Colored sl(N) homology, SU(N) representations, and the Hopf link

Joshua Wang

March 8, 2022

Outline

1. Khovanov homology and $\mathrm{SU}(2)$ representations.

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3. Colored $\mathfrak{sl}(N)$ homology and $\mathrm{SU}(N)$ representations.

Khovanov homology and SU(2) representations

Given a link $L \subset S^3$, consider the space

$$\mathscr{R}_2(L) = \left\{ \rho \colon \pi_1(S^3 \setminus L) \to \mathsf{SU}(2) \,\middle|\, \rho(\text{meridian}) \text{ is traceless} \right\}$$

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Observation (Kronheimer–Mrowka '08, Jacobsson–Rubinsztein '08)

If L is a $(2, n)$-torus knot or link, then $\mathsf{Kh}(L) \cong H^*(\mathscr{R}_2(L))$ as abelian groups.

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- $\mathscr{R}_2(\text{(2,4)-torus link}) = S^2 \sqcup S^2 \sqcup \mathrm{SO}(3)$ — $\mathrm{Kh} = \mathbf{Z}^6 \oplus \mathbf{Z}/2$
- $\mathscr{R}_2(\text{cinquefoil}) = S^2 \sqcup \mathrm{SO}(3) \sqcup \mathrm{SO}(3)$ — $\mathrm{Kh} = \mathbf{Z}^6 \oplus (\mathbf{Z}/2)^2$

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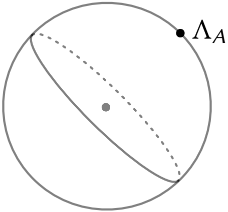

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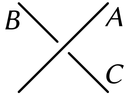

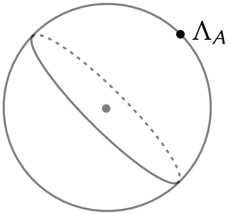

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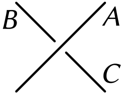

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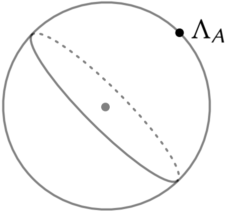

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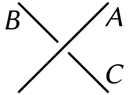

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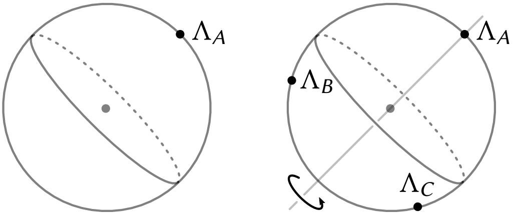

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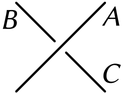

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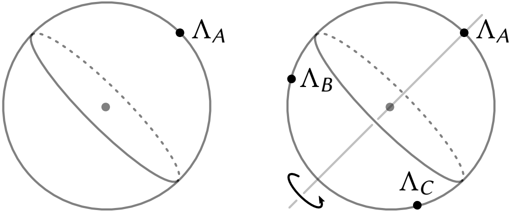
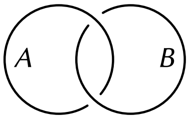
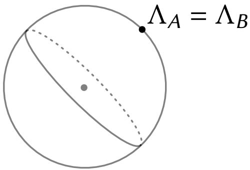
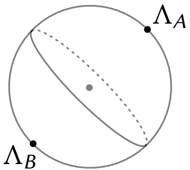


$\mathscr{R}_2(L)$ = set of all such configurations of points on S^2.

Khovanov homology and SU(2) representations







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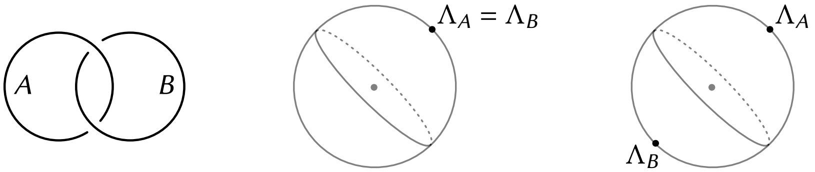

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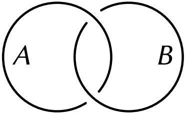

