What is a stochastic Hamiltonian process on finite graph? An optimal transport answer

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Partially Supported by NSF and ONR

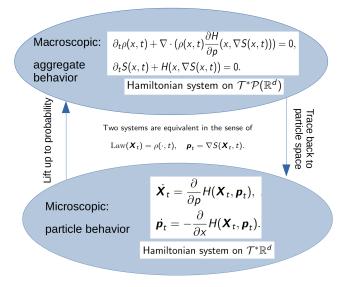
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Motivation

Can we define a stochastic process X_t on graph that behaves like a Hamiltonian system?

- 1. Curiosity.
- 2. The notion of *gradient flow on graph* has been investigated extensively, Maas'11, Mielke'11, Chow-Huang-Li-Zhou'12, and many more.
- Recent developments on discrete optimal transport (OT) (Gangbo-Li-Mou'19), Schrödinger equations (SE) (Chow-Li-Zhou'19) as well as Schrödinger Bridge Problem (SBP)(Leonard'14, Leonard'16) have demonstrated Hamiltonian principles on graph.

Motivation



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Stochastic Hamiltonian process on a finite graph

Definition

A stochastic process $\{X_t\}$ is called a Hamiltonian process on the graph G = (V, E) if

1. The density ρ of X_t satisfies the following generalized Master equation,

$$\frac{d\rho}{dt} = \rho Q(S, \rho, t),$$

with

$$Q_{ij}(S, \rho, t) = \mathbb{1}_{(i,j) \in E} f_{ji}(S_j - S_i, \rho, t), \ \ Q_{ii}(S, \rho, t) = -\sum_{j \in \mathcal{N}(i)} Q_{ij}(S, \rho, t).$$

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The density ρ and the potential S form a Hamiltonian system on the cotangent bundle T^{*}P(G) of the density space P(G).

A key concern is whether a Markov process X_t exists for such a master equation or not.

Stochastic Hamiltonian process on a finite graph

Theorem

Suppose that the stochastic process $\{X_t\}_{t\geq 0}$ with density $\{\rho_t\}_{t\geq 0}$ and potential $\{S_t\}_{t\geq 0}$ forms a Hamiltonian process on the graph G. In addition assume that F_{ij} is the antiderivative of f_{ij} . Then the Hamiltonian always possesses the form

$$\mathscr{H}(\rho, S) = \sum_{i \in V} \sum_{j \in N(i)} \rho_i F_{ji}(S_j - S_i, \rho, t) + \mathcal{V}(\rho, t),$$
(1)

where \mathcal{V} is a function depending ρ and t. Moreover, the Hamiltonian system on $\mathcal{T}^*\mathcal{P}(\mathsf{G})$ is

$$\begin{split} &\frac{\partial}{\partial t}\rho_i(t) = \sum_{j \in \mathcal{N}(i)} f_{ij}(S_i - S_j, \rho, t)\rho_j - f_{ji}(S_j - S_i, \rho, t)\rho_i, \\ &\frac{\partial}{\partial t}S_i(t) = -\sum_{j \in \mathcal{N}(i)} \left(F_{ji}(S_j - S_i, \rho, t) + \rho_j \frac{\partial}{\partial \rho_i}F_{ji}(S_j - S_i, \rho, t)\right) - \frac{\partial}{\partial \rho_i}\mathcal{V}(\rho, t). \end{split}$$

 X_t must be a time-inhomogeneous Markov process.

Example 1: OT on graph

The OT problem on graph G,

$$\min_{\rho,\nu} \left\{ \int_{0}^{1} \langle \mathbf{v}, \mathbf{v} \rangle_{\theta(\rho)} dt \right\},$$

$$\partial_{t}\rho + \operatorname{div}_{G}^{\theta}(\rho \mathbf{v}) = 0, \ \rho(\cdot, 0) = \rho_{a}, \ \rho(\cdot, 1) = \rho_{b},$$
(2)

where we define

$$\langle \mathbf{v}, \mathbf{v} \rangle_{\theta(\rho)} = rac{1}{2} \sum_{(j,l) \in E} heta_{jl}(\rho) \mathbf{v}_{jl}^2, \quad (\operatorname{div}_G^{\theta}(\rho \mathbf{v}))_j = - \sum_{l \in \mathcal{N}(j)} heta_{jl}(\rho) \mathbf{v}_{jl},$$

with the upwind choice

$$heta_{ij}^{U}(
ho) = egin{cases}
ho_{j} & S_{j} < S_{i} \
ho_{i} & S_{i} < S_{j} \end{cases}$$

The Hamiltonian system is

$$\frac{d\rho_i}{dt} + \sum_{j \in \mathcal{N}(i)} \theta_{ij}(\rho)(S_j - S_i) = 0, \quad \frac{dS_i}{dt} + \frac{1}{2} \sum_{j \in \mathcal{N}(i)} \frac{\partial \theta_{ij}(\rho)}{\partial \rho_i} (S_i - S_j)^2 = 0.$$
(3)

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Example 2: SBP on graph

The Schrödinger Bridge Problem on G can be expressed as

$$\min_{\rho,\nu} \left\{ \int_{0}^{1} (\langle v, v \rangle_{\theta} v_{(\rho)} + \frac{1}{8} \mathcal{I}_{G}(\rho)) dt \right\},$$

$$\partial \rho + \operatorname{div}_{G}^{\theta^{U}}(\rho v) = 0, \ \rho(\cdot, 0) = \rho_{a}, \ \rho(\cdot, 1) = \rho_{b},$$
(4)

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where the discrete Fisher Information is

$$\mathcal{I}_{G}(\rho) = rac{1}{2} \sum_{(i,j)\in E} (\log(
ho_{i}) - \log(
ho_{j}))^{2} \widetilde{ heta}_{ij}(
ho).$$

Here $\tilde{\theta}$ is some weight function, not necessarily equal to θ^U before.

Reference

Jianbo Cui, Shu Liu, and Haomin Zhou, *What is a stochastic Hamiltonian process on finite graph? An optimal transport answer,* Journal of Differential Equation, Vol. 305 (2021).

Thank you!

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