Efficient resolution of Thue–Mahler equations

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Background

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  - $X, Y, z_1, \ldots, z_v$ are unknown integers
  - $\gcd(X, Y) = 1$

Our main objective

Solve $F(X, Y) = a \cdot p_1^{z_1} \cdots p_v^{z_v}$

Why?

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Theorem (Bennett, G., Rechnitzer)

Let $E/\mathbb{Q}$ be an elliptic curve of conductor $N = 2^{\alpha}3^{\beta}N_0$ where N_0 is coprime to 6.

Then there exists an integral binary cubic form F of discriminant

$$D_F = sign(\Delta_E)2^{\alpha_0}3^{\beta_0}N_1,$$

and relatively prime integers u and v with

$$F(u,v) = c_0u^3 + c_1u^2v + c_2uv^2 + c_3v^3 = 2^{\alpha_1}3^{\beta_1}\prod_{p|N_0} p^{\kappa_p}$$

such that E is isomorphic over $\mathbb{Q}$ to $E_{\mathcal{D}}$, where

$$E_{\mathcal{D}} : 3^{[\beta_0/3]}y^2 = x^3 - 27\mathcal{D}^2H_F(u,v)x + 27\mathcal{D}^3G_F(u,v).$$

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3. Check “local” conditions and output the elliptic curves that arise

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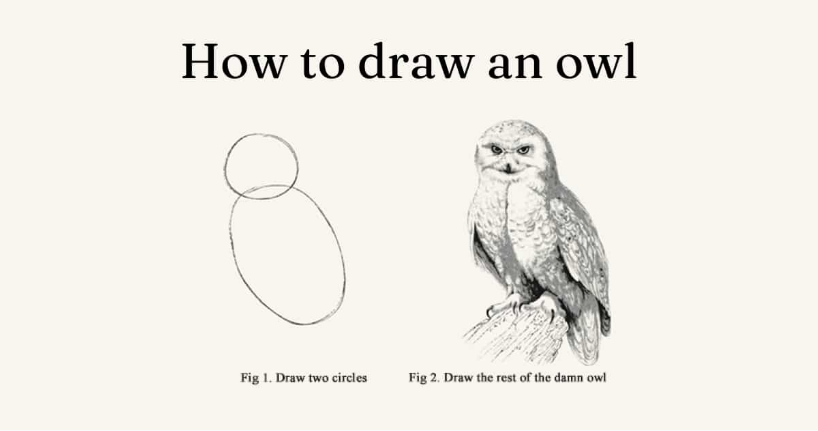
How to draw an owl
Fig 1. Draw two circles
Fig 2. Draw the rest of the damn owl

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- **Hambrook (2011)**:
  Implementation of a Thue–Mahler solver

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- At 5 seconds per form, this requires 11.55 months on a single core

How bad could it be?

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  $X^3 + 3X^2Y + 44XY^2 + 66Y^3 = 3^{z_1} \cdot 11^{z_2} \cdot 17^{z_3} \cdot 23^{z_4} \cdot 31^{z_5}$

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A new Thue–Mahler solver!

An example

Let

$$F(X, Y) = 3X^5 + 65X^4Y - 290X^3Y^2 - 2110X^2Y^3 + 975XY^4 + 3149Y^5.$$

Then $F(X, Y) = -2^5 \cdot 3^4 \cdot 5^{z_1} \cdot 11^{z_2}$ has no solutions

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  - $F(X, Y) = \pm\, 3^4 \cdot m$ has no solutions for $m \in \mathbb{Z}$, m coprime to 3

More examples

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More examples

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- $486X^{11} + 2673X^{10}Y + 8910X^9Y^2 + \cdots + 22XY^{10} + Y^{11} = 3^{z_1}$

Solving a Thue–Mahler equation

Overview

- Generate a very large upper bound for the solutions using the theory of linear forms in logarithms
- Reduce this bound via Diophantine approximation computations
- Search below this reduced bound

