Machine Learning in Banach Spaces: A Black-box or White-box Method?

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Outline

Plato,
428/424-348 BC

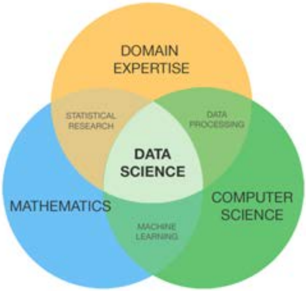
Platonism is a form of realism. Mathematics is abstract, has no space-time or causal properties, and is immutable.

--- *Platonism in the Philosophy of Mathematics*

David Hilbert,
1862-1943

Formalism states that the theory of mathematical logic is the rules for certain string operations.

--- *Formalism in the Philosophy of Mathematics*



Source: Palmer, Shelly. Data Science for the C-Suite. New York: Digital Living Press, 2015. Print.

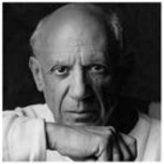
We should dramatically increase investment for research and development in privacy-enhancing technologies, encouraging cross-cutting research that involves not only computer science and mathematics, but also social science, communications, and legal disciplines.

--- *Big Data:Seizing Opportunities, Preserving Values* (Obama White House Archives)

Steve Jobs,
1955-2011

Innovation is to make all sorts of things together.

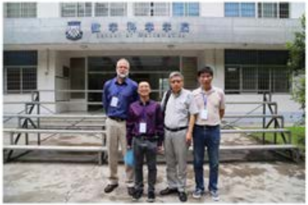
--- *Steve Jobs*



Pablo Picasso,
1881-1973

Good artists copy, great artists steal.

--- *Pablo Picasso*



Master Advisor : Liren Huang

Doctoral Advisor : Gregory E. Fasshauer

Postdoctoral Collaborator : Yuesheng Xu

Optimization

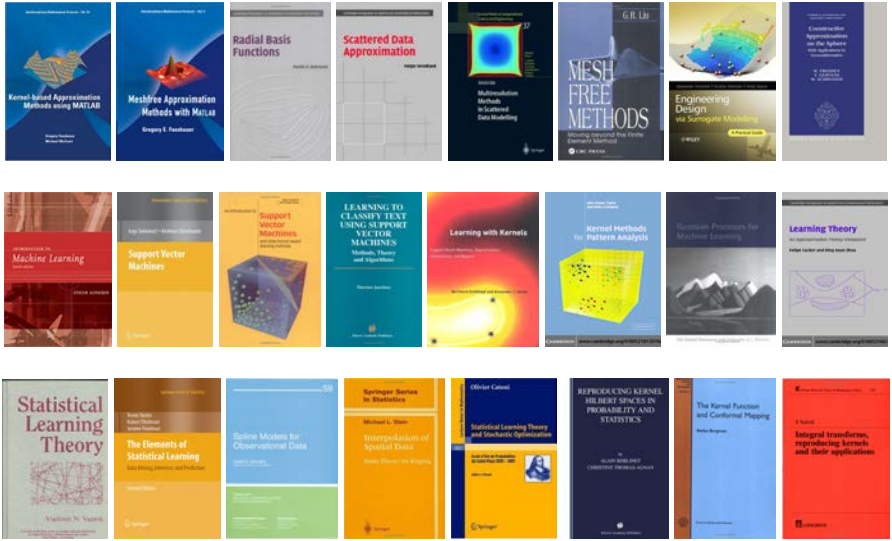
Kung Fu Ng
PhD Advisor of Huang

Approximation Theory

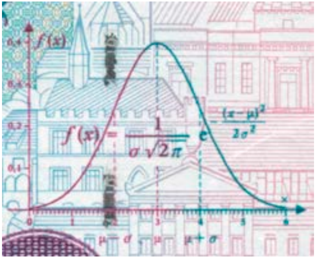
Larry L. Schumaker
PhD Advisor of Fasshauer

