Anisotropy Gauss curvature flow and L^p-Minkowski problem

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Workshop on "Applied and Computational Differential Geometry and Geometric PDEs"
BIRS, July 31-August 4, 2023

Minkowski problem: Given a Borel measure $\mu = fd\sigma_{\mathbb{S}^n}$ on $\mathbb{S}^n$, find a convex body $\Omega \subset \mathbb{R}^{n+1}$ such that its n-th area measure

$$S_n(\Omega, x) = \mu.$$

Corresponding PDE

$$\sigma_n(W_u(x)) = f(x), \quad W_u(x) > 0, \ \forall x \in \mathbb{S}^n, \tag{1.1}$$

u the support function of Ω

$$W_u(x) = (u_{ij}(x) + u\delta_{ij}(x)), \quad \forall x \in \mathbb{S}^n.$$

Necessary condition

$$\int_{\mathbb{S}^n} f(x)x_j \, d\mu_{\mathbb{S}^n} = 0, \ \forall j = 1, \cdots, n+1.$$

L^p-Minkowski problem (Lutwak), try to solve

$$\sigma_n(W_u(x)) = u^{p-1}f(x). \tag{1.2}$$

The classical Minkowski problem corresponds to $p = 1$ in (1.2). We are interested in $p < 1$.

$\forall \alpha > 0$, define entropy

$$\mathscr{E}_{\alpha,f}(\Omega) := \sup_{z_0 \in \Omega} \frac{\alpha}{\alpha - 1} \log \left(\fint_{\mathbb{S}^n} u_{z_0}(x)^{1-\frac{1}{\alpha}} f(x) d\theta(x) \right).$$

A variational problem:

$$\text{Minimize } \mathscr{E}_{\alpha}(\Omega) \text{ under constraint } |\Omega| = c. \tag{1.3}$$

A minimizer is a solution to

$$\sigma_n(u_{ij} + u\delta_{ij}) = \lambda f u^{-\frac{1}{\alpha}}, \text{on } \mathbb{S}^n, \tag{1.4}$$

for some $\lambda > 0$.

Problem: solve (1.3).

Find a path to the minimizer of the constraint problem (1.3).

Candidate

$$\textit{Isotropic flow} \quad X_t = -f^{\alpha}(\nu) K^{\alpha} \nu, \ \alpha > 0. \tag{2.1}$$

Andrews:

Theorem 1

For any $\alpha > 0$ and positive $f \in C^{\infty}(\mathbb{S}^n)$ and any initial smooth, strictly convex hypersurface $\tilde{M}_0 \subset \mathbb{R}^{n+1}$, the hypersurfaces $\tilde{M}_\tau$ given by the solution of (2.1) exist for a finite time T and converge in Hausdorff distance to a point $p \in \mathbb{R}^{n+1}$ as t approaches T.

$\Omega \subset \mathbb{R}^{n+1}$ bounded convex, $M = \partial\Omega$, $K(x)$ the Gauss curvature.
Flow by power of Gauss curvature

$$X_t = -K^\alpha \nu, \quad \alpha > 0. \tag{2.2}$$

$$\textbf{Entropy} \quad \mathscr{E}_\alpha(\Omega) = \sup_{z_0 \in \Omega} \mathscr{E}_\alpha(\Omega, z_0),$$

$$\textit{where} \quad \mathscr{E}_\alpha(\Omega, z_0) := \frac{\alpha}{\alpha - 1} \log\left(\oint_{\mathbb{S}^n} u_{z_0}(x)^{1-\frac{1}{\alpha}}\, d\theta(x) \right).$$

$$\exists! z_e,\ \mathscr{E}_\alpha(\Omega) = \mathscr{E}_\alpha(\Omega, z_e), \quad dist(z_e, \partial\Omega) \geq \delta(d(\Omega), n, |\Omega|). \tag{2.3}$$

Normalized flow

$$\frac{\partial}{\partial t} X(x,t) = -\frac{K^\alpha(x,t)}{\oint_{\mathbb{S}^n} K^{\alpha-1}} \nu(x,t) + X(x,t). \tag{2.4}$$