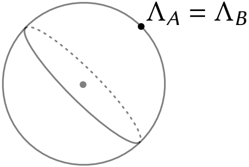

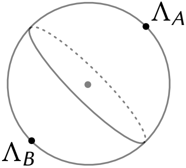

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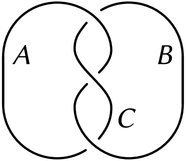

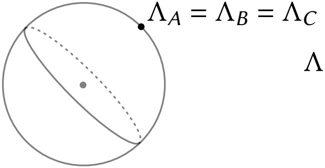

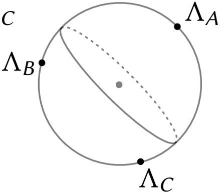

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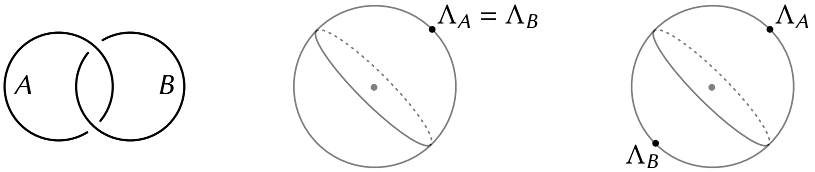

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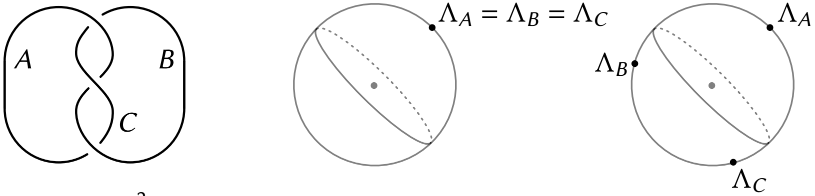

$$\mathscr{R}_2(\text{trefoil}) = S^2 \sqcup \mathrm{SO}(3)$$
$$= \{\Lambda_A = \Lambda_B = \Lambda_C\} \sqcup \{\Lambda_A, \Lambda_B, \Lambda_C \text{ equidistant on a great circle}\}$$

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Given a particular arc A in our diagram, we obtain a fiber bundle

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Observation

If L is a $(2,n)$-torus knot or link, then $\mathrm{Kh}(L) \cong H^*(\mathscr{R}_2(L))$ as modules over $\mathbf{Z}[X]/X^2$ and $\overline{\mathrm{Kh}}(L) \cong H^*(\overline{\mathscr{R}_2}(L))$.

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Also true when L is a rational link (Lewallen '09 + Shumakovitch '10), but there are alternating 3-bridge counterexamples (Zentner '11).

Khovanov homology and SU(2) representations

Kronheimer–Mrowka '11 define an SU(2) instanton homology for links $\mathrm{I}^\sharp(L)$, together with a spectral sequence $\mathrm{Kh}(L) \Longrightarrow \mathrm{I}^\sharp(L)$.

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$\mathsf{I}^\sharp$ is defined by a version of *Morse* homology for the Chern-Simons functional CS. The space $\mathscr{R}_2(L)$ is the set of critical points of CS.

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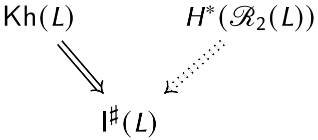

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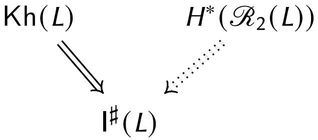

$\mathrm{I}^\sharp$ is defined by a version of Morse homology for the Chern-Simons functional CS. The space $\mathscr{R}_2(L)$ is the set of critical points of CS. If CS is Morse–Bott, there is a spectral sequence $H^*(\mathscr{R}_2(L)) \Longrightarrow \mathrm{I}^\sharp(L)$.

For rational links, both spectral sequences immediately degenerate.

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The $\mathfrak{sl}(N)$ link polynomial $P_N(L) \in \mathbf{Z}[q, q^{-1}]$ is defined by the skein relation

$q^N P_N\Big($  $\Big) - q^{-N} P_N\Big($  $\Big) = (q + q^{-1}) P_N\Big($  $\Big)$

and $P_N(\text{unknot}) = q^{N-1} + q^{N-3} + \cdots + q^{-(N-1)}$. P_2 is the Jones polynomial.

