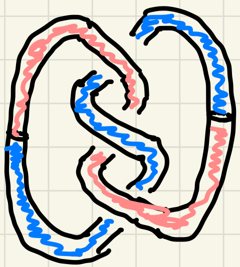
$\widehat{HFK}$ of knots in $S^1 \times D^2$

extension of
work w/ Hanselman
+ Watson

Two types of boundary:

Sutured: Juhasz



$M = S^3 \setminus \nu(K)$

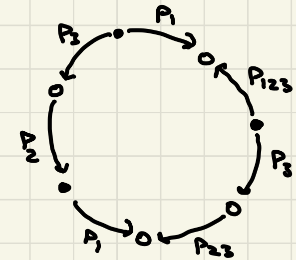
γ_μ = 2 parallel copies of μ

$\rightsquigarrow \mathrm{SFH}(M, \gamma_\mu) = \widehat{HFK}(K)$

bigraded group, 2 diff's to

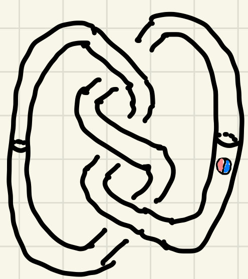
$\widehat{HF}(S^3) = \mathbb{Z}$ (Dehn fill along μ)

$$\begin{tikzcd} \mathbb{Z} \\ \mathbb{Z} \arrow[u] \arrow[d] \\ \mathbb{Z} \end{tikzcd}$$



Bordered: Lipshitz-Ozsuath-Thurston

work over $\mathbb{F} = \mathbb{F}_2$



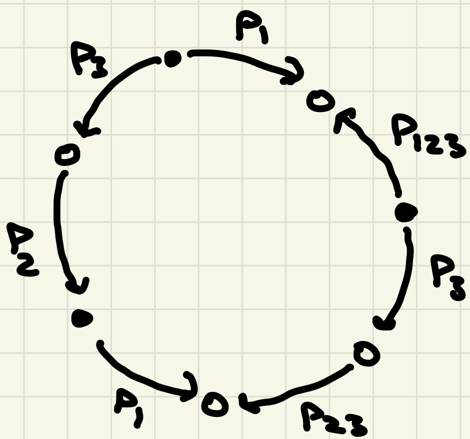
$M \rightsquigarrow \widehat{CFD}(M)$ type D structure

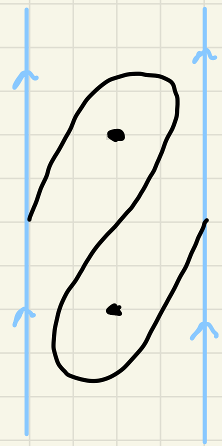
twisted cx over $A(T^2)$ = $\begin{tikzcd} \bullet \arrow[r, bend left, "\rho_1"] \arrow[r, bend right, "\rho_3"'] & \circ \arrow[l, "\rho_2"'] \end{tikzcd}$ / $\rho_2\rho_1 = \rho_3\rho_2 = 0$

Graphical Interpretation: Hanselman-R-Watson

$\widehat{CFD}(M) \rightsquigarrow \widehat{HF}(M)$ = collection of immersed closed curves (w/ local systems) in (covering space of) $\partial M - z$

$M = S^3 \setminus \nu(T(2,3))$





Auroux
Lekili-Perutz

Type D structure
$\Downarrow$
Object of $\mathrm{Fuk}(\mathrm{Sym}^g(\partial M - z))$

Haiden-Kontsevich-Katzarkov

Objects of $\mathrm{Fuk}(\Sigma - z)$ are direct sums of loops w/ local systems + chains

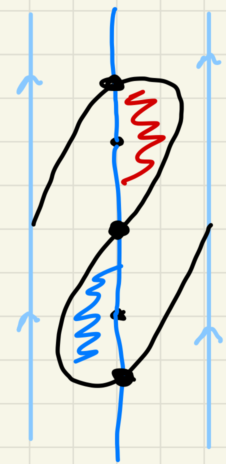
Pairing:

$$\langle L, L' \rangle = \mathrm{Hom}(L, L')$$

generated by $L \cap L'$
after pulling tight.

Knot Floer:

$$\widehat{HFK}(K) = \langle \widehat{HF}(M), L_\mu \rangle$$



$\widehat{HFK}(T(2,3))$

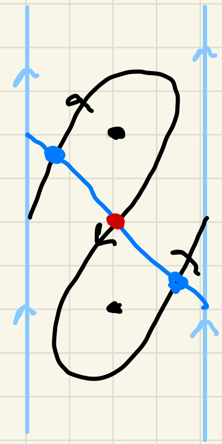
$$\begin{array}{c}\mathbb{Z}\\ \uparrow \\ \mathbb{Z} \\ \downarrow \\ \mathbb{Z}\end{array}$$

Pairing:

$$\langle L, L' \rangle = \mathrm{Hom}(L, L')$$

generated by $L \cap L'$ after pulling tight.

