CONVERGENCE FOR DISCRETE HARMONIC MAPS

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I. Main Thm

II. Discretization

III. 3 Ingredients

Applied and Computational Geometry and Geometric PDEs

BIRS

THINK: hyperbolic surfaces

I. Let X, Y be R. mflds,

$f : X \to Y$ a piecewise smooth map.

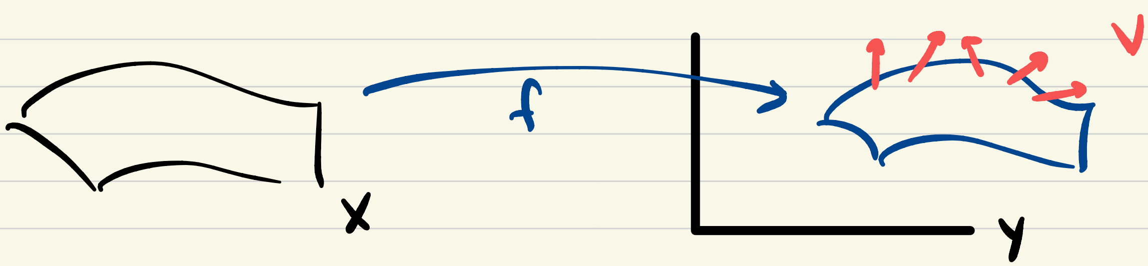
- The energy of f is $$E(f) = \int_X \| df \|^2$$

let $\mathcal{C} = \{ \text{p.smooth maps } X \text{ to } Y, \simeq f \}$.

$$E : \mathcal{C} \longrightarrow \mathbb{R}$$

- f is harmonic if f is a critical pt. of E,

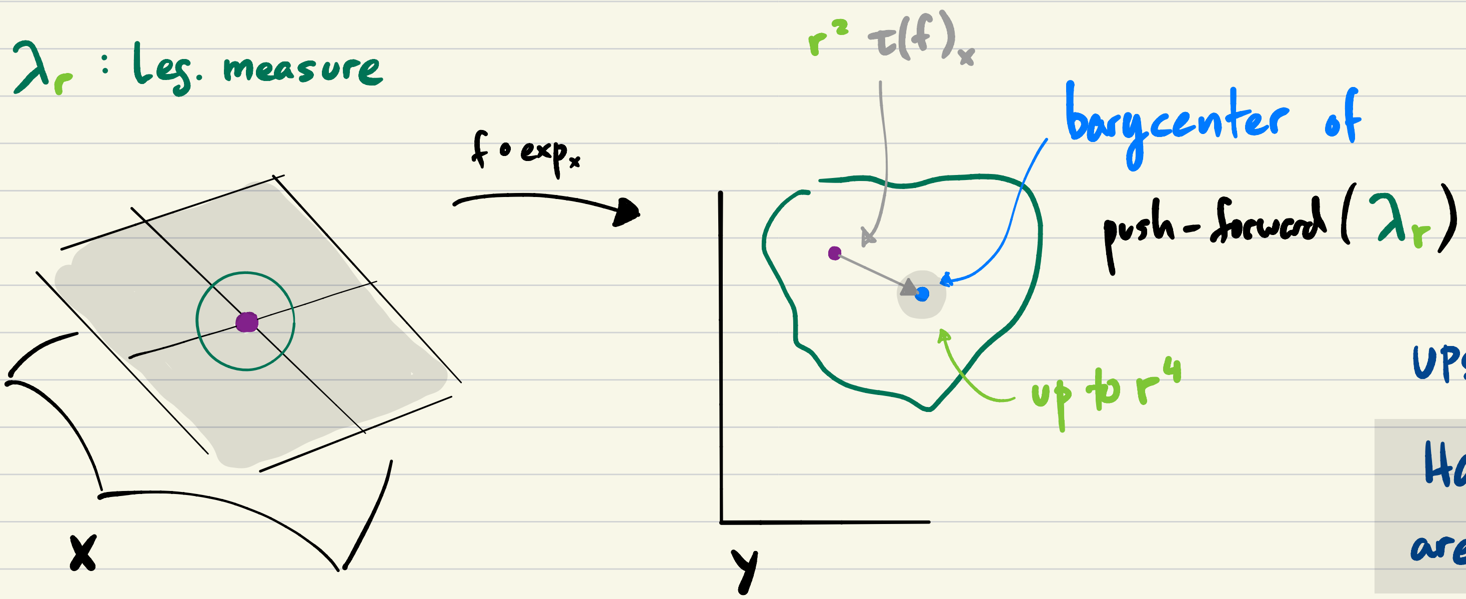
i.e. $\frac{d}{dt}\Big|_{t=0} E(\exp_f tV) = 0$, for all $V \in T_f \mathcal{C}$



- The **tension field** of f is $\tau(f) = -\operatorname{grad} E_f \in T_f \mathcal{C}$

$$\boxed{f \text{ harmonic} \iff \tau(f) = 0}$$

NB. Intuitively, $\tau(f)(x)$ points from $f(x)$ to the barycenter of the f-image of points nearby x.