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Khovanov–Rozansky homology ($\mathfrak{sl}(N)$ link homology), denoted $\mathsf{KR}_N(L)$, is a bigraded homological invariant categorifying P_N. Note: $\mathsf{KR}_2(L) \cong \mathsf{Kh}(L)$.

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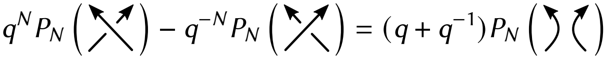

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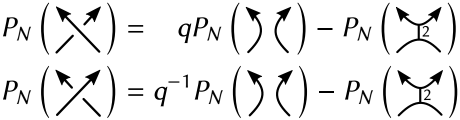
P_N extends to certain trivalent graphs in the plane (MOY graphs) so that



(+ a global shift)

$\mathfrak{sl}(N)$ link homology

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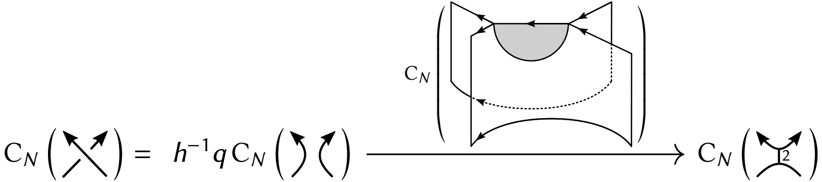

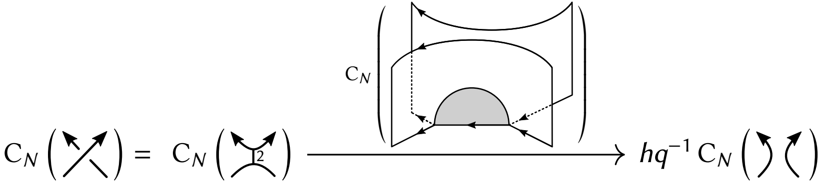

$\mathfrak{sl}(N)$ link homology and $\mathsf{SU}(N)$ representations

Associate to L the space

$$\mathscr{R}_N(L) = \left\{ \rho \colon \pi_1(S^3 \setminus L) \to \mathsf{SU}(N) \,\middle|\, \rho(\text{meridian}) \in C_1 \right\}$$

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Given a diagram of L, we can think of a point in $\mathscr{R}_N(L)$ as a choice of $\Lambda_A \in \mathbf{CP}^{N-1}$ for each arc A, subject to a constraint for each crossing.

$\mathfrak{sl}(N)$ link homology and $\mathrm{SU}(N)$ representations

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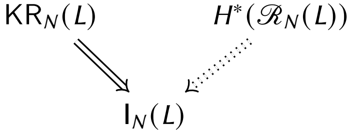

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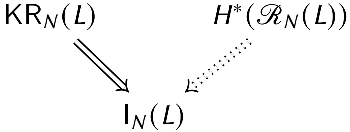

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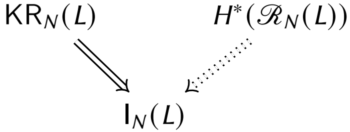

The rank of the $SU(N)$ instanton homology of Kronheimer–Mrowka '11 turns out to be invariant under crossing change. Maybe related to a Lee-type deformation of $\mathrm{KR}_N(L)$?

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If L is a $(2, n)$ torus knot or link, then $\mathrm{KR}_N(L) \cong H^*(\mathscr{R}_N(L))$ as abelian groups.

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If L is a $(2, n)$ torus knot or link, then $\mathrm{KR}_N(L) \cong H^*(\mathscr{R}_N(L))$ as abelian groups. In fact, they are isomorphic as $\mathbf{Z}[X]/X^N$-modules, and $\overline{\mathrm{KR}}_N(L) \cong H^*(\overline{\mathscr{R}_N}(L))$.

