{H^s} + \|S(t)u^+\|_{H^s} \leq \varepsilon + \|u^+\|_{H^\varepsilon}.$$

$S(t)$ is unitary!

A bound from above.

Assume $u(t,x)$ is the global smooth solution to

$$\begin{cases} i\partial_t u + \Delta u = |u|^2 u \\ u|_{t=0} = u_0 \qquad x \in \mathbb{T}^2, \; s \gg 1 \end{cases}$$

Fact 1 $*$ $\|u(t)\|_{H^s} \leq C\, |t|^{(s-1)+\varepsilon}$ $\quad |t| \geq 1$

for any torus (Bourgain, Sohinger, Planchon-Visciglia)

Better results available for irrational tori.

(Hrabski, Pan, S., Wilson)

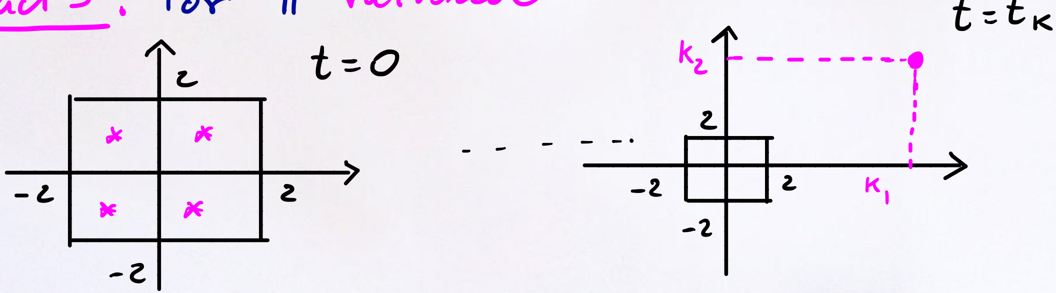
Are there solutions that grow?

Fact 2: Fix $s>1$, $0<\delta \ll 1$, $K \gg 1$, then for the cubic, defocusing NLS in $\mathbb{T}^2$ rational, $\exists$ an initial data $u_0 \in H^s$ and a time $T \gg 1$ s.t.

$$\|u_0\|_{H^s} < \delta \quad \text{and} \quad \|u(T)\|_{H^s} > K$$

(Colliander-Keel-S-Takaoka-Tao)

Fact 3: For $\mathbb{T}^2$ rational



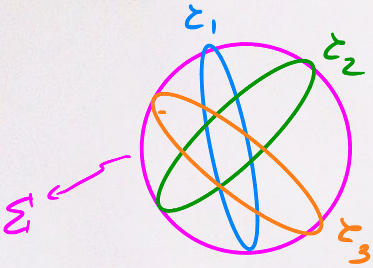
arbitrarily large modes exists.

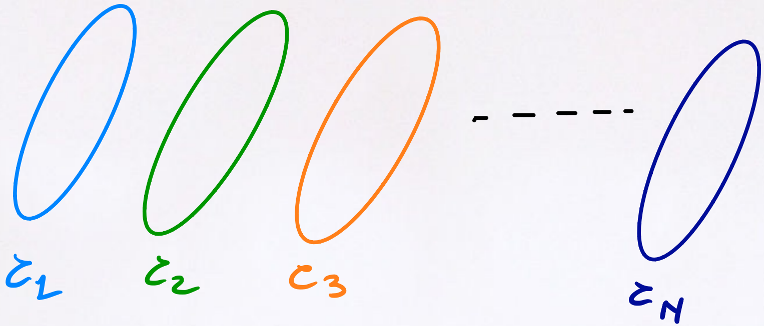
(Carles-Faou)

Fact #2: the dynamics of a toy model

Conservation of mass $\Longrightarrow$ $\Sigma' = \{ x \in \mathbb{C}^N / |x|^2 = 1 \}$