Inverse Problem

Charles A. Micchelli
Postdoctoral Collaborator of Xu


Kernel-based Approximation Methods using MATLAB
Meshfree Approximation Methods with MATLAB
Gregory E. Fasshauer
Radial Basis Functions
Scattered Data Approximation
Multiresolution Methods in Scattered Data Modelling
G.R. Liu
MESH FREE METHODS
Moving beyond the Finite Element Method
Engineering Design via Surrogate Modelling
Machine Learning
Support Vector Machines
Support Vector Machines
LEARNING TO CLASSIFY TEXT USING SUPPORT VECTOR MACHINES
Methods, Theory and Algorithms
Learning with Kernels
Kernel Methods for Pattern Analysis
Learning Theory
Statistical Learning Theory
Vladimir N. Vapnik
The Elements of Statistical Learning
Spline Models for Observational Data
Springer Series in Statistics
Michael L. Stein
Interpolation of Spatial Data
Olivier Catoni
Statistical Learning Theory and Stochastic Optimization
REPRODUCING KERNEL HILBERT SPACES IN PROBABILITY AND STATISTICS
The Kernel Function and Conformal Mapping
Integral transforms, reproducing kernels and their applications

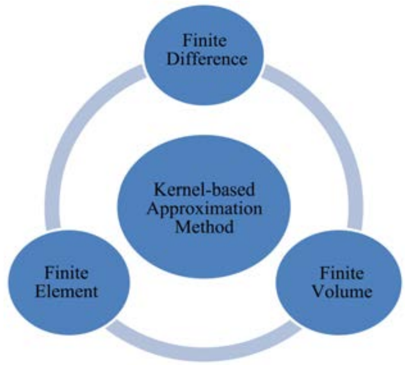
Carl Friedrich Gauß, 1777-1855



Gaussian Kernels

Gauss mentioned Gaussian kernels that now so often carry his name in 1809 in his second book [12].

By the techniques of numerical analysis, stochastic analysis, and regression analysis, we use the kernel-based approximation methods to combine the algorithms of finite difference, finite elment and finite volume in a system.

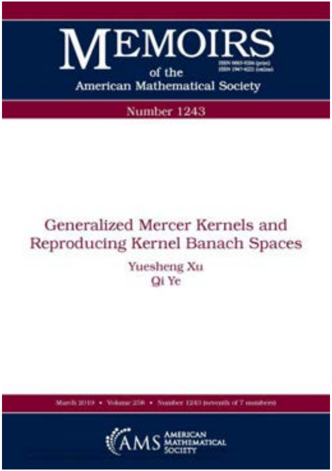


The data-driven methods are used to analyze black-box models, and the model-driven methods are used to analyze white-box models.

Remark. With the same thought of the Tai Chi diagram, we use the discrete local information of the black-box and white-box models to construct the global approximate solutions by the algorithms of regularized learning.

Outline

- The details of reproducing kernel Banach spaces (RKBS) are mentioned in my paper [29] for 122 pages.
- It studies the constructions and theorems of RKBS and verifies the learning theory in RKBS, specially, the sparse learning methods in 1-norm RKBS.



- L. Huang, C. Liu, L. Tan, and Q. Ye, Generalized representer theorems in Banach spaces, Analysis and Applications. **19** (2021), 125–146 [14].
- Qi Ye. Analysis of regularized learning for generalized data in Banach spaces. arXiv:2109. 03159 (2021), 1-36 [30].

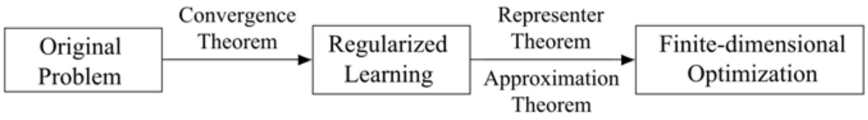
The work of regularized learning in Banach spaces provides another road to study the algorithms of machine learning including:

- the interpretability in approximation theory,
- the nonconvexity and nonsmoothness in optimization theory,
- the generalization and overfitting in regularization theory.

The original problem is to solve an exact solution from

$$\inf_{f \in \mathcal{B}} R(f),$$

where $\mathcal{B}$ is a Banach space and $R : \mathcal{B} \to [0, \infty)$ is an expected risk function.



Key to Success of Regularized Learning

Remark. We can solve the finite dimensional optimization problems by

- alternating direction multiplier methods (splitting methods),
- composite optimization methods.