$\boxed{\text{Dehn filling: } \widehat{HF}(M_K(\alpha)) = \langle \widehat{HF}(M), L_\alpha \rangle}$



Dehn filling

$K(-1) = \Sigma(2,3,7)$

$\widehat{HF}(\Sigma(2,3,7)) =$

Paring:

$\langle L, L' \rangle = \mathrm{Hom}(L, L')$

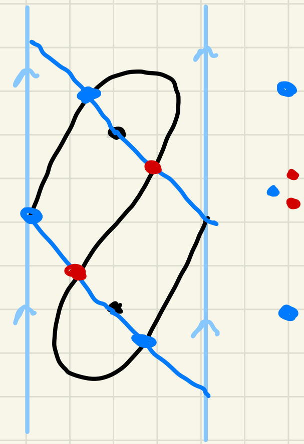
generated by $L \cap L'$
after pulling tight.

Dual Knot

$K_{-1} \subset K(-1)$

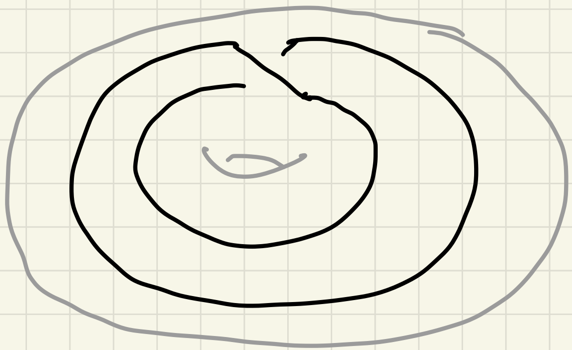
Dual knot:

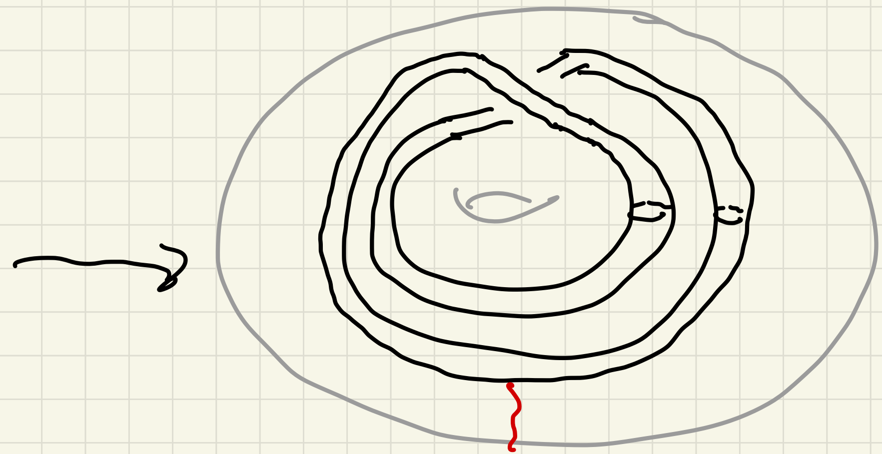
$$\widehat{HFK}(K_\alpha) = \langle \widehat{HF}(M), \overline{L}_\alpha \rangle$$



$\widehat{HFK}(K_{-1})$

$K \subset S^1 \times D^2$

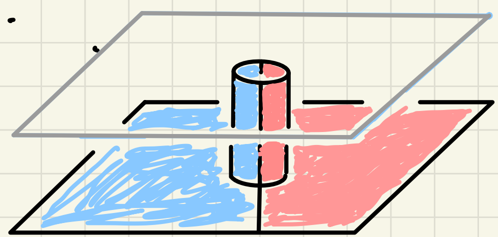


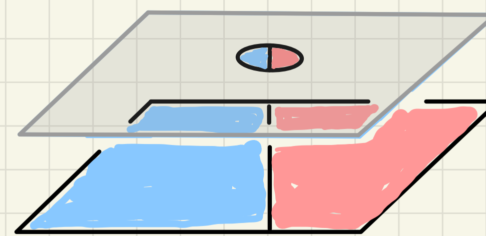


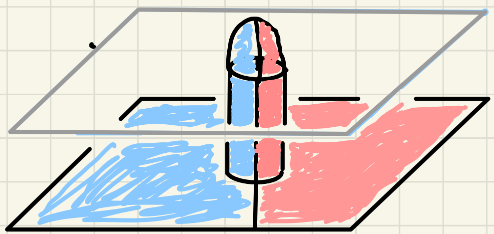
Boundary cpts T_1 (outer) bordered

T_2 (inner) sutured

2 options:





