Counterexamples to Conjectures of Thomassé and Bonato-Tardif, what is next?

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Equimorphism

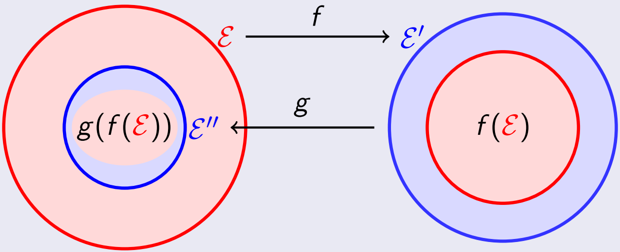
Embedding

An injective map preserving the structure.

Sibling

Two structures $\mathcal{E}$ and $\mathcal{E}'$ are called *siblings* (or equimorphic), denoted by $\mathcal{E} \approx \mathcal{E}'$, when there are mutual embeddings between them.

$\mathcal{E} \approx \mathcal{E}' \cong \mathcal{E}''$, $g(\mathcal{E}') = \mathcal{E}'' \supseteq (g \circ f)(\mathcal{E}) \cong \mathcal{E}$.



Siblings in Some Categories

Cantor-Schröder-Bernstein Theorem (Sets)

If there exist injective maps $f : A \to B$ and $g : B \to A$ between two sets A and B, then there exists a bijection (isomorphism) $h : A \to B$.

Vector Spaces

If there are mutual injective linear transformations between two vector spaces over a fixed field, then they are isomorphic.

Rational Numbers

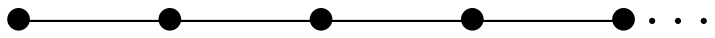
$\mathbb{Q}$ as a chain: there are mutual injective order preserving maps between $\mathbb{Q}$ and $\mathbb{Q} + \infty$, nonetheless, $\mathbb{Q} \ncong \mathbb{Q} + \infty$.

Thomassé's Conjecture

Sibling Number

The number of isomorphism classes of siblings of a relation $\mathcal{E}$, denoted by $Sib(\mathcal{E})$.

If $\mathcal{E}$ is a ray, $Sib(\mathcal{E}) = 1$ in the category of trees,



and $Sib(\mathcal{E}) = \aleph_0$ as a binary relation.

Thomassé's Conjecture (2000)

For a countable relation $\mathcal{E}$, $Sib(\mathcal{E}) = 1$, $\aleph_0$ or $2^{\aleph_0}$.

The Alternate Thomassé Conjecture

For a relation $\mathcal{E}$ of any cardinality, $Sib(\mathcal{E}) = 1$ or ∞.

The Bonato-Tardif Conjecture

Conjectures about Trees

- The Bonato-Tardif Conjecture (BT) (2006): If T is a tree, then $Sib(T) = 1$ or ∞ in the category of trees.
- Tyomkyn's Conjecture (2009): If a locally finite tree T has a non-surjective embedding, then $Sib(T) = \infty$, unless T is a ray.

Note [Pouzet]

If for a tree T we have $T \oplus 1 \not\approx T$, then the conjectures of BT and Thomassé are equivalent.

Results for the conjectures of Bonato-Tardif and Tyomkyn

The BT conjecture holds for

- rayless trees [Bonato, Tardif] (2006)
- rooted trees [Tyomkyn] (2009)
- scattered trees and stable trees [Laflamme, Pouzet, Sauer] (2017)

Tyomkyn's conjecture holds for

- locally finite scattered trees [Laflamme, Pouzet, Sauer] (2017)

Thomassé and the Alternate Thomassé Conjecture

Positive Results

Thomassé's conjecture holds for

- countable chains [Laflamme, Pouzet, Woodrow] (2017) and countable direct sums of chains [Abdi] (arXiv, 2022^{+})

The Alternate Thomassé conjecture holds for

- rayless graphs [Bonato, Bruhn, Diestel, Sprüssel] (2011)
- chains [Laflamme, Pouzet, Woodrow] (2017)
- countable $\aleph_0$-categorical relational structures [Laflamme, Pouzet, Sauer, Woodrow] (2021)
- countable universal theories [Braunfeld, Laskowski] (arXiv, 2022^{+})
- countable cographs [Hahn, Pouzet, Woodrow] (arXiv, 2022^{+})
- direct sums of chains [Abdi] (arXiv, 2022^{+})
- countable N-free posets [Abdi] (arXiv, 2022^{+})

Counterexample to the Conjectures of Bonato-Tardif, Tyomkyn, Thomassé

A Claim

Tateno (2008) in his thesis claimed that there were locally finite trees with an arbitrary finite number of siblings.

The ideas were never published and after receiving Tateno's manuscript, we revisited the ideas and could give a rigorous exposition.