- We have generalized data

$$(\xi_1^*, y_1), (\xi_2^*, y_2), \ldots, (\xi_{N_n}^*, y_{N_n}) \in \mathcal{B}_* \times Y, \quad \text{for } n \in \mathbb{N},$$

which can be written as

$$(\boldsymbol{\xi}_n^*, \boldsymbol{y}_n) \in \mathcal{B}_*^{N_n} \times Y^{N_n}, \quad \text{for } n \in \mathbb{N},$$

where $\mathcal{B}_*$ is a predual space of $\mathcal{B}$ and Y is an output range.

- For each nth approximate step, we have a multi-loss function as

$$\mathbb{L}_n : \mathcal{B}_*^{N_n} \times Y^{N_n} \times \mathbb{R}^{N_n} \to [0, \infty),$$

for example,

$$\mathbb{L}_n(\boldsymbol{\xi}, \boldsymbol{y}, \boldsymbol{t}) = \frac{1}{N_n} \sum_{k=1}^{N_n} L(\xi_k^*, y_k, t_k),$$

where L is a classical loss function.

- The regularized learning is to solve an approximate solution from

$$\inf_{f\in\mathcal{B}} R_n(f)+\lambda\Phi(\|f\|),$$

where R_n is an empirical risk function, that is,

$$R_n(f):=\mathbb{L}_n(\boldsymbol{\xi}_n^*,\boldsymbol{y}_n,\langle f,\boldsymbol{\xi}_n^*\rangle),\quad \text{for } f\in\mathcal{B},$$

and $\lambda>0$, $\Phi:[0,\infty)\to[0,\infty)$ is an increasing function.

- The approximate solution can be equivalently or approximately solved by a finite-dimensional optimization, that is,

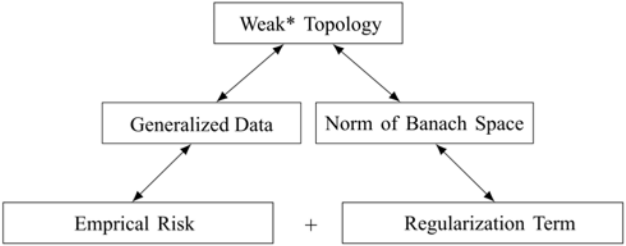
$$\inf_{f\in\mathcal{U}_m} R_n(f)+\lambda\Phi(\|f\|),$$

where $\mathcal{U}_m$ is a subset of $\mathcal{B}$ such that there exists an surjection Γ_m from Ω_m onto $\mathcal{U}_m$, where $\Omega_m\subset\mathbb{R}^m$ and $m\in\mathbb{N}$.

The representer theorems, approximation theorems, and convergence theorems of regularized learning are guaranteed by the assumptions:

1. The R_n converges pointwise to R when $n \to \infty$.
2. The collection of all generalized input data is relatively compact in $\mathcal{B}_*$ and the collection of all multi-loss functions is uniformly local Lipschitz continuous.

$\Rightarrow$ $\{R_n : n \in \mathbb{N}\}$ is weakly* equicontinuous on any closed ball of origin.



Regularized Learning for Generalized Data

- Representer Theorems in [30]
  - Theorem 4.2 shows that the approximate solutions are equivalently solved by the finite-dimensional optimization.
- Approximation Theorems in [30]
  - Theorem 4.5 shows that the approximate solutions are approximately solved by the finite-dimensional optimization.
- Convergence Theorems in [30]
  - Theorem 4.6 shows the existence of a weakly* convergent subnet of approximate solutions to an exact solution from the orignial problem.
  - Theorem 4.7 shows that any element, to which a subnet of approximate solutions weakly* converges, is an exact solution.
  - Theorem 4.8 shows that the subsequence of approximate solutions for special regularization parameters is weakly* convergent.

- Suppose that an exact solution belonging to $\mathcal{B}^b \cap \mathcal{B}^w$ is solved by the both models, that is,

Black-box Model

$$\inf_{f\in\mathcal{B}^b} R^b(f),$$
$$\inf_{f\in\mathcal{B}^b} R_n^b(f) + \lambda \|f\|_b .$$

White-box Model

$$\inf_{f\in\mathcal{B}^w} R^w(f),$$
$$\inf_{f\in\mathcal{B}^w} R_n^w(f) + \lambda \|f\|_w .$$

- Let $\mathcal{B} := \mathcal{B}^b \cap \mathcal{B}^w$ with $\|f\| := \|f\|_b + \|f\|_w$. Thus, the composite model can be written as

$$\inf_{f\in\mathcal{B}} R(f), \quad \inf_{f\in\mathcal{B}} R_n(f) + \lambda \|f\| ,$$
where $R := R^b + R^w,\ R_n := R_n^b + R_n^w .$

