Optimal Hardy-weights for elliptic operators with mixed boundary conditions

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Mathematical aspects of the physics with non-self-adjoint operators

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The operator (P, B)

Let P be a second-order, linear, elliptic operator in divergence form with real and locally regular coefficients defined on a domain $\Omega \subset \mathbb{R}^n$

$$Pu := -\mathrm{div}\, \left[A(x)\nabla u + u\tilde{\mathbf{b}}(x)\right] + \bar{\mathbf{b}}(x)\cdot\nabla u + c(x)u \qquad x \in \Omega.$$

Let $\partial\Omega_{\mathrm{Rob}}$ be a relatively open C^1-portion of $\partial\Omega$, and consider the oblique boundary operator

$$Bu := \left(A(x)\nabla u + u\tilde{\mathbf{b}}(x)\right)\cdot\vec{n}(x) + \gamma(x)u \qquad x \in \partial\Omega_{\mathrm{Rob}},$$

where $\vec{n}(x)$ is the outward unit normal vector to $\partial\Omega$ at $x \in \partial\Omega_{\mathrm{Rob}}$, and γ is a real measurable function defined on $\partial\Omega_{\mathrm{Rob}}$. Let $\partial\Omega_{\mathrm{Dir}} := \partial\Omega \setminus \partial\Omega_{\mathrm{Rob}}$ be the Dirichlet part of $\partial\Omega$.

If further $\bar{\mathbf{b}} = \tilde{\mathbf{b}}$ in Ω, we say that (P, B) is symmetric in Ω.

Weak solutions

Definition

We say that $u \in H^1_{\mathrm{loc}}(\overline{\Omega} \setminus \partial\Omega_{\mathrm{Dir}})$ *is a* *weak solution (resp., supersolution)* *of the boundary value problem*

$$\begin{cases} Pu = 0 & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega_{\mathrm{Rob}}. \end{cases} \qquad \textbf{(P,B)}$$

if for any (resp., nonnegative) $\phi \in C_0^\infty(\bar{\Omega} \setminus \partial\Omega_{\mathrm{Dir}})$ *we have*

$$\int_\Omega [(a^{ij} D_j u + u\tilde{\mathbf{b}}^i) D_i\phi + (\bar{\mathbf{b}}^i D_i u + cu)\phi] \mathrm{d}x + \int_{\partial\Omega_{\mathrm{Rob}}} \gamma u \phi \mathrm{d}\sigma = \begin{cases} 0, & \\ \geq 0, & \text{resp.} \end{cases}$$

In this case we write $(P, B)u = 0$ *(resp.,* $(P, B)u \geq 0$*).*

Hardy-weight of (P, B)

Definition

- *We say that (P, B) is nonnegative in Ω (in short $(P, B) \geq 0$ in Ω) if there exists a positive weak solution to the boundary value problem*

$$\begin{cases} Pu = 0 & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega_{\mathrm{Rob}}. \end{cases} \qquad \textbf{(P,B)}$$

- *We say that $W \gneqq 0$ is a Hardy-weight for (P, B) in Ω if $(P - W, B) \geq 0$ in Ω.*
- *A nonnegative operator (P, B) in Ω is said to be subcritical (resp., critical) in Ω if (P, B) admits (resp., does not admit) a Hardy-weight for (P, B) in Ω.*

Agmon-Allegretto-Piepenbrink (AAP) theorem

Theorem

Suppose that (P, B) is a symmetric operator (i.e., $\bar{\mathbf{b}} = \tilde{\mathbf{b}}$ in Ω). Then $(P, B) \geq 0$ in Ω iff the corresponding quadratic form is nonnegative on $C_0^\infty(\bar{\Omega} \setminus \partial\Omega_{\mathrm{Dir}})$.

Hence, in the symmetric case, the inequality $(P - W, B) \geq 0$ in Ω is equivalent to the validity of the following Hardy-type inequality

$$\int_\Omega \left(|\nabla \phi|_A^2 + (c - \mathrm{div}\, \bar{b})|\phi|^2\right) \mathrm{d}x + \int_{\partial\Omega_{\mathrm{Rob}}} \gamma |\phi|^2 \mathrm{d}\sigma \geq \int_\Omega W |\phi|^2 \mathrm{d}x$$

for all $\phi \in C_0^\infty(\bar{\Omega} \setminus \partial\Omega_{\mathrm{Dir}})$.