[Abdi, Laflamme, Tateno, Woodrow] (arXiv, 2022^+)

There are locally finite trees having an arbitrary finite number of siblings disproving all conjectures of Bonato-Tardif, Tyomkyn and Thomassé.

https://arxiv.org/abs/2205.14679

The Counterexamples

Properties of the Tree Examples

For each $n > 0$ there is a locally finite tree T with exactly n siblings such that

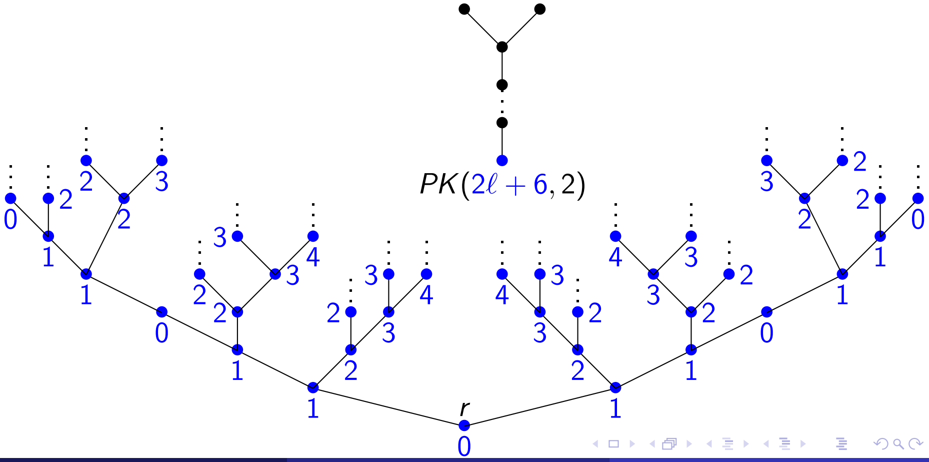
- $T \oplus 1 \not\leftrightarrow T$;
- there is a self-embedding ϕ of T such that $T \setminus \phi(T)$ is non-empty and finite.

Partial Order Counterexamples

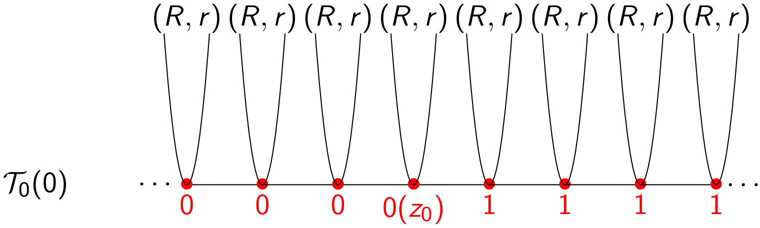
The tree examples can be adapted to construct partial orders with an arbitrary finite number of siblings.

A Labelled Tree

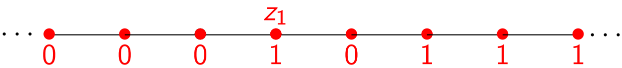
We construct $\mathcal{R} = (R, r)$ as follows. If $l(v) = 0$, then v has exactly two neighbours. If $l(v) \neq 0$, then v has exactly three neighbours of $l(v) - 1$, $l(v)$ and $l(v) + 1$.

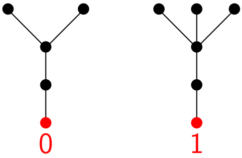


$\mathcal{T}_0(0)$ and $\mathcal{T}_1(0)$



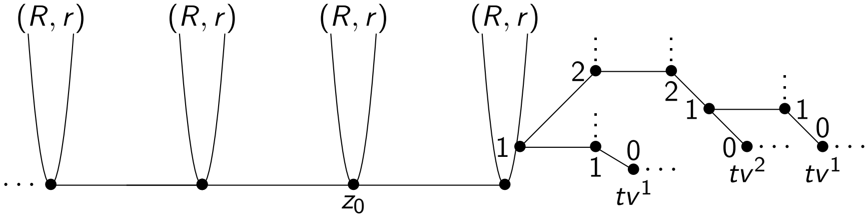
$\mathcal{T}_1(0)$ is similarly constructed on the following double ray





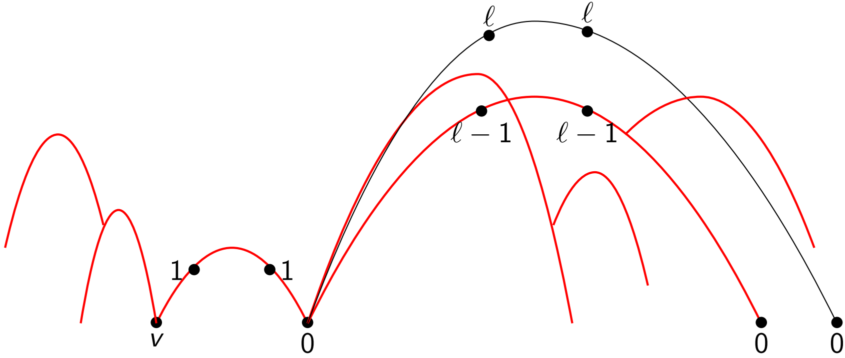
Target Vertex

target vertex tv^{ℓ} of height ℓ : a tree vertex $v \in \mathcal{T}_0(k)$ such that the label of the last consecutive pair $w, w' \in P_{z_0,v}$ is ℓ.



