

# Anisotropy Gauss curvature flow and $L^p$ -Minkowski problem

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**Minkowski problem:** Given a Borel measure  $\mu = fd\sigma_{\mathbb{S}^n}$  on  $\mathbb{S}^n$ , find a convex body  $\Omega \subset \mathbb{R}^{n+1}$  such that its  $n$ -th area measure

$$S_n(\Omega, x) = \mu.$$

Corresponding PDE

$$\sigma_n(W_u(x)) = f(x), \quad W_u(x) > 0, \quad \forall x \in \mathbb{S}^n, \quad (1.1)$$

$u$  the support function of  $\Omega$

$$W_u(x) = (u_{ij}(x) + u\delta_{ij}(x)), \quad \forall x \in \mathbb{S}^n.$$

Necessary condition

$$\int_{\mathbb{S}^n} f(x)x_j d\mu_{\mathbb{S}^n} = 0, \quad \forall j = 1, \dots, n+1.$$

$L^p$ -Minkowski problem (Lutwak), try to solve

$$\sigma_n(W_u(x)) = u^{p-1}f(x). \quad (1.2)$$

The classical Minkowski problem corresponds to  $p = 1$  in (1.2).  
 We are interested in  $p < 1$ .

$\forall \alpha > 0$ , define entropy

$$\mathcal{E}_{\alpha, f}(\Omega) := \sup_{z_0 \in \Omega} \frac{\alpha}{\alpha - 1} \log \left( \int_{\mathbb{S}^n} u_{z_0}(x)^{1 - \frac{1}{\alpha}} f(x) d\theta(x) \right).$$

**A variational problem:**

$$\text{Minimize } \mathcal{E}_{\alpha}(\Omega) \text{ under constraint } |\Omega| = c. \quad (1.3)$$

A minimizer is a solution to

$$\sigma_n(u_{ij} + u\delta_{ij}) = \lambda f u^{-\frac{1}{\alpha}}, \text{ on } \mathbb{S}^n, \quad (1.4)$$

for some  $\lambda > 0$ .

**Problem:** solve (1.3).

Find a path to the minimizer of the constraint problem (1.3).

## Candidate

$$\text{Isotropic flow } X_t = -f^\alpha(\mathbf{v})K^\alpha \mathbf{v}, \quad \alpha > 0. \quad (2.1)$$

Andrews:

### Theorem 1

*For any  $\alpha > 0$  and positive  $f \in C^\infty(\mathbb{S}^n)$  and any initial smooth, strictly convex hypersurface  $\tilde{M}_0 \subset \mathbb{R}^{n+1}$ , the hypersurfaces  $\tilde{M}_\tau$  given by the solution of (2.1) exist for a finite time  $T$  and converge in Hausdorff distance to a point  $p \in \mathbb{R}^{n+1}$  as  $t$  approaches  $T$ .*

$\Omega \subset \mathbb{R}^{n+1}$  bounded convex,  $M = \partial\Omega$ ,  $K(x)$  the Gauss curvature.  
 Flow by power of Gauss curvature

$$X_t = -K^\alpha \nu, \quad \alpha > 0. \quad (2.2)$$

**Entropy**  $\mathcal{E}_\alpha(\Omega) = \sup_{z_0 \in \Omega} \mathcal{E}_\alpha(\Omega, z_0),$

where  $\mathcal{E}_\alpha(\Omega, z_0) := \frac{\alpha}{\alpha-1} \log \left( \int_{\mathbb{S}^n} u_{z_0}(x)^{1-\frac{1}{\alpha}} d\theta(x) \right).$

$$\exists! z_e, \mathcal{E}_\alpha(\Omega) = \mathcal{E}_\alpha(\Omega, z_e), \quad \text{dist}(z_e, \partial\Omega) \geq \delta(d(\Omega), n, |\Omega|). \quad (2.3)$$

Normalized flow

$$\frac{\partial}{\partial t} X(x, t) = - \frac{K^\alpha(x, t)}{\int_{\mathbb{S}^n} K^{\alpha-1}} \nu(x, t) + X(x, t). \quad (2.4)$$