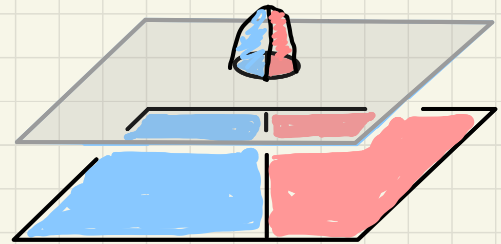


$\widehat{hfk}(K) \in Fuk(T^2 - z)$

noncompact (loops + arcs)

choose an arc from T_1 to T_2

BUT: If we glue to a manifold with $S = D^2$, dependence on arc vanishes

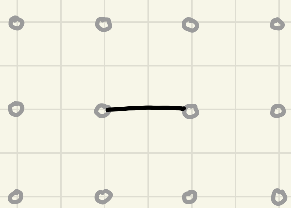


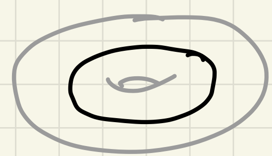
$\widehat{HFK}(K) \in Fuk(T^2 - z)$

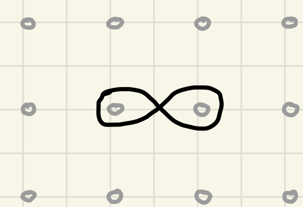
compact object (loops only)

no choices

BUT: If we glue to a manifold with $S = D^2$, get extra factor of $H_*(S^1)$.







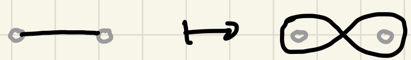
$\widehat{hfk}$ and $\widehat{HFK}$:

$\widehat{hfk}(K) \in \mathrm{Fuk}_z(T^2-0)$.

Functor TC:
$$\begin{tikzcd} \mathrm{Fuk}_z(T^2-0) \arrow[r] \arrow[dr] & \mathrm{Fuk}_z(T^2-0) \arrow[d, phantom, "\cup"] \\ & \mathrm{Fuk}_{cs}(T^2-z) \end{tikzcd}$$

Lipshitz - Treumann
Hanselman

$$\widehat{hfk}(K) \mapsto \widehat{HFK}(K)$$



view $\widehat{hfk}(K) \in \mathrm{Fuk}_z(T^2-0)/\sim$

where $L_1 \sim L_2$ if $\langle L_1, \tilde{L}\rangle \simeq \langle L_2, \tilde{L}\rangle$ forall compact L

Spin^c Structures:

$$\widehat{hfk}(K) = \bigoplus_s \widehat{hfk}(K,s)$$

$s \in \mathrm{Spin}^c(M, T_1)$

$\sim \mathrm{coker}\left(H_1(T_1) \to H_1(M)\right)$

$= \mathbb{Z}/n \qquad n = [K] \in H_1(S^1 \times D^2)$

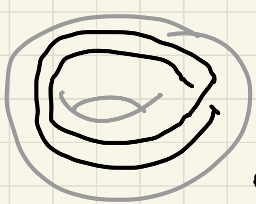
$$\widehat{hfk}(K,s) \in \mathrm{Fuk}(\widetilde{T_1 - z})$$

$\pi_1(\widetilde{T_1}) = \ker\left(\pi_1(T_1) \to H_1(T_1) \to H_1(M)\right)$

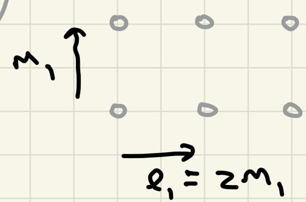
$= \begin{cases} 0 & n \neq 0 \\ [\partial D^2] & n = 0 \end{cases}$