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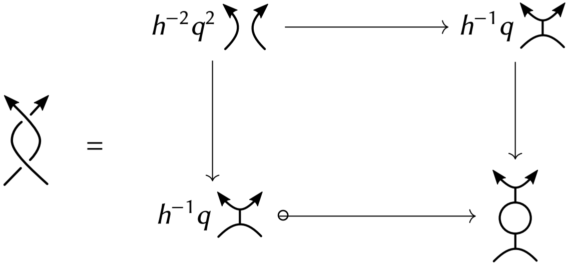
Simplification of the full twist complex (e.g. Krasner '09)



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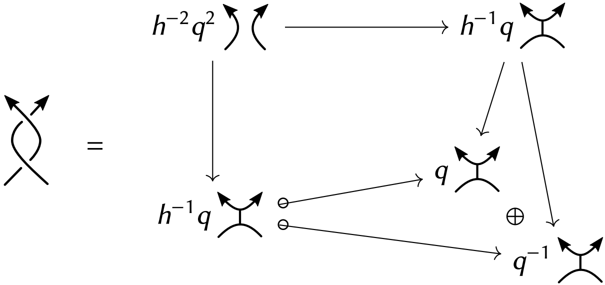

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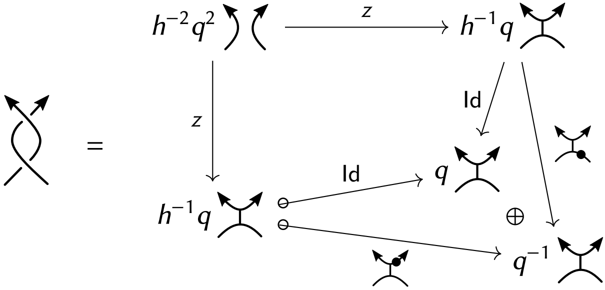

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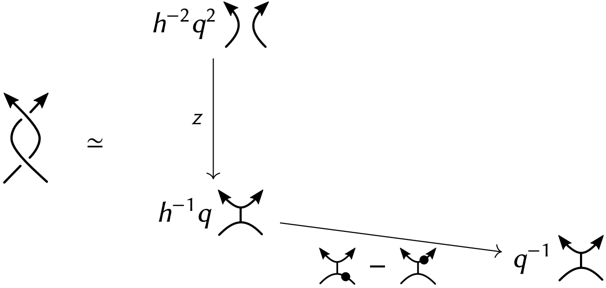

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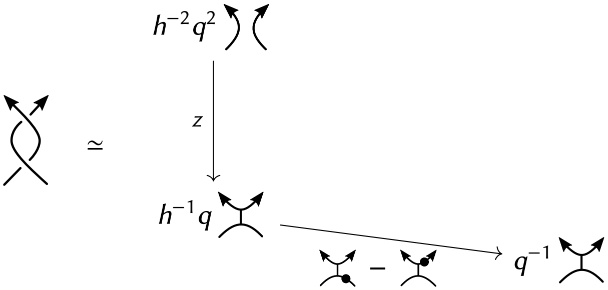

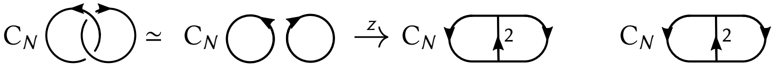

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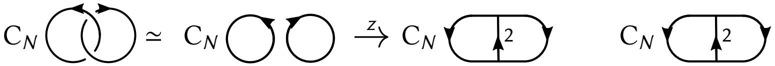
Khovanov–Rozansky complex of the Hopf link:



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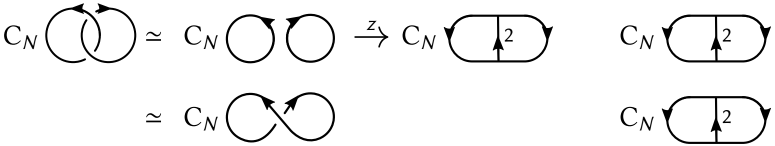

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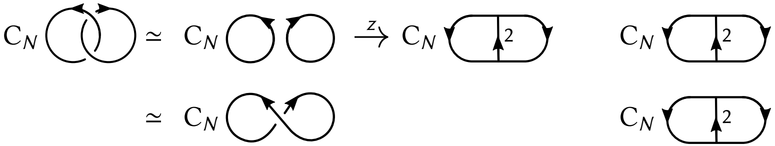

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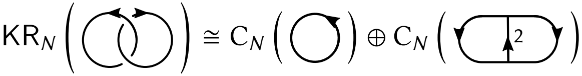

