L^p-Bounds for Eigenfunctions of Analytic Non Self-Adjoint Operators with Double Characteristics

Francis White

University of Paris XIII (LAGA)

Banff International Research Station
July 11th, 2022

Overview

Table of Contents

Pseudodifferential Operators with Double-Characteristics

- Let $0 < h \leq 1$ be a semiclassical parameter (Planck's constant).
- If $a = a(x, \xi; h)$ is a symbol on the classical phase space $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_\xi$, we denote the semiclassical Weyl quantization of a on $\mathbb{R}^n$ by

$$\mathrm{Op}_h^w(a)u(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}(x-y)\cdot\xi} a\left(\frac{x+y}{2}, \xi; h\right) u(y)\, dy\, d\xi.$$

- Consider semiclassical pseudodifferential operators on $\mathbb{R}^n$ of the form

$$P(h) = \mathrm{Op}_h^w(p_0 + hp_1),$$

  where $p_0, p_1 \in \mathbb{C}^\infty(\mathbb{R}^{2n})$ belong to a suitable symbol class, p_0 is independent of h, and $p_1 = p_1(x, \xi; h)$ is a well-behaved h-dependent subprincipal part.
- Assume that
  - $\mathbf{Re}\, p_0 \geq 0$ **with** $(\mathbf{Re}\, p_0)^{-1}(0) = \{0\}$**, and**
  - $\mathbf{Im}\, p_0(0) = \nabla(\mathbf{Im}\, p_0)(0) = 0$**.**
- Note that $\nabla(\mathrm{Re}\, p_0)(0) = 0$ and hence $p_0(0) = \nabla p_0(0) = 0$.
- *Example*: $P(h) = -h^2\Delta + V(x)$, where $V \in C^\infty(\mathbb{R}^n)$ is such that $\mathrm{Re}\, V \geq 0$ with $(\mathrm{Re}\, V)^{-1}(0) = \{0\}$ and $\mathrm{Im}\, V(0) = \nabla(\mathrm{Im}\, V)(0) = 0$.

Symbol Classes and $P(h)$ as an Unbounded Operator

- A measurable function $m : \mathbb{R}^{2n} \to (0, \infty)$ is said to be an **order function** if $\exists C > 0$, $\exists N \in \mathbb{R}$, such that $m(X) \leq C\langle X - Y\rangle^N m(Y)$ for all $X, Y \in \mathbb{R}^{2n}$. Here $\langle X\rangle := (1 + |X|^2)^{1/2}$.
- Associated to m is the symbol class:

$$S(m) = \{a \in C^\infty(\mathbb{R}^{2n}) : \forall\alpha \in \mathbb{N}^{2n},\ \exists C = C_\alpha > 0 \text{ such that } |\partial_X^\alpha a(X)| \leq Cm(X) \text{ for all } X \in \mathbb{R}^{2n}\}.$$

- We assume that there exists an order function m on $\mathbb{R}^{2n}$ with $m \geq 1$ and $m \in S(m)$ such that $p_0, p_1 \in S(m)$.
- We also assume that $\operatorname{Re} p_0$ is elliptic at infinity in the sense that $\exists C > 0$ such that

$$\operatorname{Re} p_0(X) \geq \frac{1}{C} m(X), \quad |X| \geq C.$$

- We may view $P(h)$ as a closed, unbounded operator on $L^2(\mathbb{R}^n)$ with domain the semiclassical Sobolev space

$$\mathcal{D}(P(h)) = H_h(m) := \operatorname{Op}_h^w(m)^{-1}\left(L^2(\mathbb{R}^n)\right).$$

Low-Lying Eigenvalues and Eigenfunctions

- For $0 < \epsilon \ll 1$ sufficiently small,
$$\mathrm{Spec}(P(h)) \cap \{\mathrm{Re}\, z < \epsilon\}$$
is discrete consisting entirely of eigenvalues. By Gårding's inequality, there is $C > 0$ such that
$$\mathrm{Spec}(P(h)) \cap \{\mathrm{Re}\, z < \epsilon\} = \mathrm{Spec}(P(h)) \cap \{-Ch < \mathrm{Re}\, z < \epsilon\}.$$
- We say $z(h) \in \mathbb{C}$ is a **low-lying eigenvalue** of $P(h)$ if $\exists C > 0$ such that
$$z(h) \in \mathrm{Spec}(P(h)), \quad |z(h)| \leq Ch, \quad 0 < h \ll 1.$$
- *Hitrik-Pravda-Starov '13*: If $p_1 \sim \Sigma_{j=0}^{\infty} h^j p_{1,j}$ in $S(m)$ and the quadratic approximation q to p_0 at $0 \in \mathbb{R}^{2n}$ satisfies a partial ellipticity condition, then $\exists$ a complete semiclassical asymptotic expansion for the low-lying eigenvalues $z(h)$ of $P(h)$.
- Not as much is known about the corresponding low-lying eigenfunctions! For example, it is unknown if they possess WKB expansions.
- In this talk: discuss the problem of obtaining optimal L^p-bounds for low-lying eigenfunctions in the case p_0 and p_1 extend holomorphically to neighborhood of $\mathbb{R}^{2n}$ in $\mathbb{C}^{2n}$.