## Andrews-Guan-Ni:

- Under the normalized flow (2.4),  $\mathcal{C}_\alpha(\Omega(t))$  and  $\mathcal{E}_\alpha(\Omega(t))$  are non-increasing.
- $\forall \alpha \geq \frac{1}{n+2}$ ,  $\mathcal{E}_\alpha^\infty := \lim_{t \rightarrow \infty} \mathcal{E}_\alpha(\Omega_t)$  exists,
- 

$$\mathcal{E}_\alpha^\infty - \mathcal{E}_\alpha(\Omega(t_0)) \leq - \int_{t_0}^{\infty} \left[ \frac{\int_{\mathbb{S}^n} f^{1+\frac{1}{\alpha}} d\sigma_t \cdot \int_{\mathbb{S}^n} d\sigma_t}{\int_{\mathbb{S}^n} f^{\frac{1}{\alpha}} d\sigma_t \cdot \int_{\mathbb{S}^n} f d\sigma_t} - 1 \right] dt \leq 0.$$

Here  $f(x, t) = \frac{K^\alpha(x, t)}{u(x, t)}$ ,  $d\sigma_t(x) = \frac{u(x, t)}{K(x, t)} d\theta(x)$ .

- Flow (2.4) converges to a soliton  $\eta u = K^\alpha$ ,  $\eta = \int_{\mathbb{S}^n} K^{\alpha-1}$ .
- The soliton is a critical point of  $\mathcal{E}_\alpha(\Omega)$  under the constraint  $|\Omega| = |B_1|$ .

## Classification of solitons:

- $\alpha = \frac{1}{n+2}$ , solitons are ellipsoids. (Andrews)
- $\forall \alpha > \frac{1}{n+2}$ , soliton is the sphere. (Brendle-Choi-Daskopolous)

$\int_{\mathbb{S}^n} f = 1$ ,  $|\Omega| = |B(1)|$ , the normalized flow

$$X_t = - \frac{f^\alpha(\mathbf{v})K^\alpha}{\int_{\mathbb{S}^n} f^\alpha K^{\alpha-1}} \mathbf{v} + X. \quad (2.5)$$

**Monotonicity:** along (2.5),

$$\mathcal{E}_{\alpha,f}(\Omega_{t_2}, z) - \mathcal{E}_{\alpha,f}(\Omega_{t_1}, z) = \int_{t_1}^{t_2} \left( \frac{\int_{\mathbb{S}^n} h^{\alpha+1}(x,t) d\sigma_t}{\int_{\mathbb{S}^n} h(x,t) d\sigma_t \cdot \int_{\mathbb{S}^n} h^\alpha(x,t) d\sigma_t} - 1 \right) dt \leq 0,$$

$$h(x,t) \doteq f(x) u_z^{-\frac{1}{\alpha}}(x,t) K(x,t), \quad d\sigma_t(x) = \frac{u_z(x,t)}{K(x,t)} d\theta(x).$$

Convergence of (2.5)? Entropy point estimate (2.3) fails for  $\mathcal{E}_{\alpha,f}$ !

## Theorem 2

For  $\alpha > \frac{1}{n+2}$  and finite non-trivial Borel measure  $\mu$  on  $\mathbb{S}^n$ ,  $n \geq 1$ , there exists a weak solution of (1.4) provided the following holds:

- (i) If  $\alpha > 1$  and  $\mu = f d\sigma_{\mathbb{S}^n}$  is not concentrated onto any great subsphere  $x^\perp \cap \mathbb{S}^n$ ,  $x \in \mathbb{S}^n$ .
- (ii) If  $\alpha = 1$  and  $\mu$  satisfies that for any linear  $\ell$ -subspace  $L \subset \mathbb{R}^{n+1}$  with  $1 \leq \ell \leq n$ , we have
  - (a)  $\mu(L \cap \mathbb{S}^n) \leq \frac{\ell}{n+1} \cdot \mu(\mathbb{S}^n)$ ;
  - (b) equality in (a) for a linear  $\ell$ -subspace  $L \subset \mathbb{R}^{n+1}$  with  $1 \leq \ell \leq n$  implies the existence of a complementary linear  $(n+1-\ell)$ -subspace  $\tilde{L} \subset \mathbb{R}^{n+1}$  such that  $\text{supp } \mu \subset L \cup \tilde{L}$ .
- (iii) If  $\frac{1}{n+2} < \alpha < 1$  and  $d\mu = f d\theta$  for non-negative  $f \in L^{\frac{n+1}{n+2-\frac{1}{\alpha}}}(\mathbb{S}^n)$ .