is where the dynamics happens

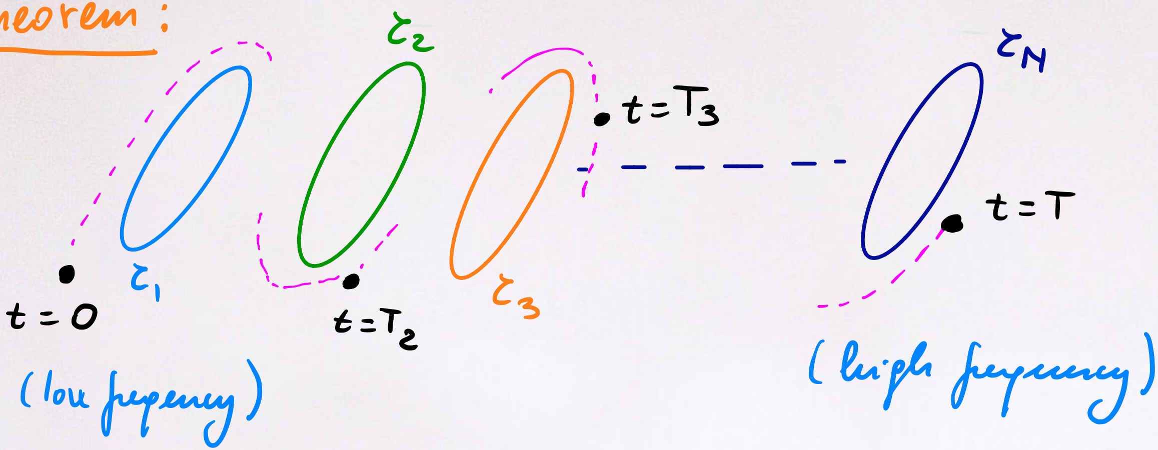




On Σ' there are c_j, $j = 1, \dots, N$, great circles that are invariant.

The heart of the matter

Theorem:



See also a more Dynamical System approach from Guardia - Kaloshin, Haus - Procesi ...

Some Remarks

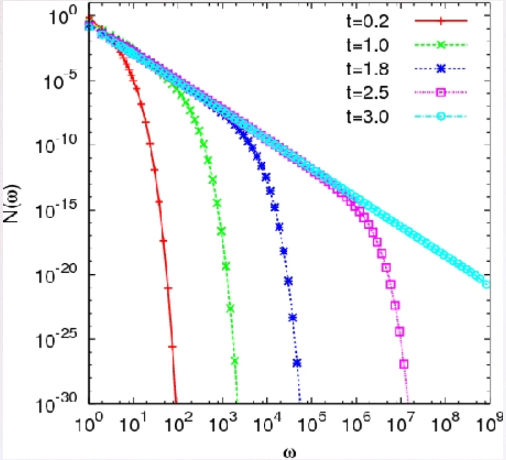
* We do not know what happens after time T.
* In the work of Carles-Faou the procedure is different but the same set Λ of frequencies is used.

Question: What happens when π^2 is irrational ?

Answers: With B. Wilson we prove that something different happens: Both constructions presented above cannot work to show transfer of energy!

Recently Giuliani-Guardia proved that if the torus is irrational but "close to rational" one can adjust the argument.

Approach 2: From weakly nonlinear dispersive equations to wave kinetic equations



Numerical solutions of the isotropic 3-wave kinetic equation –
C. Connaughton

From dispersive equations to wave kinetic equations

Consider the periodic NLS

weak nonlinearity

$$\partial_t u + \Delta u = \varepsilon |u|^2 u$$

$$u|_{t=0} = u_0 \qquad x \in \mathbb{T}_L^d$$

size of torus

What one wants to study, after assuming an initial distribution for $|\hat{u}_0|^2$, is:

$$\lim_{\varepsilon \to 0,\, L \to \infty} \mathbb{E}\left(|\hat{u}(\varepsilon^{-2}\tau, k)|^2 \right) =: n_k(\tau)$$

and show that $\boxed{\partial_\tau n_k = Q(n_k)}$ Wave kinetic equation

Can we derive the wave kinetic equation?

Fundamental original work on this topic by:

Peierls, Hasselman, Benney-Saffman-Newell, Zakharov,
L'vov, Pomeau, Nazarenko, ...

In these works one starts from a certain weakly nonlinear dispersive equation (NLS, KdV, ...) with parameters ε, L and a background probability, then various types of formal approximations and limits are taken

$\Longrightarrow$ WKE is obtained!