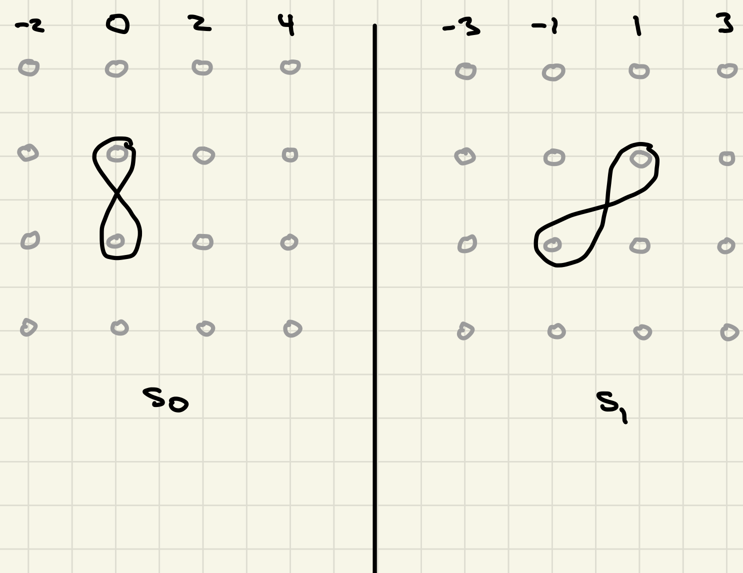
$H_1(M) = \langle m_1, m_2 \rangle \qquad \ell_1 = n m_1$

Ex: $K = \widehat{\sigma_1}$



Spin^c structures
s_0, s_1





Properties:

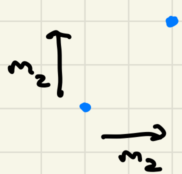
1) Euler characteristic

$$\partial[\widehat{hfk}(K)] = [\widehat{HFK}(K)] = \Delta_{K \dot\cup A}(1-m_2)$$

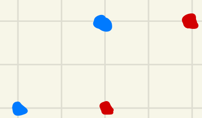
$$\Delta_L \in \mathbb{Z}[H_1(M)] = \mathbb{Z}[m_1^{\pm 1}, m_2^{\pm 1}]$$

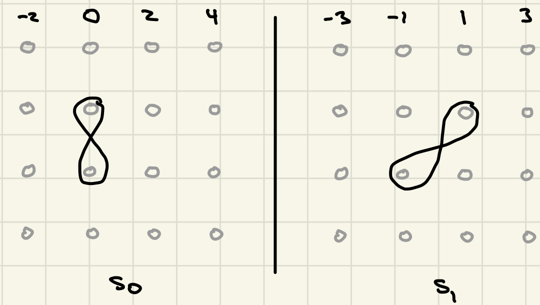
$L = K \cup A$

$\Delta_L \sim 1 + m_1 m_2$



$(1-m_2)\,\Delta_L$





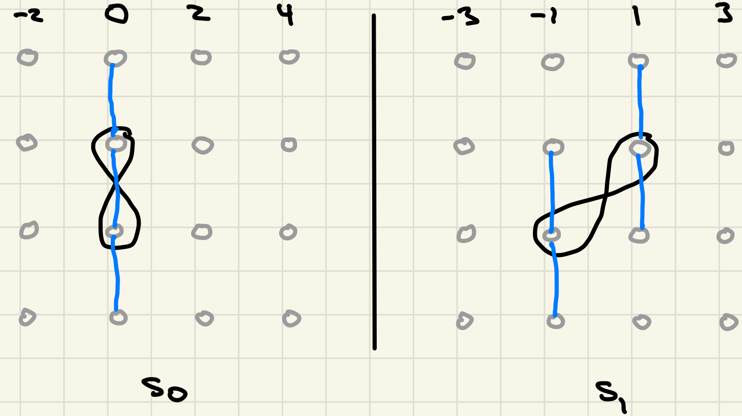
2) Conjugation Symmetry

3) Compute: $\widehat{HFL}(K \cup A)$, $\widehat{HFK}(K)$
Dehn twists, satellites w/ pattern K