Andrews-Guan-Ni:

- Under the normalized flow (2.4), $\mathscr{C}_\alpha(\Omega(t))$ and $\mathscr{E}_\alpha(\Omega(t))$ are non-increasing.
- $\forall \alpha \geq \frac{1}{n+2}$, $\mathscr{E}_\alpha^\infty := \lim_{t\to\infty} \mathscr{E}_\alpha(\Omega_t)$ exists,
-

$$\mathscr{E}_\alpha^\infty - \mathscr{E}_\alpha(\Omega(t_0)) \leq -\int_{t_0}^{\infty} \left[\frac{\int_{\mathbb{S}^n} f^{1+\frac{1}{\alpha}}\, d\sigma_t \cdot \int_{\mathbb{S}^n} d\sigma_t}{\int_{\mathbb{S}^n} f^{\frac{1}{\alpha}}\, d\sigma_t \cdot \int_{\mathbb{S}^n} f\, d\sigma_t} - 1 \right] dt \leq 0.$$

  Here $f(x,t) = \frac{K^\alpha(x,t)}{u(x,t)}$, $d\sigma_t(x) = \frac{u(x,t)}{K(x,t)}\, d\theta(x)$.
- Flow (2.4) converges to a soliton $\eta u = K^\alpha$, $\eta = \int_{\mathbb{S}^n} K^{\alpha-1}$.
- The soliton is a critical point of $\mathscr{E}_\alpha(\Omega)$ under the constraint $|\Omega| = |B_1|$.

Classification of solitons:

- $\alpha = \frac{1}{n+2}$, solitons are ellipsoids. (Andrews)
- $\forall \alpha > \frac{1}{n+2}$, soliton is the sphere. (Brendle-Choi-Daskopolous)

$\fint_{\mathbb{S}^n} f = 1$, $|\Omega| = |B(1)|$, the normalized flow

$$X_t = -\frac{f^\alpha(\nu)K^\alpha}{\fint_{\mathbb{S}^n} f^\alpha K^{\alpha-1}}\nu + X. \qquad (2.5)$$

Monotonicity: along (2.5),

$$\mathscr{E}_{\alpha,f}(\Omega_{t_2}, z) - \mathscr{E}_{\alpha,f}(\Omega_{t_1}, z) = \int_{t_1}^{t_2} \left(\frac{\fint_{\mathbb{S}^n} h^{\alpha+1}(x,t)\, d\sigma_t}{\fint_{\mathbb{S}^n} h(x,t)\, d\sigma_t \cdot \fint_{\mathbb{S}^n} h^\alpha(x,t)\, d\sigma_t} - 1 \right) dt \le 0,$$

$h(x,t) \doteqdot f(x) u_z^{-\frac{1}{\alpha}}(x,t) K(x,t)$, $d\sigma_t(x) = \frac{u_z(x,t)}{K(x,t)} d\theta(x)$.

Convergence of (2.5)? Entropy point estimate (2.3) fails for $\mathscr{E}_{\alpha,f}$!

Theorem 2

For $\alpha > \frac{1}{n+2}$ and finite non-trivial Borel measure μ on $\mathbb{S}^n$, $n \geq 1$, there exists a weak solution of (1.4) *provided the following holds:*

(i) *If $\alpha > 1$ and $\mu = f\, d\sigma_{\mathbb{S}^n}$ is not concentrated onto any great subsphere $x^{\perp} \cap \mathbb{S}^n$, $x \in \mathbb{S}^n$.*

(ii) *If $\alpha = 1$ and μ satisfies that for any linear ℓ-subspace $L \subset \mathbb{R}^{n+1}$ with $1 \leq \ell \leq n$, we have*

  (a) $\mu(L \cap \mathbb{S}^n) \leq \dfrac{\ell}{n+1} \cdot \mu(\mathbb{S}^n)$;

  (b) *equality in (a) for a linear ℓ-subspace $L \subset \mathbb{R}^{n+1}$ with $1 \leq d \leq n$ implies the existence of a complementary linear $(n+1-\ell)$-subspace $\widetilde{L} \subset \mathbb{R}^{n+1}$ such that* $\operatorname{supp} \mu \subset L \cup \widetilde{L}$.

