Anderson operator

Joint work with V.N. Dang & A. Mouzard

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1. Anderson operator as an unbounded operator
2. Precise heat kernel description
3. Anderson Gaussian free field

1. Anderson operator as an unbounded operator

1.1 Anderson operator

Let $\mathcal{S}$ be a 2-dimensional closed Riemannian manifold.

▶ **Space white noise** – *A Gaussian random distribution* ξ *with null mean and covariance* $\mathbb{E}[\xi(f_1)\xi(f_2)] = \langle f_1, f_2 \rangle_{L^2(\mathcal{S})}$, *for all smooth test functions* f_1, f_2. *It is almost surely of Hölder regularity* $\alpha - 2$, *for any* $\alpha < 1$, *i.e.* $(-1)^-$.

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- **Anderson operator** – $Hu := \Delta u + \xi u$.

In the discrete 2-dimensional torus $\mathbb{T}_n^2 := (\mathbb{Z}/n \bmod \mathbb{Z})^2$, the large scale limit of the operator

$$n\Delta_{\rm discr} + \frac{1}{n}\xi_i\delta_i$$

for the discrete Laplace operator $\Delta_{\rm discr}$ and a random iid potential $(\xi_i)_{i\in\mathbb{T}_n^2}$ with common law with finite second moment.

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To get an unbounded operator on $L^2(\mathcal{S})$ one needs a domain $D(H)$ with $Hu \in L^2(\mathcal{S})$ when $u \in D(H)$.

The multiplication problem

- Pick u β-Hölder. Then ξu well-defined iff $(\alpha - 2) + \beta > 0$, i.e. $\beta > 1^+$.
- For such u the term ξu is $(\alpha - 2)$-regular while Δu is just $(\beta - 2) > (\alpha - 2)$ regular. No compensation to get $\Delta u + \xi u \in L^2(\mathcal{S})$.

1.2 Anderson operator: previous works

• **Allez & Chouk** [15'] construct the operator on $\mathbb{T}^2$ as a symmetric closed unbounded operator on $L^2(\mathbb{T}^2)$ using paracontrolled calculus. It has compact resolvent, hence a nice spectral theory. They prove tail estimates for smallest eigenvalue.

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• **Mouzard** [20'] further simplifies the construction of [GUZ], in a 2-dimensional manifold setting, using high order paracontrolled calculus. Proves an almost sure Weyl law

$$\sharp\{\text{eigenvalues} \leq \lambda\} \sim \frac{\text{Vol}(\mathcal{S})}{4\pi}\,\lambda, \qquad (\lambda \to +\infty).$$

1.3 A glimpse at paracontrolled calculus for defining H

• **The paracontrolled structure** – Regularity is not sufficient for making sense of $Hu \in L^2(\mathcal{S})$. Impose finer paracontrolled structure

$$u = P_{u'}X + u^\sharp$$

where is a P bilinear operator called paraproduct, $u', X \in C^\alpha(\mathcal{S})$ and a remainder term $u^\sharp \in C^{2\alpha}(\mathcal{S})$, with $X := -\Delta^{-1}(\xi)$.

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$\rightsquigarrow$ Define $(X\xi)(\omega)$ as a random variable!... after regularizing ξ into $\xi_r \in C^\infty(\mathcal{S})$, setting $X_r := -\Delta^{-1}(\xi_r)$, and renormalizing

$$X_r\xi_r - \mathbb{E}[X_r\xi_r] =: X_r\xi_r + c_r \simeq X_r\xi_r - \frac{\log r}{4\pi}.$$

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• Working with $X_r\xi_r + c_r$ instead of $X_r\xi_r$ is equivalent to working with renormalized operator $\Delta + \xi_r + c_r$. One has

$$H^{-1} = \lim_{r\downarrow 0} \left(\Delta + \xi_r + c_r\right)^{-1} : L^2(\mathcal{S}) \to L^2(\mathcal{S}).$$

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An operator that depends continuously on the enhanced noise $\widehat{\xi} := (\xi, X\xi) \in C^{\alpha-2}(\mathcal{S}) \times C^{2\alpha-2}(\mathcal{S})$, with a discrete random real spectrum $\big(\lambda_n(\widehat{\xi})\big)_{n\geq 0}$ going to $+\infty$.

2. Precise heat kernel description

2.1 A fine heat kernel description

Recall $\alpha = 1^-$. For $\gamma > 0$ set

$$t^{-\gamma} C\big((0,T],E\big) := \Big\{ v \in C\big((0,T],E\big)\, ; \sup_{0<s\leq t\leq T} s^{\gamma} |v(t)| < \infty \Big\}.$$

Write p_t^{Δ} for heat kernel of Laplace-Beltrami operator; it behaves as t^{-1} for small t. Set formally

$$(\star) : v_0 \mapsto \Big\{ (t,x) \mapsto \big\langle v_0(\cdot), (p_t - p_t^{\Delta})(x,\cdot) \big\rangle \Big\}.$$

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- **Theorem** – *Almost surely the map* $(\star)$ *sends continuously*
  - *the Besov space* $B^{-\epsilon}_{1,\infty}(\mathcal{S})$ *into* $t^{(-1/2)^-}C\big((0,T],C^{\alpha}(\mathcal{S})\big)$,
  - *the Sobolev space* $H^{-2\alpha}(\mathcal{S})$ *into* $t^{-\alpha}C\big((0,T],H^{\alpha}(\mathcal{S})\big)$.