Example of a formal derivation of a WKE

Consider the Zakharov-Kuznetsov (ZK) equation

$$\partial_t \phi(x,t) = -\Delta \partial_{x_1} \phi(x,t) + \varepsilon \partial_{x_1} (\phi^2(x,t)) \qquad x \in [-L,L]^d$$

Let $n_k(t) = \mathbb{E}(|\hat{\phi}(k,t)|^2)$. At the kinetic time $t = \varepsilon^{-2}\tau$

taking $L \to \infty$ then $\varepsilon \to 0$ "$\Longrightarrow$" $\partial_\tau n_k(\tau) = Q(n_k(\tau))$

$$\boxed{\begin{aligned} Q(n_{k_1}) = \int dk_2\, dk_3\, |k_1' k_2' k_3'|^2 \,\delta(\omega(k_3) + \omega(k_2) - \omega(k_3)) \\ \times \delta(k_2 + k_3 - k_1)\,[n_{k_2} n_{k_3} - n_{k_1} n_{k_2} \operatorname{sign}(k_1') \operatorname{sign}(k_3') \\ - n_{k_1} n_{k_3} \operatorname{sig}(k_1') \operatorname{sng}(k_2^1)] \end{aligned}}$$

collision operator

$\omega(k) = k^1 |k|^2$

Define $a_k(t) := \hat{\phi}(t,k)/\sqrt{|k'|}$

Assume $a_k(t)$ are Random Phase Amplitude (RPA) fields. We want to write:

$$a_k(t) = a_k^{(0)}(t) + \varepsilon\, a_k^{(1)}(t) + \varepsilon^2 a_k^{(2)}(t) + \dots$$

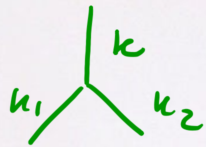
We derive $a_k^{(i)}$ $i = 0, 1, 2$ from the $\widehat{(ZK)}$:

$$\boxed{\dot{a}_k = i\,\omega(k)\, a_k + i\,\varepsilon \sum_{k = k_1 + k_2} \operatorname{sign}(k')\, a_{k_1} a_{k_2}}$$

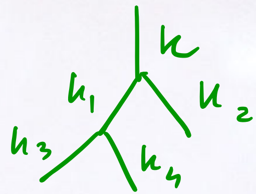
$$a_n^{(0)} = a_n(0) = \hat{\phi}_0(n)$$ (initial datum)

$$a_n^{(1)} = -i\,\mathrm{sign}(k') \sum_{n=n_1+n_2} V_{n_1 n_2 n}\, a_{n_1}^{(0)} a_{n_2}^{(0)} \int_0^t e^{i\omega_{12}^{n} s}\, ds$$

$$\omega_{12}^{n} = \omega(n_1) + \omega(n_2) - \omega(n)$$



$$a_n^{(2)} = -2 \sum_{\substack{n=n_1+n_2 \\ n_1=n_2+n_3}} \mathrm{sign}(k' n_1')\, V_{n n_1 n_2} V_{n_3 n_4 n_1} \cdot$$



$$\cdot\, a_{n_2}^{(0)} a_{n_3}^{(0)} a_{n_4}^{(0)} \int_0^t \int_0^s e^{i(\omega'_{34}\sigma + \omega_{12}^{n} s)}\, d\sigma\, ds$$

Finally one writes

ignore terms with ε^k, $k > 2$.

$$n_k(t) = \mathbb{E}(|a_k(t)|^2) \cong \langle (a_k^{(0)} + \varepsilon a_k^{(1)} + \varepsilon^2 a_k^{(2)})(a_k^{(0)} + \varepsilon a_k^{(1)} + \varepsilon^2 a_k^{(2)}) \rangle$$

replaces the expressions for $a_k^{(i)}$, $i = 0, 1, 2$,

uses RPA and keeping only up to ε^2 and taking

$$L \to \infty \quad \text{then} \quad \varepsilon \to 0$$

obtain

$$\boxed{\partial_t n_k = Q(n_k)}$$

Mathematical literature: rigorous derivation

- Erdos-Yau, Erdos-Salmhofer-Yau:

  Random $\boxed{\text{linear}}$ Schrödinger on a lattice setting

  → linear Boltzmann (kinetic time) → heat equation (diffusion time $t = \lambda^{-2-\varepsilon}$)

- Lukkarinen-Spohn: Random $\boxed{\text{cubic}}$ NLS at equilibrium and on a lattice setting

  → (linearized) wave kinetic equation at kinetic time.