Theorem 2 is known. If  $f$  is bounded from below and above, it's a result of Chou-Wang.

- 1  $\alpha > 1$ , Chen-Li-Zhu.
- 2  $\alpha = 1$ , The paper Böröczky-Lutwak-Yang-Zhang characterized even measures  $\mu$ . Chen-Li-Zhu for sufficient condition.
- 3  $\frac{1}{n+2} < \alpha < 1$ , Bianchi-Böröczky-Colesanti-Yang.

Anisotropic approach was discussed in Andrews-Böröczky-Guan-Ni under some symmetry assumptions.

## Control diameter by entropy.

$\forall \delta, \tau \in (0, 1)$ , set  $D = \text{diam}(\Omega)$ ,

$$\Psi(L \cap \mathbb{S}^n, \delta) = \{x \in \mathbb{S}^n : \langle x, y \rangle \leq \delta \text{ for } y \in L^\perp \cap \mathbb{S}^n\},$$

If  $\alpha > 1$ , and

$$\oint_{\Psi(z^\perp \cap \mathbb{S}^n, \delta)} f \leq 1 - \tau, \quad \forall z \in \mathbb{S}^n,$$

then

$$\exp\left(\frac{\alpha - 1}{\alpha} \mathcal{E}_{\alpha, f}(\Omega)\right) \geq \gamma_1(n, \alpha) \tau \delta^{1 - \frac{1}{\alpha}} D^{1 - \frac{1}{\alpha}}. \quad (3.1)$$

If  $\alpha = 1$ , and

$$\oint_{\Psi(L \cap \mathbb{S}^n, \delta)} f < \frac{(1 - \tau)i}{n + 1}$$

for any linear  $i$ -subspace  $L$  of  $\mathbb{R}^{n+1}$ ,  $i = 1, \dots, n$ , then

$$\mathcal{E}_{1, f}(\Omega) \geq \tau \log D + \log \delta - 4 \log(n + 1). \quad (3.2)$$

If  $\frac{1}{n+2} < \alpha < 1$ ,  $p = 1 - \frac{1}{\alpha}$ ,  $\tau \leq \frac{1}{2} \int_{\mathbb{S}^n} f \cdot u^{1-\frac{1}{\alpha}}$  and

$$\int_{\Psi(z^\perp \cap \mathbb{S}^n, \delta)} f^{\frac{n+1}{n+1+p}} \leq \tau^{\frac{n+1}{n+1+p}}, \quad \forall z \in \mathbb{S}^{n-1},$$

then

$$\text{either } D \leq 16n^2/\delta^2, \quad \text{or } D \leq \left( \frac{1}{2} \int_{\mathbb{S}^n} f \cdot u^{1-\frac{1}{\alpha}} \right)^{\frac{2}{p}}. \quad (3.3)$$

Moreover, if  $\tau \leq \frac{1}{2} \exp\left(\frac{\alpha-1}{\alpha} \mathcal{E}_{\alpha,f}(\Omega)\right)$ , then

$$\text{either } D \leq 16n^2/\delta^2, \quad \text{or } D \leq \left( \frac{1}{2} \exp\left(\frac{\alpha-1}{\alpha} \mathcal{E}_{\alpha,f}(\Omega)\right) \right)^{\frac{2}{p}}. \quad (3.4)$$

As (2.1) contract to a point, we assume it's the origin.