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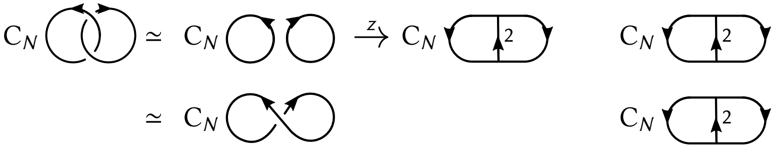

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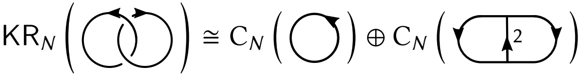

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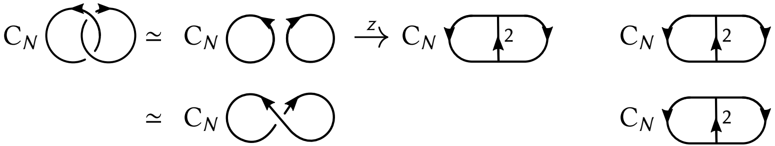

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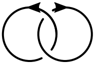
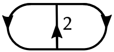
KR_N () $\cong \mathrm{C}_N$ () $\oplus\, \mathrm{C}_N$ ()

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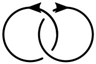
$\cong H^*\mathscr{R}_N$ ()

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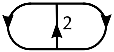

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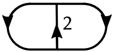

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There are two special line bundles over $\mathbf{F}(1,2,N)$

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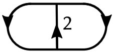

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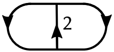
Their first Chern classes $c_1(\mathcal{S}_A), c_1(\mathcal{S}_B)$ form a basis for $H^2(\mathbf{F}(1,2,N))$.

$\mathfrak{sl}(N)$ link homology and $\mathrm{SU}(N)$ representations

There is an explicit isomorphism (Khovanov '04, Khovanov–Rozansky '08)

$$H^*(\mathbf{F}(1,2,N)) \cong \mathrm{C}_N\left(\right).$$

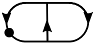
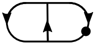


There are two special line bundles over $\mathbf{F}(1,2,N)$

$$\mathcal{S}_A \searrow \quad \swarrow \mathcal{S}_B$$

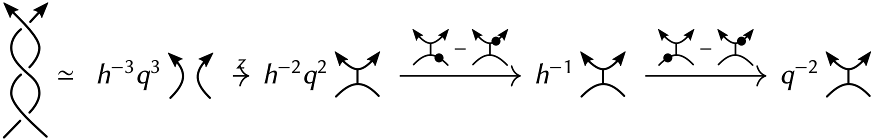
$$\{\text{orthogonal lines } \Lambda_A, \Lambda_B \text{ in } \mathbf{C}^N\}$$

Their first Chern classes $c_1(\mathcal{S}_A), c_1(\mathcal{S}_B)$ form a basis for $H^2(\mathbf{F}(1,2,N))$. The isomorphism intertwines the maps

cup with $c_1(\mathcal{S}_A) \leftrightarrow$  cup with $c_1(\mathcal{S}_B) \leftrightarrow$ 

$\mathfrak{sl}(N)$ link homology and $SU(N)$ representations

Simplified three twist complex:



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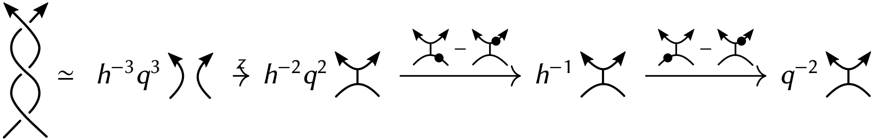

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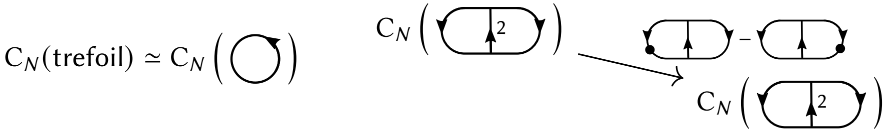