Setup

Initial steps

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- Let $K = \mathbb{Q}(\theta)$ with $f(\theta) = 0$
- Solving $F(X,Y) = ap_1^{z_1} \cdots p_v^{z_v}$ is equivalent to solving

$$\mathrm{Norm}_{K/\mathbb{Q}}(a_0X - \theta Y) = a_0^{d-1} \cdot a \cdot p_1^{z_1} \cdots p_v^{z_v}$$

An equivalent problem

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- There is a finite computable set of equations of the form

$$(a_0X - \theta Y)\mathcal{O}_K = \mathfrak{a}\mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_s^{n_s}, \qquad S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$$

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- Obtain a number of equations of the form

$$a_0X - Y\theta \ = \ \tau \cdot \delta_1^{b_1} \cdots \delta_r^{b_r}, \qquad b_i \in \mathbb{Z},$$

  where $\delta_1, \ldots, \delta_r$ is a basis for $\mathcal{O}_S^\times/\text{torsion}$.

An example

$$5X^{11} + X^{10}Y + 4X^9Y^2 + X^8Y^3 + 6X^7Y^4 + X^6Y^5 + 6X^5Y^6 + \\ 6X^3Y^8 + 4XY^{10} - 2Y^{11} = 2^{z_1} \cdot 3^{z_2} \cdot 5^{z_3} \cdot 7^{z_4} \cdot 11^{z_5}$$

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- For one such ideal equation:

$$\mathfrak{p}_1 = \langle 11, 3+\theta \rangle, \quad \mathfrak{p}_2 = \langle 7, 1+\theta \rangle,$$
$$\mathfrak{p}_3 = \langle 5, \phi \rangle, \quad \mathfrak{p}_4 = \langle 3, 5+\theta \rangle, \quad \mathfrak{p}_5 = \langle 2, 1+\theta \rangle,$$

where

$$\phi = \frac{1}{5^9}(4\theta^{10} + 9\theta^9 + 185\theta^8 + 425\theta^7 + 4625\theta^6 + 13750\theta^5 + 131250\theta^4 + 750000\theta^3 + 3203125\theta^2 + 26953125\theta + 5859375)$$

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An example - continued

- The corresponding equation $5X - Y\theta = \tau \cdot \delta_1^{b_1} \cdots \delta_{10}^{b_{10}}$ has

$$\tau = \frac{1}{5^8}(11114\theta^{10} - 156626\theta^9 - 3960\theta^8 + 713050\theta^7 + 3733000\theta^6 - 1296637500\theta^5 + 175803125\theta^4$$
$$- 1846875000\theta^3 + 1457890625\theta^2 - 701687500000\theta + 134298828125)$$

$$\delta_1 = \frac{1}{5^9}(62639\theta^{10} - 748196\theta^9 - 4621980\theta^8 - 22207025\theta^7 + 38965000\theta^6 - 34195000\theta^5 - 449543750\theta^4$$
$$- 212713125000\theta^3 - 517657031250\theta^2 - 2098097656250\theta + 942912109375),$$

$$\delta_2 = \frac{1}{5^8}(-304507\theta^{10} - 1286200\theta^9 - 8286278\theta^8 - 147444530\theta^7 - 120138150\theta^6 + 2957350000\theta^5$$
$$+ 317693750\theta^4 + 196456718750\theta^3 - 1856078125\theta^2 + 1597415625000\theta - 1543269140625),$$

$$\delta_3 = \frac{1}{5^9}(-506181269733\theta^{10} - 151990813790480\theta^9 + 34170399960000\theta^8 + 206312637308500\theta^7$$
$$- 862101634598875\theta^6 - 112487612450893750\theta^5 + 132779534749000000\theta^4 - 479693441045625000\theta^3$$
$$- 481688292060625000\theta^2 - 552604241339570312500\theta + 1323149949666210937500),$$