- Let $T^b_{n,\lambda} := R^b_n + \lambda\|\cdot\|_b$, $T^w_{n,\lambda} := R^w_n + \lambda\|\cdot\|_w$, "prox" represent the proximal operator, and $\theta > 0$ be a stepsize. Thus, two iterative algorithms can be used to solve the two models above, respectively,

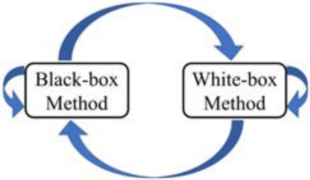
Black-box Method

$$g_{k+1} \in \mathrm{prox}_{\theta T^b_{n,\lambda}}(g_k).$$

White-box Method

$$g_{k+1} \in \mathrm{prox}_{\theta T^w_{n,\lambda}}(g_k).$$

- By the Douglas-Rochford splitting methods, the composite algorithms can be written as

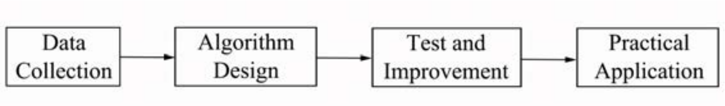


$$\begin{aligned} h^b_k &\in \mathrm{prox}_{\theta T^b_{n,\lambda}}(g_k), \\ h^w_k &\in \mathrm{prox}_{\theta T^w_{n,\lambda}}(2h^b_k - g_k), \\ g_{k+1} &:= g_k + \sigma(h^w_k - h^b_k). \end{aligned}$$

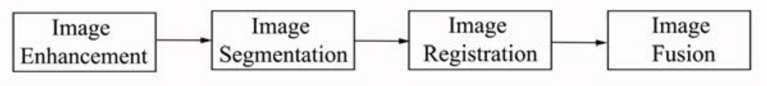
Outline

By the theory of regulizaed learning, we analyze the big data in education and medicine.

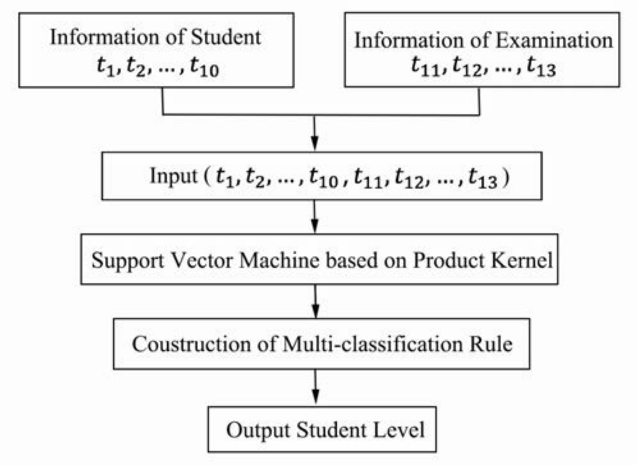
- The big data analysis of mathematics education of middle school represents:



- The big data analysis of digital images of pancreatic cancer represents:



We use the black-box methods of support vector machines to analyze the data of mathematics education for hierarchical teaching.



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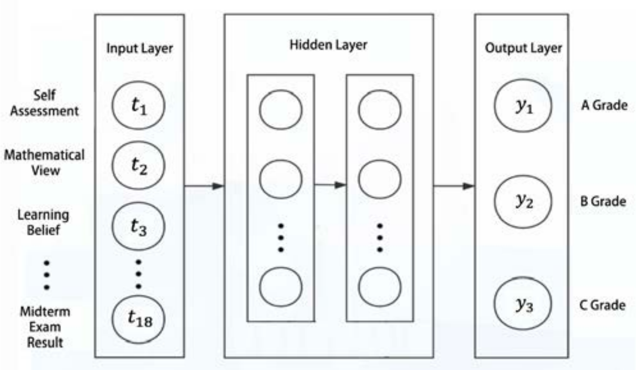
We use the black-box methods of artificial neural networks to analyze the data of mathematics education for hierarchical teaching.