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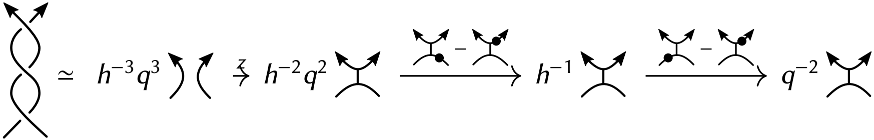

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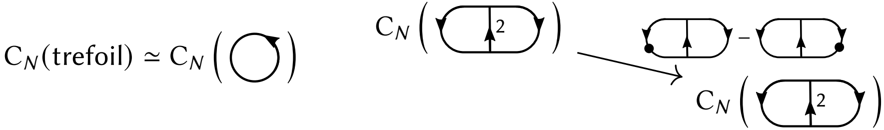

$$\simeq H^*(\mathbf{CP}^{N-1}) \qquad H^*(\mathbf{F}(1,2,N)) \xrightarrow{c_1(\mathcal{S}_A)-c_1(\mathcal{S}_B)} H^*(\mathbf{F}(1,2,N))$$

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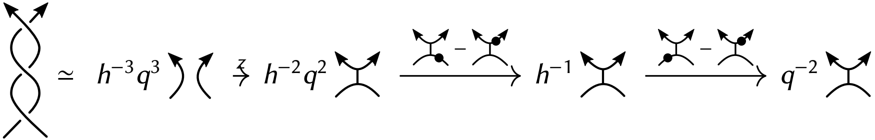

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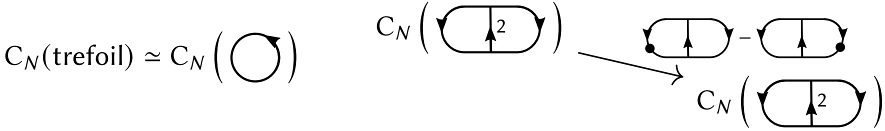

$$\simeq H^*(\mathbf{CP}^{N-1}) \qquad H^*(\mathbf{F}(1,2,N)) \xrightarrow{c_1(\mathcal{S}_A)-c_1(\mathcal{S}_B)} H^*(\mathbf{F}(1,2,N))$$

$\mathscr{R}_N(\text{trefoil}) = \mathbf{CP}^{N-1} \sqcup$ unit tangent bundle of $\mathbf{CP}^{N-1}$

$\mathfrak{sl}(N)$ link homology and $\mathrm{SU}(N)$ representations

It suffices to show that the homology of the complex

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$$\begin{array}{ccc} X & \longleftarrow & S^1 \\ \big\downarrow & & \\ \mathbf{F}(1,2,N) & & \end{array}$$

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Colored $\mathfrak{sl}(N)$ link homology

Colored $\mathfrak{sl}(N)$ homology $\mathrm{KR}_N(L)$ of a *labeled* oriented link L: every component is labeled by an integer k satisfying $0 \le k \le N$.

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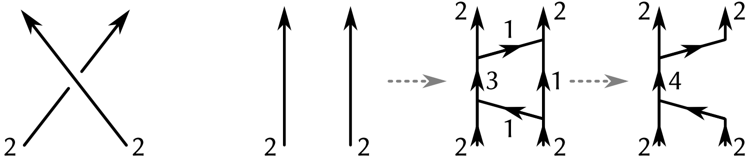

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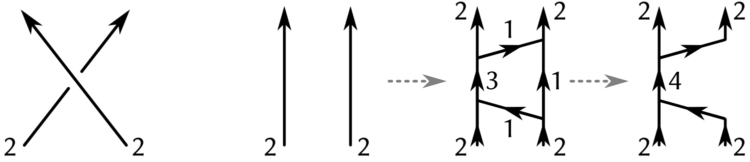
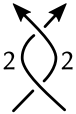
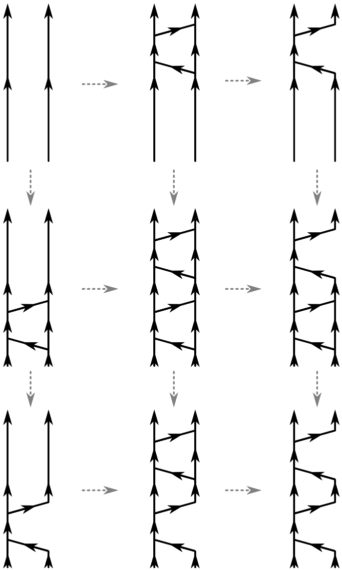


A crossing between strands labeled i, j is given a complex of $\min(i, j) + 1$ resolutions.