1) Link Floer homology

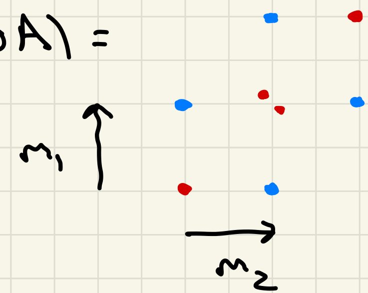
$\widehat{HFL}(K \cup A)$

$= \langle \widehat{HFK}(K), \bar{\iota}_{m_1} \rangle$

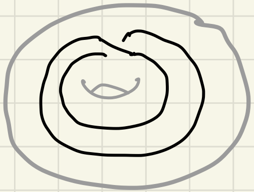


$\widehat{HFL}(\bar{\sigma}_1 \cup A)$

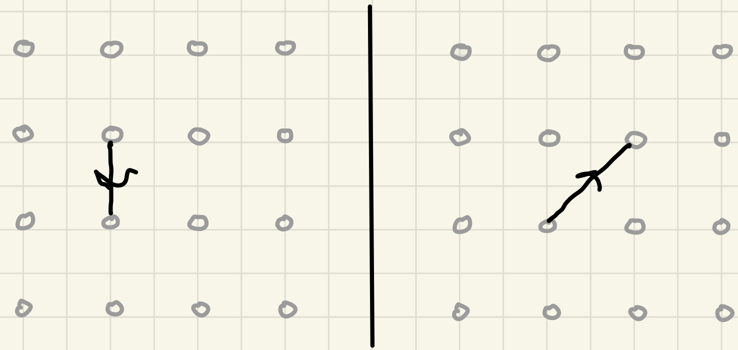
$\widehat{HFL}(\bar{\sigma}_1 \cup A) =$