### Lemma 3

Along (2.5),

(a). The entropy  $\mathcal{E}_{\alpha,f}(\Omega_t)$  is monotonically decreasing,

$$\mathcal{E}_{\alpha,f}(\Omega_{t_2}) \leq \mathcal{E}_{\alpha,f}(\Omega_{t_1}), \quad \forall t_1 \leq t_2 \in [0, \infty). \quad (4.1)$$

(b).  $\forall t_0 \in [0, \infty)$ ,

$$\mathcal{E}_{\alpha,f}(\Omega_{t_0}, 0) \geq \mathcal{E}_{\alpha,f,\infty} + \int_{t_0}^{\infty} \left( \frac{\int_{\mathbb{S}^n} h^{\alpha+1}(x,t) d\sigma_t}{\int_{\mathbb{S}^n} h(x,t) d\sigma_t \cdot \int_{\mathbb{S}^n} h^{\alpha}(x,t) d\sigma_t} - 1 \right) dt, \quad (4.2)$$

where

$$h(x,t) \doteq f(x) u_0^{-\frac{1}{\alpha}}(x,t) K(x,t), \quad \mathcal{E}_{\alpha,f,\infty} \doteq \lim_{t \rightarrow \infty} \mathcal{E}_{\alpha,f}(\Omega_t).$$

Assume  $f$  satisfies condition in Theorem 2, it follows from (4.1) and (3.1)-(3.4),

$$D(\Omega(t)) \leq C. \quad (4.3)$$

Since  $|\Omega(t)| = |B(1)|$ , we have non-collapsing estimate

$$\frac{\rho_+(\Omega(t))}{\rho_-(\Omega(t))} \leq C,$$

where  $\rho_+$  and  $\rho_-$  are the outer and inner radii of the convex body.

Set

$$\eta(t) = \oint_{\mathbb{S}^n} h(x, t) d\sigma_t. \quad (4.4)$$

As  $\oint_{\mathbb{S}^n} h(x, t) d\sigma_t$  is monotone and bounded from below and above by diameter estimates,

$$C \geq \lim_{t \rightarrow \infty} \oint_{\mathbb{S}^n} h(x, t) = \eta \geq \frac{1}{C} \quad (4.5)$$

## Holder's room

(4.2) implies that  $\exists \{t_k\}_{k=1}^\infty$ ,  $\lim_{k \rightarrow \infty} t_k = \infty$  and

$$\lim_{k \rightarrow \infty} \frac{\int_{\mathbb{S}^n} h^{\alpha+1}(x, t) d\sigma_t}{\int_{\mathbb{S}^n} h(x, t) d\sigma_t \cdot \int_{\mathbb{S}^n} h^\alpha(x, t) d\sigma_t} = 1. \quad (4.6)$$

### Lemma 4

Let  $p, q \in \mathbb{R}^+$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , set  $\beta = \min\{\frac{1}{p}, \frac{1}{q}\}$ . Let  $(M, \mu)$  be a measurable space,  $\forall F \in L^p, G \in L^q, \|F\|_{L^p} \|G\|_{L^q} > 0$ ,

$$\frac{\int_M |FG| d\mu}{\|F\|_{L^p} \|G\|_{L^q}} - 1 \leq -\beta \int_M \left( \frac{|F|^{\frac{p}{2}}}{(\int_M |F|^p d\mu)^{\frac{1}{2}}} - \frac{|G|^{\frac{q}{2}}}{(\int_M |G|^q d\mu)^{\frac{1}{2}}} \right)^2. \quad (4.7)$$

We use the extra **room** in above improved Holder inequality to deduce weak convergence of flow (2.5).

## Proposition 1

Denote  $u_k = u(x, t_k)$ ,  $\sigma_{n,k} = \sigma_n(u_{ij}(x, t_k) + u(x, t_k)\delta_{ij})$ . Then

$$\lim_{k \rightarrow \infty} \oint_{\mathbb{S}^n} |u_k^{\frac{1}{\alpha}} \sigma_{n,k} - \frac{f}{\eta}| d\theta = 0, \quad (4.8)$$

where  $\eta$  is defined in (4.4) which is bounded from below and above in (4.5). As a consequence, there is convex body  $\Omega \subset \mathbb{R}^{n+1}$  with  $o \in \Omega$ ,

$$|\Omega| = |B(1)|, \quad \mathcal{E}_{\alpha,f}(\Omega_t) \leq \mathcal{E}_{\alpha,f}(B(1)),$$