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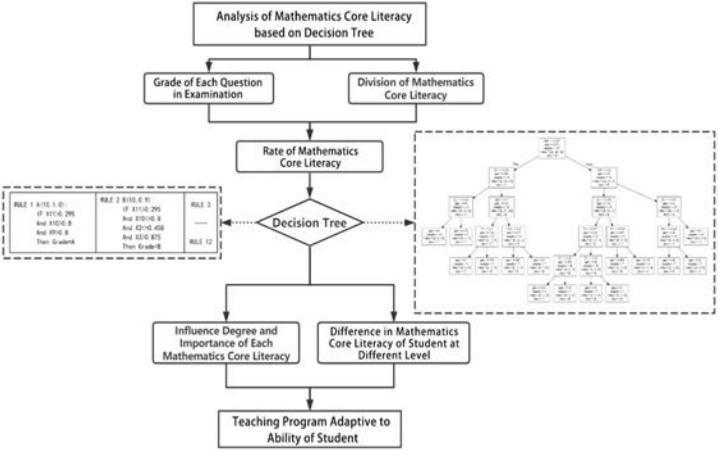
We use the white-box methods of decision trees to analyze the data of mathematics education for hierarchical teaching.



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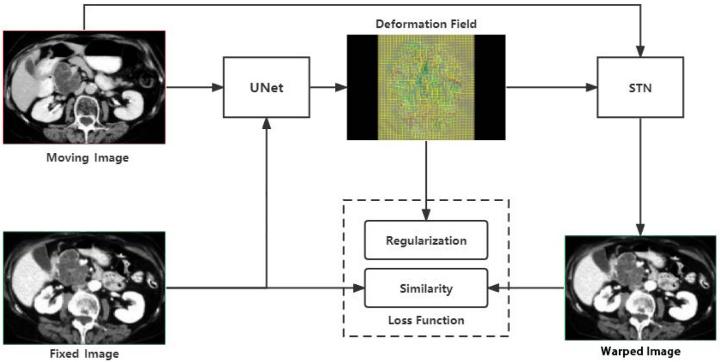
We use the model-driven methods to solve the image registration of pancreatic cancer.

- Diffeomorphic Demons Model (Tom Vercauteren, 2009 [26])

$$L(F, M, s, u) = \|F - M \circ s\|^2 + \lambda\|s - u\| + \mu\|\nabla u\|^2.$$

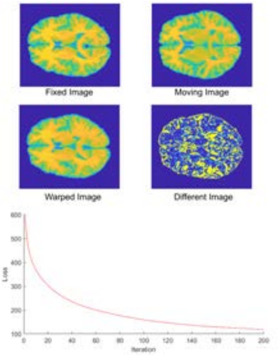
Remark. The F is a fixed image, M is a moving image, s is the approximate transformation, u is the real transformation, λ is the parameter of the spatial uncertainties term, and μ is the parameter of the regularization term.

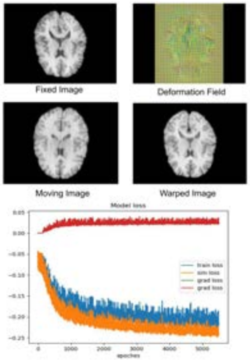
We use the data-driven methods to solve the image registration of pancreatic cancer.



Deep Learning based on Registration: VoxelMorph

Based on the LPBA40 data set, we use the model-driven and data-driven methods to register the brain CT images as follows:



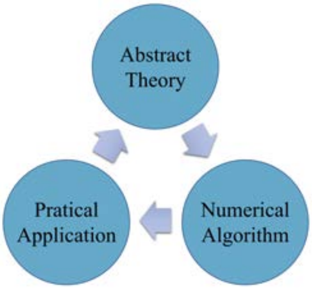


Numerical Results of Diffeomorphic Demons Model and VoxelMorph Model

Remark. The computational time, accuracy, and process of the model-driven method are longer, higher, and smoother than the data-driven method.

By the theory of regularized learning, we will combine the black-box and white-box methods to construct the composite algorithms to solve the problems of the big data analysis in education and medicine. Our original ideas are inspired by the eastern philosophy such as golden mean and Tai Chi diagram.

Plato and Aristotle



I hope someday you'll join us. And the world will live as one.

--- John Winston Lennon

John Winston Lennon,
1940-1980

Thank You!

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华南师范大学
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http://mlopt.scnu.edu.cn/en/

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