UPSHOT :

$\boxed{\text{Harmonic maps are well-balanced}}$

Thm (Eells - Sampson): Suppose X is compact, and Y is non-pos. curved.

Then $\begin{cases} f_0 = f \\ \frac{d}{dt} f_t = \tau(f_t) \end{cases}$ converges as $t \longrightarrow \infty$ to a harmonic map $\phi \simeq f$.

THINK: Gradient flow for E!

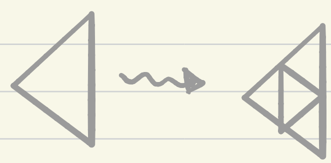
Thm (Hartman). If $\operatorname{rank}(f) > 1$ and Y is neg. curved, ϕ is unique and smooth.

GOAL: DISCRETIZATION + CONVERGENCE

MAIN THM (GLM):

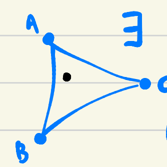
Let X, Y be compact hyperbolizable NPC surfaces, $f: X \to Y$, $\deg(f) \neq 0$, $\phi \simeq f$ harmonic.

If $\{G_n\}$ is a seq. of triangulations of X obtained by midpt. refinement which is strongly acute, then the barycentric interpolation of the corresponding discrete harmonic maps converge to ϕ in L^2.

- (midpoint refinement) 

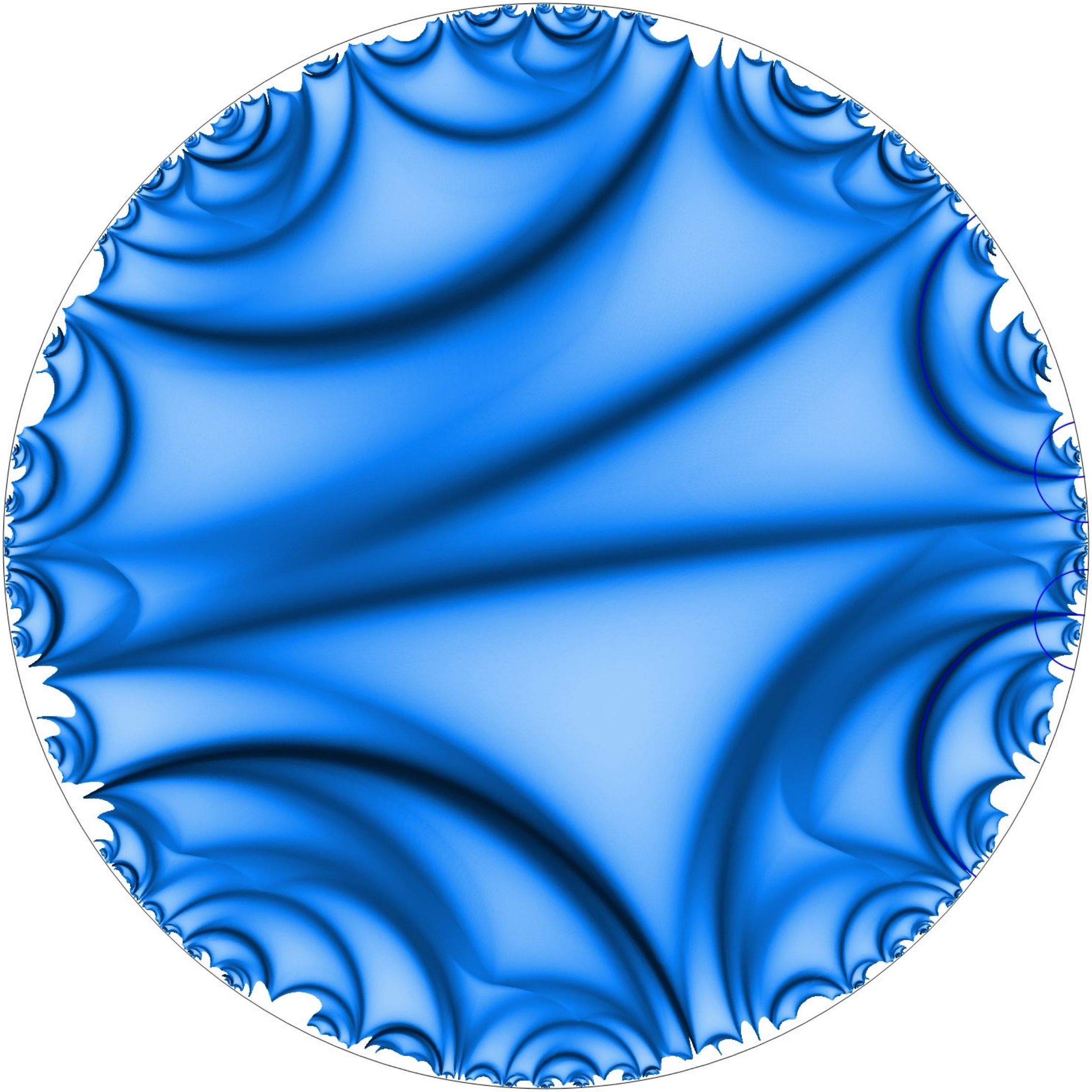
- (strongly acute) all angles of $\{G_n\}$ are $\gg 0$, $\ll \frac{\pi}{2}$