Colored $\mathfrak{sl}(N)$ link homology





Colored $\mathfrak{sl}(N)$ link homology and $\mathrm{SU}(N)$ representations

For a labeled link L,

$$\mathscr{R}_N(L) = \left\{ \rho \colon \pi_1(S^3 \setminus L) \to \mathrm{SU}(N) \,\middle|\, \rho\left(\substack{\text{meridian of a} \\ \text{component labeled } k} \right) \in C_k \right\}$$

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We identify $C_k = \mathbf{G}(k, N)$ by sending $A \in C_k$ to its $(-\mathrm{e}^{k\pi i/N})$-eigenspace Λ_A.

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Very few computations appear in the literature.

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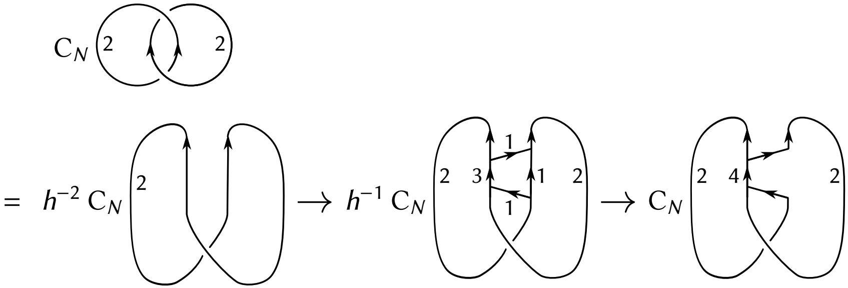
$\mathrm{KR}_N(H)$ is supported only in even homological degrees and has no torsion.

Colored $\mathfrak{sl}(N)$ link homology and $SU(N)$ representations



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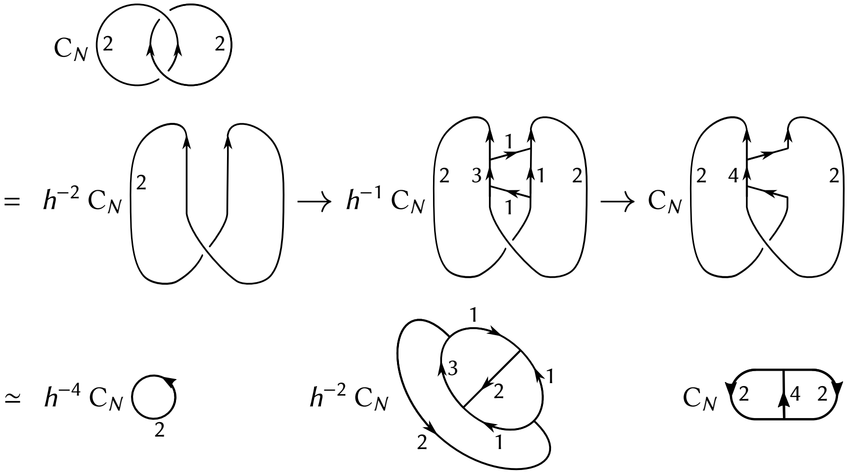

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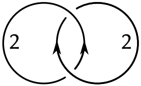
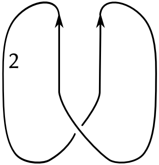
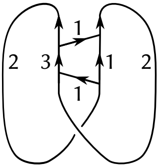
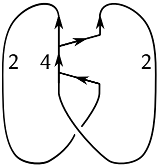
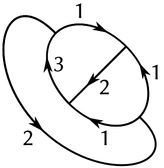
C_N 

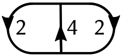
$= \; h^{-2}\, C_N$ 

$\longrightarrow h^{-1}\, C_N$ 

$\longrightarrow C_N$ 

$\simeq \; h^{-4}\, C_N$ 

$h^{-2}\, C_N$ 

C_N 

$= \; H^*(\mathbf{G}(2, N)) \qquad H^*(\mathbf{F}(1, 2, 3, N)) \qquad H^*(\mathbf{F}(2, 4, N))$

Thanks!

Thanks for listening!