Previous results for the case $\partial\Omega_{\mathrm{Rob}} \neq \emptyset$ are by Kovařík-Laptev (2012), Kovařík-Mugnolo (2018), and references therein.

Criticality theory

- (P, B) is subcritical in Ω iff (P, B) admits a minimal positive Green function $G^{\Omega}_{P,B}(x, y)$.
- (P, B) is critical in Ω iff the equation $(P, B)u = 0$ in Ω admits (up to a multiplicative constant) a unique positive supersolution ϕ.
- In fact, ϕ is a minimal positive solution of $(P, B)u = 0$ in Ω, called the (Agmon) ground state.
- (P, B) is critical in Ω if and only if (P^*, B^*) is critical in Ω, where (P^*, B^*) is the formal adjoint of (P, B) in $L^2(\Omega)$.

Aim: Find as large as possible Hardy-weight for subcritical (P, B).

Optimal Hardy weights

Definition

A Hardy-weight W of (P,B) in Ω is said to be optimal if $(P-W,B)$ is critical in Ω and $\int_\Omega \phi\phi^* W\,\mathrm{dx}=\infty$, where ϕ and ϕ^* are the ground states of $(P-W,B)$ and (P^*-W,B^*) in Ω, respectively. In this case, we say that $(P-W,B)$ is null-critical in Ω with respect to the weight W.

Definition

We say that a Hardy-weight W is optimal at infinity in Ω if for any $K \Subset \overline{\Omega}$, $\partial K \cap \partial\Omega_{\mathrm{Dir}} = \emptyset$, and $\partial K \cap \partial\Omega_{\mathrm{Rob}} \Subset \partial\Omega_{\mathrm{Rob}}$ with respect to the relative topology on $\partial\Omega_{\mathrm{Rob}}$ (in short, $K \Subset_R \Omega$), we have

$$\sup\{\lambda \in \mathbb{R} \mid (P-\lambda W, B) \geq 0 \text{ in } \Omega \setminus K\} = 1.$$

Remark: Any optimal Hardy-weight in Ω is also optimal at infinity in Ω.

Definition (Exhaustion of $\overline{\Omega} \setminus \partial\Omega_{\mathrm{Dir}}$)

A sequence $\{\Omega_k\}_{k\in\mathbb{N}} \subset \Omega$ is called an exhaustion of $\overline{\Omega} \setminus \partial\Omega_{\mathrm{Dir}}$ if it is an increasing sequence of Lipschitz subdomains s.t. $\Omega_k \Subset_R \Omega_{k+1} \Subset_R \Omega$, and

$$\bigcup_{k\in\mathbb{N}} \overline{\Omega_k} = \overline{\Omega} \setminus \partial\Omega_{\mathrm{Dir}}.$$

Definition

Let $K \Subset \Omega$ and $f \in C((\overline{\Omega \setminus K}) \setminus \partial\Omega_{\mathrm{Dir}})$. We say that

$$\lim_{x\to\infty_{\mathrm{Dir}}} f(x) = 0$$

if for any $\varepsilon > 0$ and any exhaustion $\{\Omega_k\}_{k\in\mathbb{N}}$ of $\overline{\Omega} \setminus \partial\Omega_{\mathrm{Dir}}$, there exists k_0 such that $|f(x)| < \varepsilon$ in $\Omega \setminus \Omega_{k_0}$.

Green potential

Definition

Let (P,B) be a subcritical operator in Ω, and let $G(x,y) := G^{\Omega}_{P,B}(x,y)$ the corresponding minimal positive Green function. Fix $0 \lneqq \varphi \in C_0^\infty(\Omega)$. The Green potential with a density φ is the function

$$G_\varphi(x) := \int_\Omega G(x,y)\varphi(y)\,\mathrm{d}y.$$

Theorem

Let (P, B) be a subcritical operator in Ω and let G_φ be the Green potential with a density $0 \lneqq \varphi \in C_0^\infty(\Omega)$. Assume that a positive solution $u > 0$ satisfies $(P, B)u = 0$ and Ancona condition:

$$\lim_{x \to \infty_{\mathrm{Dir}}} \frac{G_\varphi(x)}{u(x)} = 0.$$

Then

$$W := \frac{P(\sqrt{G_\varphi u})}{\sqrt{G_\varphi u}} \geq 0 \text{ is a Hardy-weight.}$$

Moreover, $(P - W, B)$ is critical in Ω with a ground state $\sqrt{G_\varphi u}$, and

$$W = \frac{|\nabla(G_\varphi/u)|_A^2}{4(G_\varphi/u)^2} \qquad \text{in } \Omega \setminus \operatorname{supp}(\varphi).$$

Theorem (Continue)

Furthermore, assume that one of the following regularity conditions are satisfied.