NB. cf. Bartels: $\{G_n\}$ is "logarithmically right-angled"

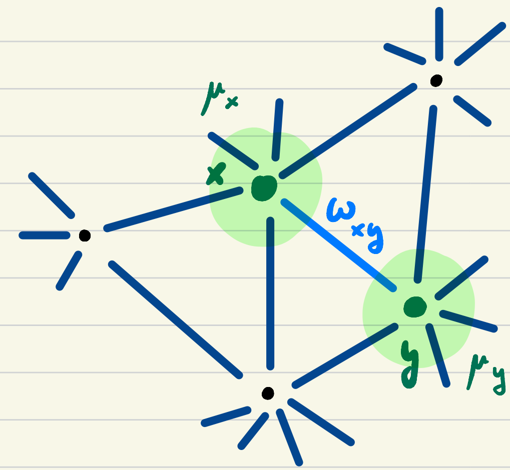
- (barycentric interpolation) $u_n \rightsquigarrow \hat{u}_n$ 

$\exists$ bary$(r\delta_A + s\delta_B + t\delta_C)$ b/c NPC
$\updownarrow$
$(r,s,t) \in \Delta^2 \subseteq \mathbb{R}^3$

- (convergence in L^2) $d_2^2(\phi, \hat{u}_n) = \int_X d^2(\phi(x), \hat{u}_n(x))\, d\mathrm{vol}_X(x)$



II. Discretization



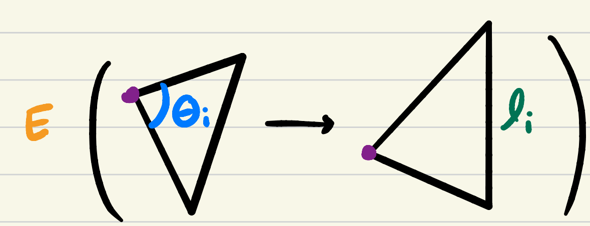
- Let G = a triangulation of X, endowed with positive edge weights $= \{\omega_{xy} : x \sim y\}$
  vertex weights $= \{\mu_x\}$
- Let $\mathcal{C}_G$ = "discrete $\mathcal{C}$" $\cong \tilde{Y}^{V(G)}$ ← $< \infty$ dim'l R. mfld !

  with $T_u \mathcal{C}_G = \bigoplus_x T_{u(x)} \tilde{Y}$ for $u \in \mathcal{C}_G$,

  and inner product $\langle \cdot, \cdot \rangle_u = \bigoplus_x \mu_x \langle \cdot, \cdot \rangle_{u(x)}$

For us, ω_{xy} = "cot wts"

μ_x = "area wts"