These two functions depend continuously on the enhanced noise $\widehat{\xi} = (\xi, X\xi)$.

2.2 A fine heat kernel description: benefits

- **Theorem (Bounds for the eigenvalues)** – Small time asymptotics for $\mathrm{tr}_{L^2}(e^{-tH})$ and Tauberian theorem give direct short proof of Weyl law

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Write u_n for eigenfunction associated with eigenvalue λ_n and define for $\lambda \in \mathbb{R}$ the spectral projector on $L^2(\mathcal{S})$

$$\pi_{\leq\lambda}(f) := \sum_{\lambda_n \leq \lambda} (f, u_n)_{L^2}\, u_n.$$

- **Theorem (Bounds for the eigenfunctions of H)** – *One has for all* $n \geq 0$ *such that* $|\lambda_n| \geq 1$ *the* n*-uniform estimate*

$$\|u_n\|_{C^{2\alpha-1}} \lesssim \big|\lambda_n(\widehat{\xi}\,)\big|^{(1/2)^+}, \qquad \|u_n\|_{L^p} \lesssim \big|\lambda_n(\widehat{\xi}\,)\big|^{(\frac{1}{2}-\frac{1}{p})^+},$$

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and for all $\lambda \in \mathbb{R}_+$ the upper bound

$$\|\pi_{\leq\lambda}(f)\|_{H^\alpha} \lesssim \lambda^{\frac{1}{2}}\,\|f\|_{L^2}, \qquad \|\pi_{\leq\lambda}(f)\|_{L^p} \lesssim \lambda^{(\frac{1}{2}-\frac{1}{p})^+}\,\|f\|_{L^2}.$$

3. Anderson Gaussian free field

3.1 Anderson GFF: Definition and elementary properties

Gaussian free field (GFF): Random field ϕ_{GFF} with centered Gaussian law and covariance

$$\mathbb{E}\left[\phi_{GFF}(f_1)\,\phi_{GFF}(f_2)\right] = \int_{S\times S} f_1(x) G_\Delta(x,y) f_2(y)\, dxdy,$$

with G_Δ Green function of Δ – i.e. kernel of Δ^{-1}. One has almost surely $\phi_{GFF} \in H^{-\epsilon}(\mathcal{S})$, for all $\epsilon > 0$.

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Anderson Gaussian free field: A doubly random field ϕ with centered Gaussian law and covariance

$$\mathbb{E}'\big[\phi(f_1)\,\phi(f_2)\big] = \int_{S\times S} f_1(x) G(x,y) f_2(y)\,dxdy,$$

with G Green function of the *random* operator $H + c$ – i.e. kernel of $(H+c)^{-1}$, with c random big enough for $H+c$ to be positive. Write $\phi = \phi(\omega, \omega')$, with ω the randomness from H and ω' the additionnal 'field' randomness, and $\mathbb{E}'$ expectation wrt ω'.

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▶ **Theorem** – *One has* (ω,ω')*-almost surely* $\phi \in H^{-\epsilon}(\mathcal{S})$*, for all* $\epsilon > 0$*, and the Cameron-Martin space of the* ω'*-law of* ϕ *is continuously embedded into* $H^{1^-}(\mathcal{S})$.

3.2 Anderson GFF: Wick square

Even though ϕ is only a distribution one can make sense of its square using a renormalization process after regularization $\phi_r := e^{-r\Delta}(\phi)$

$$:\phi_r^2: := \phi_r^2 - \mathbb{E}'\left[\phi_r^2\right].$$

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▶ **Theorem** – *Almost surely in $\omega \in \Omega$, the regularized Wick square $:\phi_r^2:$ converges in law as r goes to 0, as a random variable on Ω' with values in $H^{-2\epsilon}(\mathcal{S})$, to a limit random variable $:\phi^2:$, and one has for all $\lambda \in \mathbb{C}$ sufficiently small*

$$Z(\lambda) := \mathbb{E}'\left[e^{-\lambda:\phi^2:(\mathbf{1})}\right] = \det_2\Big(Id + \lambda(H+c)^{-1}\Big)^{-1/2}.$$

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▶ **Theorem (The distribution of Z characterizes the distribution of the spectrum of H)** – *Let (S_1, g_1) and (S_2, g_2) be two Riemannian closed surfaces. Then the spectra of the operators $H(S_1, g_1)$ and $H(S_2, g_2)$ have the same law iff the random holomorphic functions $Z(S_1, g_1)$ and $Z(S_2, g_2)$ have the same law.*

Two open questions

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From our analysis of p_t the zeta function of the Anderson operator

$$\zeta_H(s) := \sum_{n\geq 0} \lambda_n(\widehat{\xi})^{-s}$$

has almost surely a meromorphic extension to the half plane $\{\mathrm{Re}(s) > 1/2\}$.

- *Prove the function $\mathbb{E}[\zeta_H(\cdot)]$ has a meromorphic extension to all of $\mathbb{C}$.*

Thank you for your attention!