(iii) *If $\frac{1}{n+2} < \alpha < 1$ and $d\mu = f\, d\theta$ for non-negative $f \in L^{\frac{n+1}{n+2-\frac{1}{\alpha}}}(\mathbb{S}^n)$.*

Theorem 2 is known. If f is bounded from below and above, it's a result of Chou-Wang.

1. $\alpha > 1$, Chen-Li-Zhu.
2. $\alpha = 1$, The paper Böröczky-Lutwak-Yang-Zhang characterized even measures μ. Chen-Li-Zhu for sufficient condition.
3. $\frac{1}{n+2} < \alpha < 1$, Bianchi-Böröczky-Colesanti-Yang.

Anisotropic approach was discussed in Andrews-Böröczky-Guan-Ni under some symmetry assumptions.

Control diameter by entropy.

$\forall \delta, \tau \in (0,1)$, set $D = diam(\Omega)$,

$$\Psi(L \cap \mathbb{S}^n, \delta) = \{x \in \mathbb{S}^n : \langle x, y\rangle \le \delta \text{ for } y \in L^{\perp} \cap \mathbb{S}^n\},$$

If $\alpha > 1$, and

$$\oint_{\Psi(z^{\perp}\cap \mathbb{S}^n, \delta)} f \le 1 - \tau, \ \forall z \in S^n,$$

then

$$\exp\left(\frac{\alpha - 1}{\alpha} \mathscr{E}_{\alpha,f}(\Omega)\right) \ge \gamma_1(n,\alpha)\tau\delta^{1-\frac{1}{\alpha}} D^{1-\frac{1}{\alpha}}. \tag{3.1}$$

If $\alpha = 1$, and

$$\oint_{\Psi(L\cap \mathbb{S}^n, \delta)} f < \frac{(1-\tau)i}{n+1}$$

for any linear i-subspace L of $\mathbb{R}^{n+1}$, $i = 1, \dots, n$, then

$$\mathscr{E}_{1,f}(\Omega) \ge \tau \log D + \log \delta - 4\log(n+1). \tag{3.2}$$

If $\frac{1}{n+2} < \alpha < 1$, $p = 1 - \frac{1}{\alpha}$, $\tau \leq \frac{1}{2} \fint_{\mathbb{S}^n} f \cdot u^{1-\frac{1}{\alpha}}$ and

$$\fint_{\Psi(z^\perp \cap \mathbb{S}^n, \delta)} f^{\frac{n+1}{n+1+p}} \leq \tau^{\frac{n+1}{n+1+p}}, \ \forall z \in \mathbb{S}^{n-1},$$

then

$$\text{either } D \leq 16n^2/\delta^2, \quad \text{or} \quad D \leq \left(\frac{1}{2} \fint_{\mathbb{S}^n} f \cdot u^{1-\frac{1}{\alpha}} \right)^{\frac{2}{p}}. \tag{3.3}$$

Moreover, if $\tau \leq \frac{1}{2} \exp\left(\frac{\alpha-1}{\alpha} \mathscr{E}_{\alpha,f}(\Omega) \right)$, then

$$\text{either } D \leq 16n^2/\delta^2, \quad \text{or} \quad D \leq \left(\frac{1}{2} \exp\left(\frac{\alpha-1}{\alpha} \mathscr{E}_{\alpha,f}(\Omega) \right) \right)^{\frac{2}{p}}. \tag{3.4}$$

As (2.1) contract to a point, we assume it's the origin.