Random Initial Data:

- Buckmaster-Germain-Hani-Shatah: NLS in continuum case
  $\longrightarrow$ below kinetic time (linear kinetic equation)
- Collot-Germain, Deng-Hani: NLS in continuum case
  $\longrightarrow$ strictly below kinetic time (linear kinetic equation)
- Deng-Hani: NLS in continuum case
  $\longrightarrow$ at kinetic time (nonlinear kinetic equation)

$i\partial_t \phi + \Delta \phi = \lambda |\phi|^2 \phi$, on periodic torus $[0,L]^d$ $d \geq 3$, L, λ linked.

- Lukkarinen-Vuoksenmaa: NLS in lattice case
  $\longrightarrow$ at Kinetic time $d \geq 4$.
- Ma: ZK equation with dissipation and in continuum.
  WKE before kinetic time

Recent work by S.-Tran

We consider the stochastic ZK equation

$$\begin{cases} d\phi(x,t) = -\Delta\partial_x \phi(x,t)\,dt + \varepsilon\partial_{x_1}(\phi^2(x,t))\,dt + \varepsilon^{\theta}\underbrace{\partial_{x_1}\phi \odot dW(t)}_{\text{Stochastic term}} \\ \phi(x,0) = \phi_0(x) \xrightarrow{\quad} \text{randomly distributed} \end{cases}$$

$\varepsilon \ll 1, \quad 0 < \theta \ll 1$

The equation is considered on a lattice

$\Lambda = \{0, 1, \dots 2L\}^d$ $\quad d \geq 2$ (dimension)

L in $\mathbb{N}$.

Passing to frequency space

We write $k = (k^1, \dots, k^d) \in \Lambda_* = \left\{ -\frac{L}{2L-1}, \dots, 0, \dots, \frac{L}{2L-1} \right\}^d$

$$\boxed{\omega_k = \omega(k) = \sin(2\pi k^1)\left[\sin^2(2\pi k^1) + \dots + \sin^2(2\pi k^d)\right]}$$

[dispersive relation]

$$\overline{\omega_k} = \sin(2\pi k^1)$$

$$W(x,t) = \sum_{k \neq 0} \frac{W_k(t)}{\overline{\omega}(k)} e^{i 2\pi k \cdot x}$$

[Stochastic term]

$\{W_k(t)\}$ = sequence of independent real Wiener processes on (Ω, F, P).

$$W_{-k}(t) = -W_k(t) \quad \forall \; k \in \Lambda^* = \Lambda_* \setminus \{0\}.$$

Set $a_k = \dfrac{\hat{\phi}(k)}{\sqrt{|\bar{\omega}(k)|}}$ and rewrite the equation

$$da_k = i\,\omega(k)\,a_k\,dt + i\,\varepsilon^{\theta} a_k \circ dW_k$$

$$i\,\varepsilon \int_{(\Lambda^+)^2} dk_1\,dk_2\,\mathrm{sig}(k^1)\sqrt{|\bar{\omega}(k)\,\bar{\omega}(k_1)\,\bar{\omega}(k_2)|}\;\delta(k-k_1-k_2)\,a_{k_1} a_{k_2}\,dt$$

Definition [two points correlation function] density function

$$f(a(t)) = \int |a(t)|^2\, d\rho(t) := \langle a\,\bar{a}\rangle$$

Statement of the main result

Consider the two-points correlation function

$$f(k,t) = \langle a(t,k)\, \bar{a}(t,k) \rangle = \int d\rho\, |a_k(t)|^2$$

theorem [S.-Tran] let $d \geq 2$, under suitable (but general) assumptions on the initial distribution ρ_0, if $t = \varepsilon^{-2}\tau$, $\varepsilon \ll 1$

$$\lim_{\varepsilon \to 0,\, L \to \infty} f(k, \varepsilon^{-2}\tau) = f^{\infty}(k,\tau) \quad \text{and}$$