1. (P, B) *is symmetric,* $A \in C^{0,1}_{\mathrm{loc}}(\overline{\Omega} \setminus \partial\Omega_{\mathrm{Dir}}, \mathbb{R}^{n^2})$, $\bar{\mathbf{b}} = \tilde{\mathbf{b}} \in C^{\alpha}_{\mathrm{loc}}(\overline{\Omega} \setminus \partial\Omega_{\mathrm{Dir}}, \mathbb{R}^n)$, $c \in L^{\infty}_{\mathrm{loc}}(\overline{\Omega} \setminus \partial\Omega_{\mathrm{Dir}})$, *and* $\partial\Omega_{\mathrm{Rob}} \in C^{1,\alpha}$.
2. $\partial\Omega_{\mathrm{Rob}}$, $\partial\Omega_{\mathrm{Dir}}$ *are both relatively open and closed sets,* $\partial\Omega_{\mathrm{Rob}}$ *is bounded and admits a finite number of connected components, and the coefficients of* P *are smooth enough functions in* Ω.

Then W *is an optimal Hardy-weight for* (P, B) *in* Ω.

Family of optimal Hardy-weights

Theorem

Assume that the operator (P, B), and the functions G_φ, u satisfy the assumptions of the above theorem.
Let w be an optimal (Dirichlet) Hardy-weight of $Ly := -y''$ in $\mathbb{R}_+$, and let $\psi_w(t)$ be the corresponding ground state. Suppose further that $\psi_w' \geq 0$ on $\{t = G_\varphi(x)/u(x) \mid x \in \Omega\}$, and set

$$W := \frac{P\left(u\psi_w\left(G_\varphi/u\right)\right)}{u\psi_w\left(G_\varphi/u\right)}.$$

Then, the following assertions are satisfied:

1. $W \geq 0$ *in* Ω *and* $W := |\nabla(G_\varphi/u)|_A^2\, w(G_\varphi/u)$ *in* $\Omega \setminus \operatorname{supp}(\varphi)$.
2. $(P - W, B)$ *is critical in* Ω *with ground state* $u\psi_w(G_\varphi/u)$.
3. *Under one of further assumptions of the above theorem,* W *is an optimal Hardy-weight for* (P, B) *in* Ω.

Optimal Hardy-weights for the Dirichlet Laplacian on $\mathbb{R}_+$

Proposition

Let $0 \lneqq w \in L^1_{\rm loc}(\mathbb{R}_+)$. Then w is an optimal Hardy-weight for the Dirichlet Laplacian on $\mathbb{R}_+$ with a corresponding ground state ψ_w if and only if the following three conditions are satisfied.

1. $\psi_w > 0$ *satisfies* $-\psi_w'' - w\psi_w = 0$ *in* $\mathbb{R}_+$,
2. $\displaystyle\int_0^1 \frac{1}{\psi_w^2}\,\mathrm{d}t = \int_1^\infty \frac{1}{\psi_w^2}\,\mathrm{d}t = \infty$,
3. $\displaystyle\int_0^1 \psi_w^2\, w\,\mathrm{d}t = \int_1^\infty \psi_w^2\, w\,\mathrm{d}t = \infty$.

Example

Under the assumptions on u and G_φ, let

$$0 \le a \le \frac{1}{\sup\limits_{\Omega} (G_\varphi/u)}, \quad w(t) := (2t - at^2)^{-2}, \quad \psi_w(t) := \sqrt{2t - at^2}.$$

(w and ψ_w are related to Ermakov-Pinney equation $-y'' = \frac{1}{y^3}$.)
Then

$$W := \frac{P\left(u\psi_w\left(G_\varphi/u\right)\right)}{u\psi_w\left(G_\varphi/u\right)} \quad \Big(\text{at } \infty \; W = |\nabla(G_\varphi/u)|_A^2 \, w(G_\varphi/u) \Big).$$

is an optimal Hardy weight which is larger at infinity than the "Classical" Hardy-weight $W = \frac{|\nabla(G_\varphi/u)|_A^2}{4(G_\varphi/u)^2}$.