and its support function  $u$  satisfies

$$u^{\frac{1}{\alpha}} S_{\Omega} = \frac{1}{\eta} f d\theta. \quad (4.9)$$

We only need to verify (4.8), it is equivalent to prove

$$\lim_{k \rightarrow \infty} \int_{\mathbb{S}^n} |u_k^{\frac{1}{\alpha}} \sigma_{n,k} - f \eta^{-1}(t_k)| d\theta = 0. \quad (4.10)$$

$$\int_{\mathbb{S}^n} |u_k^{\frac{1}{\alpha}} \sigma_{n,k} - \frac{f}{\eta(t_k)}| \leq \left( \int_{\mathbb{S}^n} \left| \frac{f}{\eta(t_k) u_k^{\frac{1}{\alpha}} \sigma_{n,k}} - 1 \right|^{1+\alpha} d\sigma_{t_k} \right)^{\frac{1}{1+\alpha}} \left( \int_{\mathbb{S}^n} u_k^{\frac{1}{\alpha^2}} \sigma_{n,k} \right)^{\frac{\alpha}{1+\alpha}}$$

Since  $D(t_k)$  is bounded,

$$\int_{\mathbb{S}^n} u_k^{\frac{1}{\alpha^2}} \sigma_{n,k} d\theta \leq (D(t_k))^{\frac{1}{\alpha^2}} \int_{\mathbb{S}^n} u_k^{\frac{1}{\alpha^2}} \sigma_{n,k} d\theta \leq (D(t_k))^{\frac{1}{\alpha^2}} |\partial\Omega_{t_k}| \leq C.$$



By Lemma 4, with  $p = \alpha + 1$ ,  $F^{\frac{1}{1+\alpha}} = h(x, t_k)$ ,  $G = 1$

$$\lim_{k \rightarrow \infty} \oint \left( \left( \frac{h(x, t_k)}{\eta(t_k)} \right)^{\frac{1+\alpha}{2}} - 1 \right)^2 d\sigma_{t_k} = 0. \quad (4.12)$$

For  $t_k$  fixed, let

$$\gamma_k(x) = f \eta^{-1}(t_k) u_k^{-\frac{1}{\alpha}} \sigma_{n,k}^{-1} = h(x, t_k) \eta^{-1}(t_k)$$

and set

$$\Sigma_k = \{x \in \mathbb{S}^n \mid |\gamma_k(x) - 1| \leq \frac{1}{2}\}$$

It is straightforward to check that  $\exists A_\alpha \geq 1$  depending only on  $\alpha$  such that

$$\begin{aligned} A_\alpha |\gamma_k^{\frac{1+\alpha}{2}}(x) - 1| &\geq |\gamma_k(x) - 1|, \quad \forall x \in \Sigma_k; \\ A_\alpha |\gamma_k^{\frac{1+\alpha}{2}}(x) - 1|^2 &\geq |\gamma_k(x) - 1|^{1+\alpha}, \quad \forall x \in \Sigma_k^c. \end{aligned}$$

Since  $|\gamma_k^{\frac{1+\alpha}{2}}(x) - 1| \leq 2^{1+\alpha}$ ,  $\forall x \in \Sigma_k$ , let  $\delta = \min\{1 + \alpha, 2\}$ ,

$$\int_{\mathbb{S}^n} |\gamma_k(x) - 1|^{1+\alpha} d\sigma_{t_k} \leq C \left( \left( \int_{\mathbb{S}^n} |\gamma_k^{\frac{1+\alpha}{2}}(x) - 1|^2 d\sigma_{t_k} \right)^{\frac{\delta}{2}} + \int_{\mathbb{S}^n} |\gamma_k^{\frac{1+\alpha}{2}}(x) - 1|^2 d\sigma_{t_k} \right)$$

By (4.12),

$$\lim_{k \rightarrow \infty} \int_{\mathbb{S}^n} |\gamma_k^{\frac{1+\alpha}{2}}(x) - 1|^2 d\sigma_{t_k} = 0.$$

It follows (4.10). □

**Thank you**