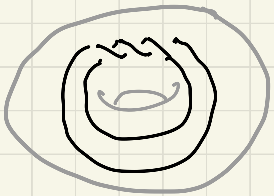
2) Dehn twist



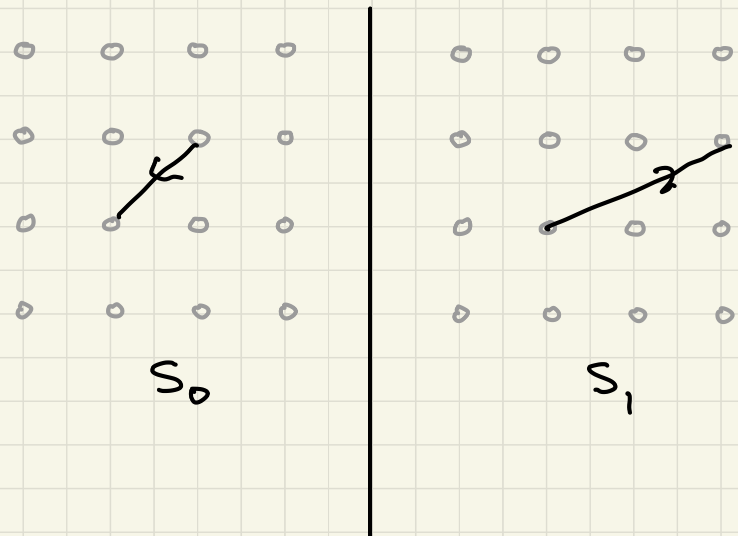
$$\mathrm{hfk}(\hat{\sigma}_1) =$$



$$\begin{tikzcd} l_1 \arrow[d] & m_1 \arrow[d] \\ l_1 & m_1 + l_1 \end{tikzcd}$$



$$\mathrm{hfk}(\sigma_1^3) =$$

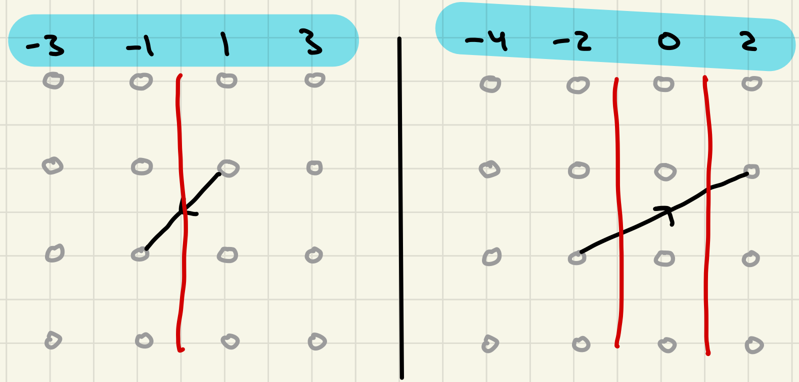


S_0 $\qquad$ S_1

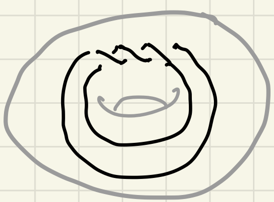
3) Dehn Fill

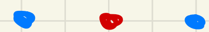
$\widehat{HFK}(K \subset S^3) =$

$\langle \widehat{HFK}(K), L_{m_1} \rangle$



$\widehat{HFK}(\bar{\sigma}_1^3)$



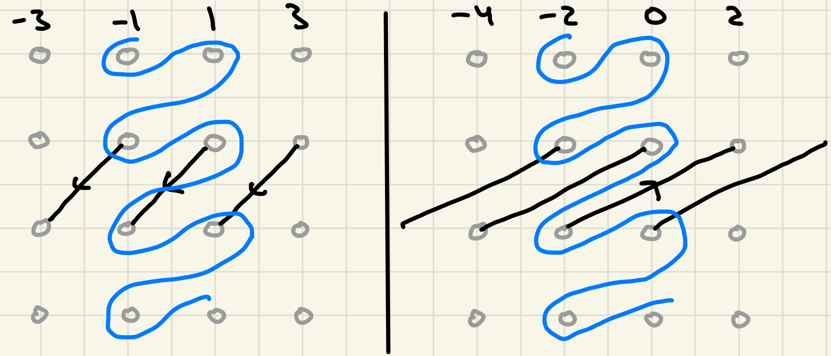


4) Satellites

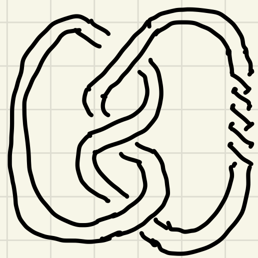
Eftekhary, Hedden, Hom, Levine, Petkova
Hanselman-Watson, W. Chen, ...

$\widehat{HFK}(S_P(C)) =$

$\langle \widehat{HF}(S^3 - C), \widehat{hfk}(P) \rangle$

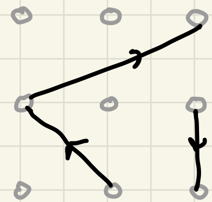


$\widehat{HFK}(S_{\sigma_1^3}(T(2,3)))$

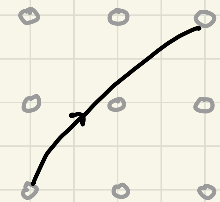


Linear Knots:

Def: K is *linear* if $\widehat{HFK}(K)$ is a union of short line segments



but not



- Preserved by Dehn twisting

Examples:

1) $L = K \cup A \subset S^3$ has $\widehat{HFK}(L)$ thin + K is L-space knot

(In progress)

? 1) above, but w/ K alternating

? 3-strand braids

$K = (1,1)$ pattern (W. Chen)

Hanselman-Watson:

cabling operator

$\Big\downarrow$

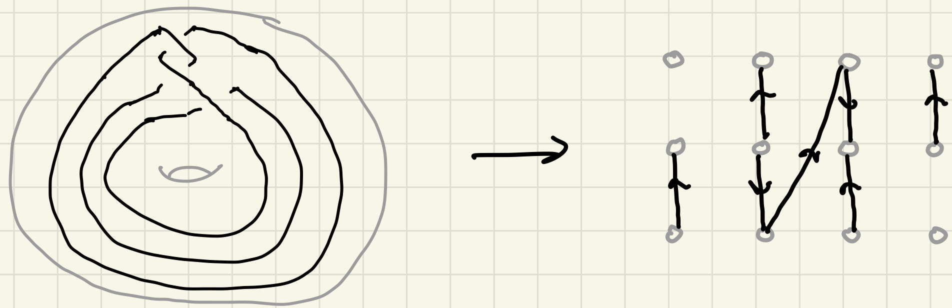
Examples of K which are not linear.

$F \subset \widetilde{I}_K$ (F. Ye)

Thm: K as in 1) $\Rightarrow$ $\widehat{hfk}(K)$ determined by $\Delta_K, \tau_K, \Delta_L, \tau_L$

Ex:

Mazur pattern



Eftekhary, Hedden: For some P, $\widehat{HFK}(S_P(C)) \cong \bigoplus_{\alpha} SFH(M_C, \gamma_\alpha)$

Thm: If K is linear, $\widehat{HFK}(S_P(C)) \cong \bigoplus_{\alpha \in \widehat{hfk}(P)} SFH(M_C, \gamma_\alpha)$

Differentials:

$K \subset S^3$: $\widehat{CFK}(S^3)$ is filtered by A-grading

associated graded $= \widehat{HFK}(K)$.

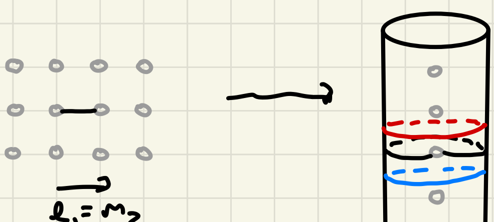
$H_*(\widehat{HFK}(K), d_z) = \widehat{HF}(S^3) = H_*(\widehat{HFK}(K), d_w)$

$K \subset S^1 \times S^2$: $\widehat{CFD}(S^1 \times D^2)$ is filtered by A.