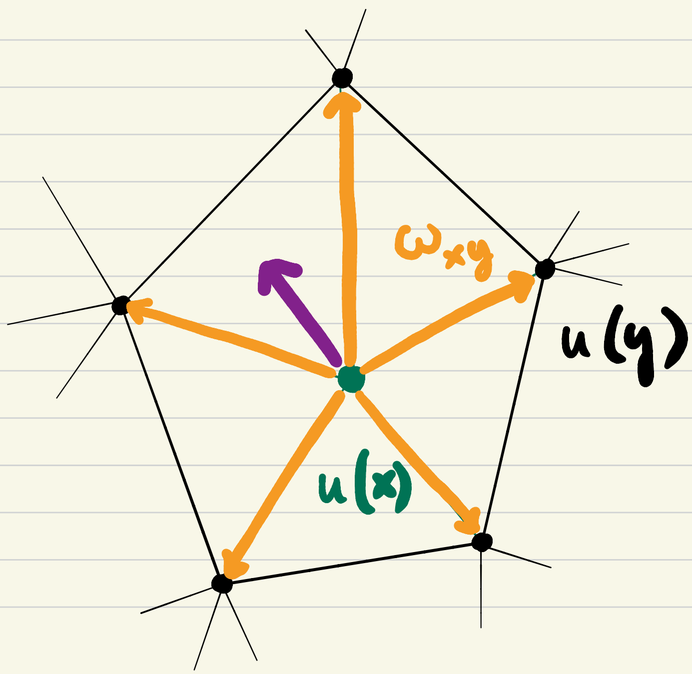
$= \sum_i \cot \theta_i \cdot \ell_i^2$

- The discrete energy of $u \in \mathcal{C}_G$ is $E_G(u) = \sum_{x \sim y} \omega_{xy} \, d^2(u(x), u(y))$

  and u is discrete harmonic if it's a crit pt of $E_G : \mathcal{C}_G \longrightarrow \mathbb{R}$

- The discrete tension field of $u \in \mathcal{C}_G$ is given by

$$\tau_G(u) = -\operatorname{grad}_u E_G \in T_u \mathcal{C}_G$$
$$= \frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy} \, \overrightarrow{u(x)\, u(y)}$$

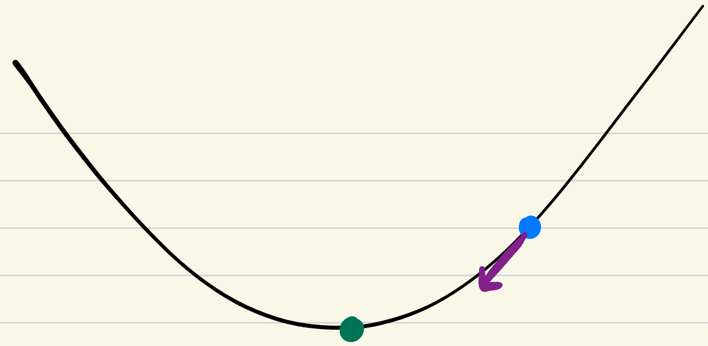


u is discrete harmonic $\iff \tau_G(u) = 0$

so,

Discrete harmonic maps are (discretely) well-balanced

III. 3 Ingredients



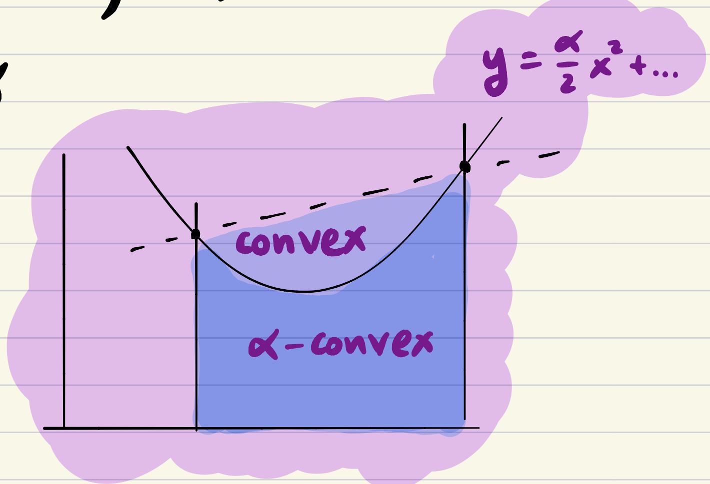
(1) The discrete energy E_G is STRONGLY convex

(2) cotan weights are BETTER than you think (to 3^{rd} order!) at most vertices.

(3) On $\mathcal{C}_G$, d_2 & d_∞ admit a STRONGER comparison (THINK: $(\mathbb{H}^2)^N$) than you expect, nearby the D.H.M.