$$\frac{\partial}{\partial \tau} f^{\infty}(k,\tau) = Q(f^{\infty})(k,\tau)$$

3-Wave Kinetic Equation

The difficulties

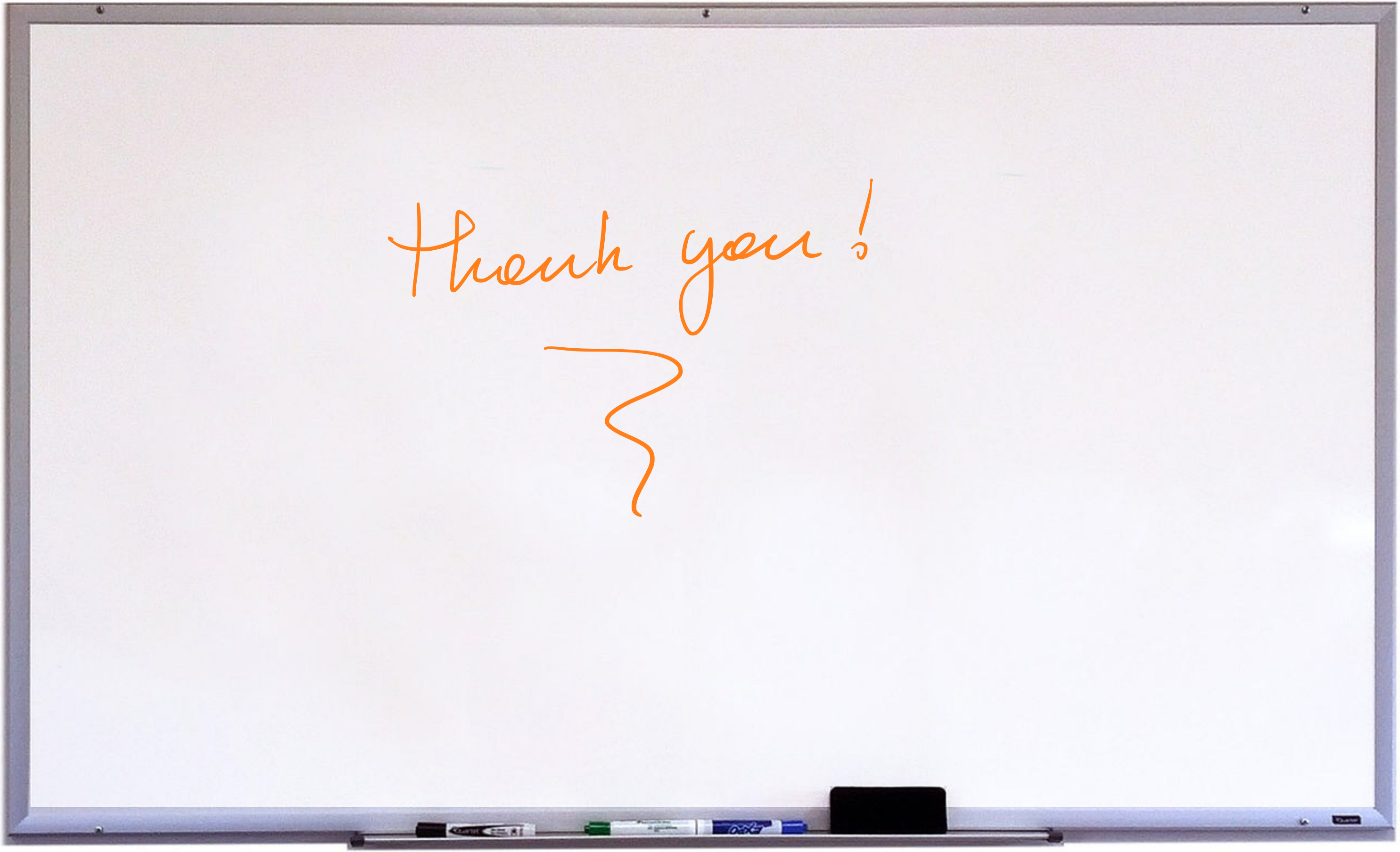
- In the rigorous derivation one needs to estimate all Feynman graphs
- The discreate setting is much more complicated than the continuum setting
- The dispersion relation is very singular
- The quadratic nonlinearity is not as good as the cubic nonlinearity

How we dealt with the obstacles

- We concentrated on the study of the equation for the density function $\rho(t)$ [Liouville Equation]
- The stochastic term acts only on angles not magnitude and gives to the Liouville equation some dissipation w.r.t. the angle variables.
- We looked for a weaker type of convergence and this allowed for L and ε not to be coupled.

Looking at the Future

1) We need better upper bounds for the H^s norms
2) We need examples of growing solutions
3) We would like to go further than the kinetic scale
4) We would like to understand better the connection between the two approaches described.
5) We would like to see more numerical work.


Thank you!