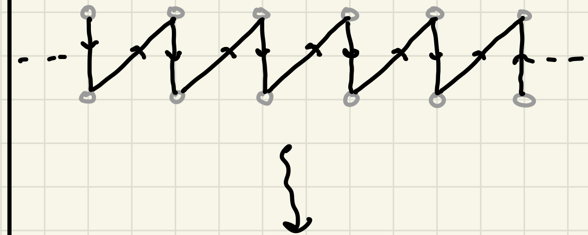
associated graded $\rightsquigarrow \widehat{hfk}(K)$.

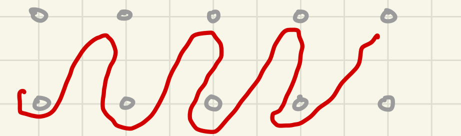
Ex: $K = S^1 \times 0 \subset S^1 \times D^2$

$\widehat{CFD}(M_K, \gamma) \rightsquigarrow \widehat{CFD}(S^1 \times D^2)$



$K = \widehat{\sigma_1}$



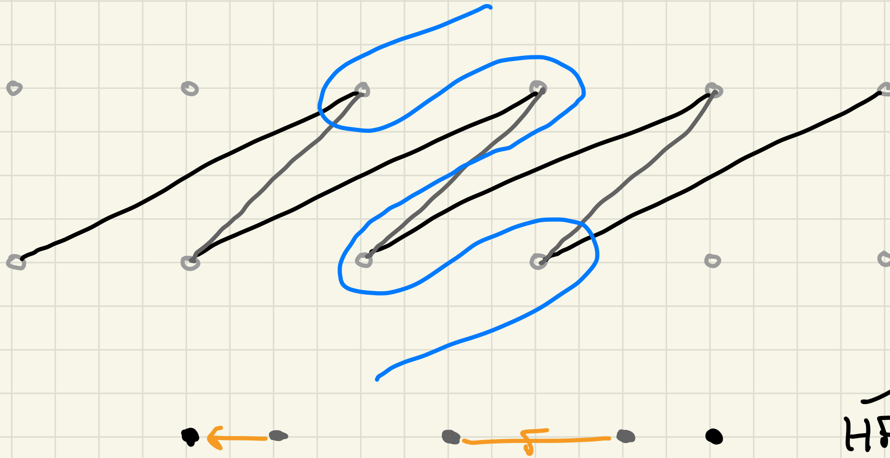


d_z

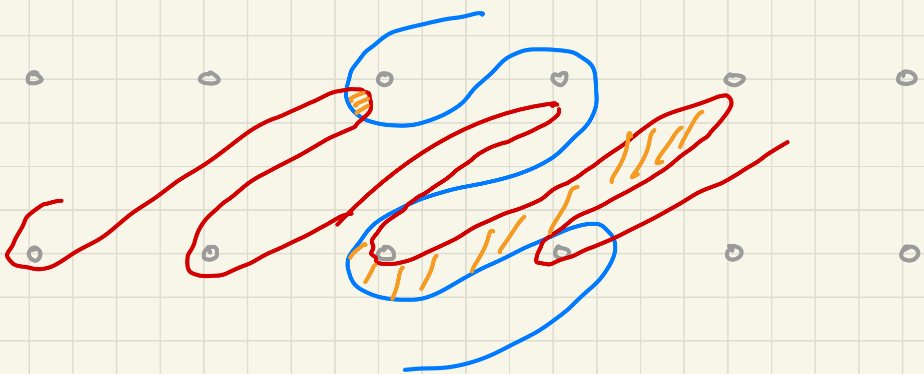
and

d_w

Cable of Trefoil:



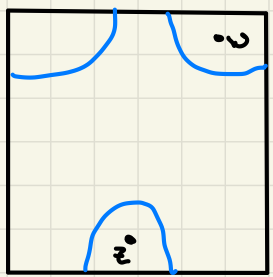
$\widehat{HFK}(\text{cable})$

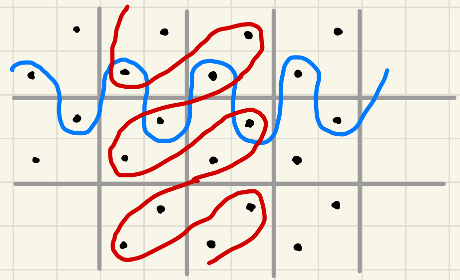


(1,1) Patterns: (W. Chen)