$$\delta_4 = \frac{1}{5^9}(375938718\theta^{10} + 11130685130\theta^9 + 97012538300\theta^8 + 284204509000\theta^7 + 3376801042500\theta^6 + 8970753712500\theta^5$$
$$+ 88078172156250\theta^4 + 172701451406250\theta^3 + 2100841248437500\theta^2 + 4119271933593750\theta + 3744720025390625),$$

$$\vdots$$

$$\delta_{10} = \frac{1}{5^9}(-173\theta^{10} - 1528\theta^9 - 4840\theta^8 + 6800\theta^7 + 54125\theta^6 - 298750\theta^5 - 4609375\theta^4$$
$$- 195468750\theta^3 - 119531250\theta^2 + 2707031250\theta + 1181640625).$$

Height Bounds

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Given $a_0X - Y\theta = \tau \cdot \delta_1^{b_1} \cdots \delta_r^{b_r}$

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$$\|\mathbf{b}\|_2 \leq \sqrt{r} \cdot c_{20} = \mathcal{B}_2$$

An example - continued

$$5X^{11} + X^{10}Y + 4X^9Y^2 + X^8Y^3 + 6X^7Y^4 + X^6Y^5 + 6X^5Y^6 + 6X^3Y^8 + 4XY^{10} - 2Y^{11} = 2^{z_1} \cdot 3^{z_2} \cdot 5^{z_3} \cdot 7^{z_4} \cdot 11^{z_5}$$

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We obtain a bound of

$$B \leq 1.33 \times 10^{222} \implies \|\mathbf{b}\|_2 \leq 4.2 \times 10^{222}$$

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$$a_0 X - Y\theta \;=\; \tau \cdot \underbrace{\delta_1^{b_1} \cdots \delta_r^{b_r}}_{\varepsilon}, \qquad B = \max\{|b_1|, \ldots, |b_r|\}$$

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- Obtain a new bound for B:

$$B \leq 2c_{17} \sum_{\nu \in M_K} \varepsilon_\nu \qquad \implies \text{ iterate!}$$

Valuations of $a_0X - \theta Y$

- Let $\mathfrak{p} \in S$ and $k \in \mathbb{Z}_{\geq 1}$

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- Let $\mathfrak{a} = (p\mathcal{O}_K)/\mathfrak{p}$, then

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If $\mathbf{w} + L$ does not contain any vectors $\mathbf{v}$ with $\|\mathbf{v}\|_2 \le \mathcal{B}_2$, then

$$\mathrm{ord}_{\mathfrak{p}}(a_0X - \theta Y) \le k - 1$$

An example - continued

For $(5X - \theta Y)\mathcal{O}_K = \mathfrak{a}\mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_5^{n_5}$, where $5X - Y\theta = \tau \cdot \delta_1^{b_1} \cdots \delta_{10}^{b_{10}}$:

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Iteration	$\mathcal{B}_0$	Bounds for $\text{ord}_{\mathfrak{p}_j}(5X - \theta Y)$ with $1 \leq j \leq 5$				
0	1.33×10^{222}	237	292	355	518	821

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$$B = \max\{|b_1|, \ldots, |b_{10}|\} \leq 179 \implies \|\mathbf{b}\|_2 \leq 567$$

Searching below the reduced bound

Sieving

$$a_0 X - Y\theta \;=\; \tau \cdot \delta_1^{b_1} \cdots \delta_r^{b_r}, \qquad \|\mathbf{b}\|_2 \leq \mathcal{B}_2$$

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L has huge index $\implies$ easy to determine $\mathbf{b}$ using Finke and Pohst!

Examples

An appeal to your generosity

An appeal to your generosity

Looking for donations

An appeal to your generosity

Looking for donations: Cores with Magma and storage!

An appeal to your generosity

Looking for donations: Cores with Magma and storage!

$\Longrightarrow$ adela.gherga@warwick.ac.uk

Thank You