Example (half ball or half space)

Let $n \geq 3$, and either

$$\Omega = B_1^+(0), \partial\Omega_{\mathrm{Rob}} = \{x \in B_1(0) \,|\, x_n = 0\}; \text{ or } \Omega = \mathbb{R}^n_+, \partial\Omega_{\mathrm{Rob}} = \{x \in \mathbb{R}^n \,|\, x_n = 0\}.$$

$$Pu := -\Delta u \quad \text{in } \Omega, \qquad Bu = \nabla u \cdot \vec{n} \text{ on } \partial\Omega_{\mathrm{Rob}}.$$

Taking $u = 1$ and the explicit Green functions $G^{\Omega}_{P,B}$ given by Schwarz reflection principle, we get an optimal Hardy-weight $W = P(G_\varphi^{1/2})/G_\varphi^{1/2}$.

For $\Omega = B_1^+(0)$, $W(x) \sim (2 \cdot \mathrm{dist}(x, \partial\Omega_{\mathrm{Dir}}))^{-2}$ as $x \to \xi$, where $\xi_n > 0$ and $|\xi| = 1$.

For $\Omega = \mathbb{R}^n_+$, $W(x) \sim \frac{(n-2)^2}{4}|x|^{-2}$ as $x \to \infty$ such that $x/|x| \to (\xi', \xi_n)$ with $\xi_n > 0$.

Example (exterior of the unit ball)

Let $n \geq 3$, and $\Omega = \{x \in \mathbb{R}^n \mid |x| > 1\}$ with $\partial\Omega_{\mathrm{Rob}} = \partial\Omega$. Assume that $Pu = -\Delta u$ and $Bu = \nabla u \cdot \vec{n} + \gamma(x) u$ on $\partial\Omega_{\mathrm{Rob}}$, where $\gamma \in L^\infty(\partial\Omega_{\mathrm{Rob}})$ satisfies $\gamma > (1-n)/2$, and take $\varepsilon > 0$ such that $\varepsilon(n + 2\gamma - 1) \geq 1$ on $\partial\Omega_{\mathrm{Rob}}$. Then,

$$v := \sqrt{(|x| - 1 + \varepsilon)|x|^{1-n}} \text{ satisfies}$$

$$\begin{cases} -\Delta v - \dfrac{(n-1)(n-3)v}{4|x|^2} - \dfrac{v}{4(|x| - 1 + \varepsilon)^2} = 0 & \text{in } \Omega, \\[2ex] \nabla v \cdot \vec{n} + \gamma v = \dfrac{-1 + \varepsilon(n + 2\gamma - 1)}{2\sqrt{\varepsilon}} \geq 0 & \text{on } \partial\Omega_{\mathrm{Rob}}. \end{cases}$$

Hence, the AAP theorem implies the Hardy-type inequality in $H^1(\Omega)$

$$\int_\Omega |\nabla\phi|^2 \mathrm{d}x + \int_{\partial\Omega_{\mathrm{Rob}}} \gamma\phi^2 \mathrm{d}\sigma \geq \int_\Omega \left[\frac{(n-1)(n-3)}{4|x|^2} + \frac{1}{4(|x| - 1 + \varepsilon)^2}\right] \phi^2 \mathrm{d}x.$$

Example (Continued)

Let's compare our result with [Kovařík-Laptev (2012)], where $\gamma \geq 0$ is constant and $\varepsilon = (2\gamma)^{-1}$. Instead, let $\varepsilon_\gamma := (n - 1 + 2\gamma)^{-1}$, we obtain an improvement of the Hardy inequality in [Kovařík-Laptev (2012)]. In particular, the function $v_\gamma := \sqrt{(|x| - 1 + \varepsilon_\gamma)|x|^{1-n}}$ is a positive solution of the equation

$$\begin{cases} -\Delta v - \dfrac{v(n-1)(n-3)}{4|x|^2} - \dfrac{v}{4(|x| - 1 + \varepsilon_\gamma)^2} = 0 & \text{in } \Omega, \\ \nabla v \cdot \vec{n} + \gamma v = 0 & \text{on } \partial\Omega_{\mathrm{Rob}}. \end{cases}$$

It follows that v_γ is a ground state and

$$W := \frac{(n-1)(n-3)}{4|x|^2} + \frac{1}{4(|x| - 1 + \varepsilon_\gamma)^2}$$

is an optimal Hardy-weight of (P, B) in Ω.

Thank you for your attention!