Thm (Chen): If P is a (1,1) pattern,

$$\widehat{CFK}(S_P(C)) = \langle \widehat{HF}(M_K), D_P \rangle$$

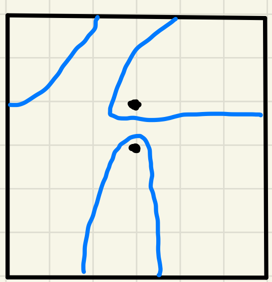


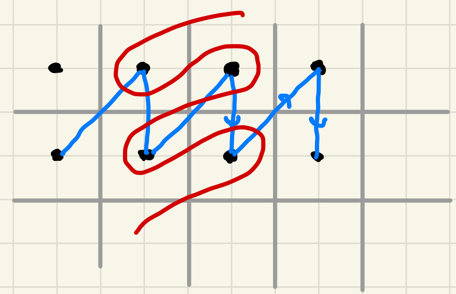


Chen - Hanselman:

similar picture for arbitrary patterns

To get back picture for $\widehat{hfk}$, pinch z, w together:





$\rightsquigarrow$ extra diff'ls in $\widehat{CFD}$ have nice diagrammatic expression.

Operators:

$\partial M = T_1 \cup T_2$ bordered.

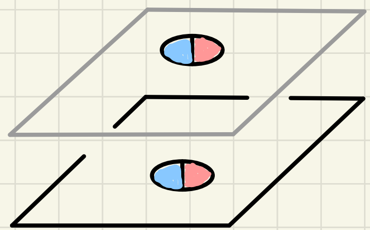
$\rightsquigarrow$ Bimodule $\widehat{CFDA}$ (LOT)

$\longrightarrow$ Functor $\mathrm{Fuk}(T^2 - s) \to \mathrm{Fuk}(T^2 - S)$

Hanselman-Watson: (p,q) cabling functor

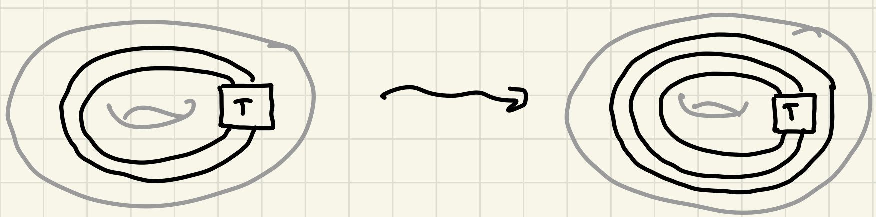
Neck cutting operator:

$M = T^2 \times I$ $S =$

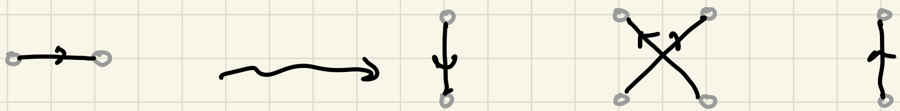


Add a strand:

$M = T^2 \times I - \nu(K)$ $K = m_1 \times \frac{1}{2} \subset T^2 \times I$

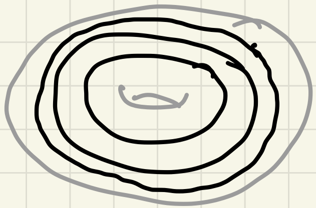


$$\operatorname{coker}\left(H_1(T_1^2) \to H_1(M)\right) \simeq \mathbb{Z}$$

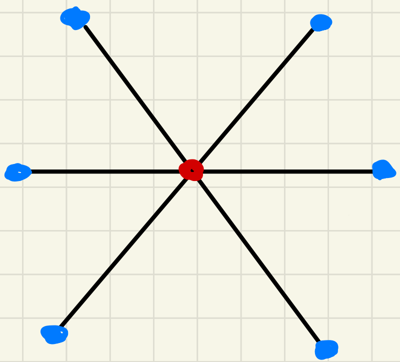


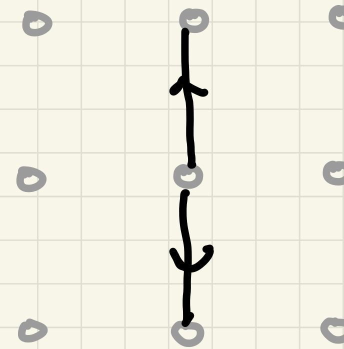
s_{-1} s_0 s_1

S_- S_+



$\widehat{hfl}(T(3,0))$: $\quad spin^c(M) \sim \langle m_1, m_2, m_3 \rangle / (m_1 + m_2 + m_3)$



 $=$ 

$\rightsquigarrow \widehat{HFL}(T(n,n))$

Licata, Gorsky-Hom

 $=$