Lemma 3

Along (2.5),

(a). *The entropy $\mathcal{E}_{\alpha,f}(\Omega_t)$ is monotonically decreasing,*

$$\mathcal{E}_{\alpha,f}(\Omega_{t_2}) \leq \mathcal{E}_{\alpha,f}(\Omega_{t_1}), \quad \forall t_1 \leq t_2 \in [0,\infty). \tag{4.1}$$

(b). $\forall t_0 \in [0,\infty)$,

$$\mathcal{E}_{\alpha,f}(\Omega_{t_0},0) \geq \mathcal{E}_{\alpha,f,\infty} + \int_{t_0}^{\infty} \left(\frac{\fint_{\mathbb{S}^n} h^{\alpha+1}(x,t)\, d\sigma_t}{\fint_{\mathbb{S}^n} h(x,t)\, d\sigma_t \cdot \fint_{\mathbb{S}^n} h^{\alpha}(x,t)\, d\sigma_t} - 1 \right) dt, \tag{4.2}$$

where

$$h(x,t) \doteqdot f(x) u_0^{-\frac{1}{\alpha}}(x,t) K(x,t), \ \mathcal{E}_{\alpha,f,\infty} \doteqdot \lim_{t\to\infty} \mathcal{E}_{\alpha,f}(\Omega_t).$$

Assume f satisfies condition in Theorem 2, it follows from (4.1) and (3.1)-(3.4),

$$D(\Omega(t)) \leq C. \tag{4.3}$$

Since $|\Omega(t)| = |B(1)|$, we have non-collapsing estimate

$$\frac{\rho_+(\Omega(t))}{\rho_-(\Omega(t))} \leq C,$$

where ρ_+ and ρ_- are the outer and inner radii of the convex body.

Set

$$\eta(t) = \oint_{\mathbb{S}^n} h(x,t)\, d\sigma_t. \tag{4.4}$$

As $\oint_{\mathbb{S}^n} h(x,t)\, d\sigma_t$ is monotone and bounded from below and above by diameter estimates,

$$C \geq \lim_{t\to\infty} \oint_{\mathbb{S}^n} h(x,t) = \eta \geq \frac{1}{C} \tag{4.5}$$

Holder's room

(4.2) implies that $\exists \{t_k\}_{k=1}^{\infty}$, $\lim_{k\to\infty} t_k = \infty$ and

$$\lim_{k\to\infty} \frac{\oint_{\mathbb{S}^n} h^{\alpha+1}(x,t)\, d\sigma_t}{\oint_{\mathbb{S}^n} h(x,t)\, d\sigma_t \cdot \oint_{\mathbb{S}^n} h^{\alpha}(x,t)\, d\sigma_t} = 1. \tag{4.6}$$

Lemma 4

Let $p,\ q \in \mathbb{R}^+$ with $\frac{1}{p} + \frac{1}{q} = 1$, set $\beta = \min\{\frac{1}{p}, \frac{1}{q}\}$. Let (M, μ) be a measurable space, $\forall F \in L^p,\ G \in L^q,\ \|F\|_{L^p}\|G\|_{L^q} > 0$,

$$\frac{\int_M |FG| d\mu}{\|F\|_{L^p}\|G\|_{L^q}} - 1 \le -\beta \int_M \left(\frac{|F|^{\frac{p}{2}}}{(\int_M |F|^p d\mu)^{\frac{1}{2}}} - \frac{|G|^{\frac{q}{2}}}{(\int_M |G|^q d\mu)^{\frac{1}{2}}} \right)^2. \tag{4.7}$$

We use the extra room in above improved Holder inequality to deduce weak convergence of flow (2.5).

Proposition 1

Denote $u_k = u(x,t_k)$, $\sigma_{n,k} = \sigma_n(u_{ij}(x,t_k) + u(x,t_k)\delta_{ij})$. *Then*

$$\lim_{k\to\infty} \oint_{\mathbb{S}^n} |u_k^{\frac{1}{\alpha}} \sigma_{n,k} - \frac{f}{\eta}| d\theta = 0, \tag{4.8}$$

where η *is defined in (4.4) which is bounded from below and above in (4.5). As a consequence, there is convex body* $\Omega \subset \mathbb{R}^{n+1}$ *with* $o \in \Omega$,