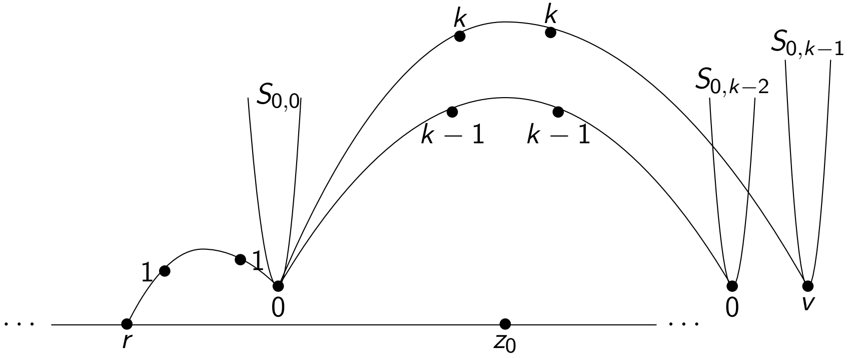
Crater

Let $v \in (R', r')$ be a target vertex of height ℓ.
$\mathcal{C}(v) = \{u \in \mathcal{T}_0(k) : ht_v(u) < \ell\}$.



$\mathcal{T}_0(k)$

Pick a target vertex $v \in \mathcal{T}_0(k-1)$ of height k



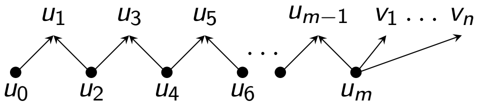
where $S_{0,0} \approx \mathcal{T}_0(0)$, $S_{0,k-2} \approx \mathcal{T}_0(k-2)$, $S_{0,k-1} \approx \mathcal{T}_0(k-1)$ and so on. Finally, let $\mathcal{T}_0 = \bigcup_k \mathcal{T}_0(k)$.

Main Property of $\mathcal{T}_0$ ($\mathcal{T}_1$)

Any non-isomorphic sibling of $\mathcal{T}_0$ is almost equal to $\mathcal{T}_0$, so has been embedded at some stage, the partial isomorphism extends to the whole structure. This implies that any sibling of $\mathcal{T}_0$ is isomorphic to either $\mathcal{T}_0$ or to $\mathcal{T}_1$

The following posets can be added to $\mathcal{T}_i$. Double rays in $\mathcal{T}_i$ can also be adapted by making them as infinite fences. Then, the siblings of each $\mathcal{T}_i$ are the same as trees (and relational structures) or as posets.

$PK(m, n)$



Conclusion

All conjectures of Bonato-Tardif, Tyomkyn and Thomassé are false.

Sibling Alternative Property

The concept of sibling under embeddability can be generalised to any relation.

Relational Sibling

Given a relation $\mathcal{R}$ (i.e. graph minor and topological minor in graphs, etc), two structures S and T are *siblings* if both $S\mathcal{R}T$ and $T\mathcal{R}S$ hold.

Sibling Alternative Property (SAP)

Given a relation $\mathcal{R}$, a collection $\mathcal{S}$ of structures has SAP if the sibling number of each element of $\mathcal{S}$ under $\mathcal{R}$ is 1 or infinite.

Quasi-Order and Well-Quasi-Order

Quasi-Order

A *quasi-order* is a reflexive and transitive binary relation.

Well-Quasi-Order

A quasi-order Q is a *well-quasi-order (WQO)* if any infinite sequence of elements of Q contains an infinite increasing subsequence.

Some Facts

- The class $\mathcal{C}$ of countable chains is WQO under embeddability [Laver] and SAP holds for $\mathcal{C}$ [Laflamme, Pouzet, Woodrow].
- The class $\mathcal{G}$ of countable cographs is WQO under embeddability [Thomassé] and SAP holds for $\mathcal{G}$ [Hahn, Pouzet, Woodrow].

Embedding is not WQO in general and SAP is false under this relation.

Research Direction

One More Fact

The class of trees is WQO under topological minor [Nash-Williams] and the class of locally finite trees has SAP under topological minor [J. Bruno, P. Szeptycki].

Question

Given a relation $\mathcal{R}$ on a collection of structures $\mathcal{S}$ so that $(\mathcal{S}, \mathcal{R})$ is WQO, does SAP hold?

Question

On the sibling end: can we characterise exactly the role of WQOs on structural sibling theorems? What are the connections? What are the boundaries?

Thank You for Your Attention