((2) + (3) are for $\boxed{\tau_G(f) \approx \tau(f)}$)

(1) **Def.** $g : M \to \mathbb{R}$ is α-convex if $(g \circ \gamma)'' \geq \alpha$ $\forall$ unit-speed geodesics γ



Thm: If Y is NPC and $\deg(f) \neq 0$ then

E_G is α-convex where $\alpha \approx \frac{\omega}{A \cdot D}$,

$\omega = \min \omega_{xy}$

$A = \sum \mu_x$

$D = \operatorname{diam}(G)$

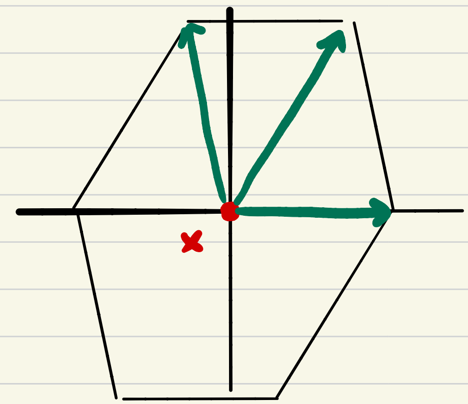
Q: smooth analogue?

Conjecture: Actually $\alpha \approx 1$, provided $A, \omega \approx 1$

(2) Lemma. If G is hexaparallel at x,

i.e. up to $GL_2\mathbb{R}$

then cotangent + area weights satisfy:

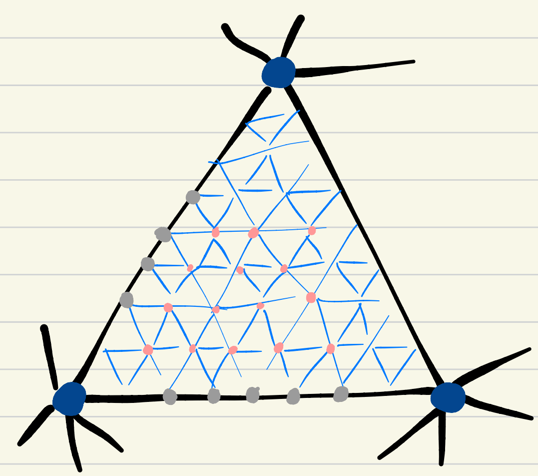


(i) $\displaystyle\sum_{y \sim x} \omega_{xy} L(\vec{xy}) = 0$ $\forall$ linear L

(ii) $\displaystyle\frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy}\, q(\vec{xy}) = 2 \operatorname{tr} q$ $\forall$ quadratic q

(iii) $\displaystyle\sum_{y \sim x} \omega_{xy}\, \sigma(\vec{xy}) = 0$ $\forall$ cubic σ

IDEA: If $\{G_n\}$ is obtained via midpoint refinement then most vertices are approximately hexaparallel.

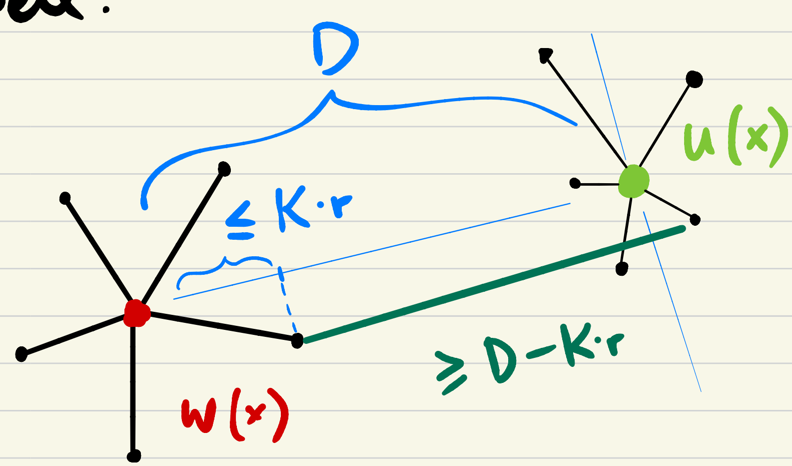


(3) "For free": $d_\infty(u,w) \lesssim \frac{1}{r} d_2(u,w)$

where $r \approx$ edge length of G

<u>Thm</u>: Let $K>0$. $\exists\, C=C(K)$ s.th.: If $w \in \mathcal{C}$ has $\mathrm{Lip}(w) \le K$, and u is DHM, $d_\infty(u,w) \le C \cdot \frac{1}{r} d_2(u,w) \cdot \left(\log \frac{1}{r}\right)^{-1/2}$

<u>"Proof"</u>: u is balanced, so going from w to u, if some pt. x moves a lot, then many pts nearby move a lot as well.



Q: smooth analogue?

THANKS!