$$|\Omega| = |B(1)|, \quad \mathscr{E}_{\alpha,f}(\Omega_t) \leq \mathscr{E}_{\alpha,f}(B(1)),$$

and its support function u *satisfies*

$$u^{\frac{1}{\alpha}} S_\Omega = \frac{1}{\eta} f d\theta. \tag{4.9}$$

We only need to verify (4.8), it is equivalent to prove

$$\lim_{k\to\infty} \oint_{\mathbb{S}^n} |u_k^{\frac{1}{\alpha}} \sigma_{n,k} - f\eta^{-1}(t_k)| d\theta = 0. \tag{4.10}$$

$$\oint_{\mathbb{S}^n} |u_k^{\frac{1}{\alpha}} \sigma_{n,k} - \frac{f}{\eta(t_k)}| \leq \left(\oint_{\mathbb{S}^n} |\frac{f}{\eta(t_k) u_k^{\frac{1}{\alpha}} \sigma_{n,k}} - 1|^{1+\alpha} d\sigma_{t_k} \right)^{\frac{1}{1+\alpha}} \left(\oint_{\mathbb{S}^n} u_k^{\frac{1}{\alpha^2}} \sigma_{n,k} \right)$$

Since $D(t_k)$ is bounded,

$$\oint_{\mathbb{S}^n} u_k^{\frac{1}{\alpha^2}} \sigma_{n,k} d\theta \leq (D(t_k))^{\frac{1}{\alpha^2}} \oint_{\mathbb{S}^n} u_k^{\frac{1}{\alpha^2}} \sigma_{n,k} d\theta \leq (D(t_k))^{\frac{1}{\alpha^2}} |\partial \Omega_{t_k}| \leq C.$$

By Lemma 4, with $p = \alpha + 1$, $F^{\frac{1}{1+\alpha}} = h(x, t_k)$, $G = 1$

$$\lim_{k\to\infty} \oint \left((\frac{h(x,t_k)}{\eta(t_k)})^{\frac{1+\alpha}{2}} - 1 \right)^2 d\sigma_{t_k} = 0. \tag{4.12}$$

For t_k fixed, let

$$\gamma_k(x) = f\eta^{-1}(t_k)u_k^{-\frac{1}{\alpha}}\sigma_{n,k}^{-1} = h(x,t_k)\eta^{-1}(t_k)$$

and set

$$\Sigma_k = \{x \in \mathbb{S}^n \mid |\gamma_k(x) - 1| \leq \frac{1}{2}.\}$$

It is straightforward to check that $\exists A_\alpha \geq 1$ depending only on α such that

$$\begin{aligned} A_\alpha|\gamma_k^{\frac{1+\alpha}{2}}(x) - 1| &\geq |\gamma_k(x) - 1|, \ \forall x \in \Sigma_k; \\ A_\alpha|\gamma_k^{\frac{1+\alpha}{2}}(x) - 1|^2 &\geq |\gamma_k(x) - 1|^{1+\alpha}, \ \forall x \in \Sigma_k^c. \end{aligned}$$

Since $|\gamma_k^{\frac{1+\alpha}{2}}(x)-1| \leq 2^{1+\alpha}$, $\forall x \in \Sigma_k$, let $\delta = \min\{1+\alpha, 2\}$,

$$\oint_{\mathbb{S}^n} |\gamma_k(x)-1|^{1+\alpha} d\sigma_{t_k} \leq C\left((\oint_{\mathbb{S}^n} |\gamma_k^{\frac{1+\alpha}{2}}(x)-1|^2 d\sigma_{t_k})^{\frac{\delta}{2}} + \oint_{\mathbb{S}^n} |\gamma_k^{\frac{1+\alpha}{2}}(x)-1|^2 d\sigma\right.$$

By (4.12),

$$\lim_{k\to\infty} \oint_{\mathbb{S}^n} |\gamma_k^{\frac{1+\alpha}{2}}(x)-1|^2 d\sigma_{t_k} = 0.$$

It follows (4.10). □

Thank you