The Singular Space of the Quadratic Approximation to p_0

- Let
$$q(X) = \frac{1}{2} p_0''(0) X \cdot X, \quad X \in \mathbb{R}^{2n},$$
be the quadratic approximation to p_0 at $0 \in \mathbb{R}^{2n}$.
- Note that $\mathrm{Re}\, p_0 \geq 0 \implies \mathrm{Re}\, q \geq 0$.
- There exists a unique $F \in \mathrm{M}_{2n \times 2n}(\mathbb{C})$, called the **Hamilton matrix** of q, such that
$$q(X; Y) = \sigma(X, FY), \quad X, Y \in \mathbb{R}^{2n}.$$
Here $q(\cdot\,; \cdot)$ denotes the unique $\mathbb{C}$-bilinear polarization of q and $\sigma = d\xi \wedge dx$.

Definition (Singular Space of q)

Let q be a complex-valued quadratic form on $\mathbb{R}^{2n}$ with non-negative real part $Re\, q \geq 0$ and let F be the Hamilton matrix of q. The **singular space** of q is

$$S = \bigcap_{j=0}^{2n-1} \ker \left[(\mathrm{Re}\, F)(\mathrm{Im}\, F)^j \right] \cap \mathbb{R}^{2n}.$$

- *Hitrik-Pravda-Starov '08*: Let $q = q(x,\xi)$ be a complex-valued quadratic form on $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_\xi$ with $\mathrm{Re}\, q \geq 0$. If q is elliptic along its singular space S, i.e.

$$q(X) = 0, \quad X \in S \implies X = 0,$$

then the quadratic differential operator $\mathrm{Op}^w_1(q)$, viewed as an unbounded operator on $L^2(\mathbb{R}^n)$ equipped with its maximal domain, has a discrete spectrum consisting entirely of eigenvalues of finite algebraic multiplicity. Furthermore,

$$\mathrm{Spec}(\mathrm{Op}^w_1(q)) = \left\{ \sum_{\substack{\lambda \in \mathrm{Spec}(F) \\ -i\lambda \in \mathbb{C}_+ \cup \Sigma(q|_S) \setminus \{0\}}} (r_\lambda + 2k_\lambda)(-i\lambda) : k_\lambda \in \mathbb{N} \right\},$$

where F is the Hamilton matrix of q, r_λ is the dimension of the space of generalized eigenvectors of F belonging to the eigenvalue $\lambda \in \mathbb{C}$, and

$$\Sigma(q|_S) = \overline{q(S)} \text{ and } \mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}.$$

Spectral Results for Operators with Double Characteristics

- *Hitrik-Pravda-Starov '13*: Let $P(h) = \mathrm{Op}_h^w(p_0 + hp_1)$, where p_0 and p_1 are as before, and assume

$$p_1 \sim \sum_{j=1}^{\infty} h^j p_{1,j} \text{ in } S(m),$$

for some $p_{1,j} \in S(m)$, $j \in \mathbb{N}$. If the quadratic approximation

$$q(X) = \frac{1}{2} p_0''(0) X \cdot X, \quad X \in \mathbb{R}^{2n},$$

to p_0 at $0 \in \mathbb{R}^{2n}$ is elliptic along its singular space S, then, for any $C > 0$, there exists $h_0 > 0$ such that for all $0 < h \leq h_0$ the spectrum of $P(h)$ in $D(0, Ch)$ is given by eigenvalues of the form

$$z_k \sim h(\lambda_k + p_{1,0}(0) + h^{1/N_k}\lambda_{k,1} + h^{2/N_k}\lambda_{k,2} + \cdots),$$

where λ_k are the eigenvalues of $\mathrm{Op}_1^w(q)$ in $D(0, C)$, and N_k is the dimension of the space of generalized eigenvectors of $\mathrm{Op}_1^w(q)$ corresponding to $\lambda_k \in \mathbb{C}$.

Precise Assumptions on the Symbol of $P(h)$

- If m is an order function on $\mathbb{R}^{2n}$, we define $S_{\mathrm{Hol}}(m)$ as the set of all $a : \mathbb{R}^{2n} \times (0,1]_h \to \mathbb{C}$ for which there exists a bounded open neighborhood W of 0 in $\mathbb{C}^{2n}$ and a function $\tilde{a} : (\mathbb{R}^{2n} + W) \times (0,1]_h \to \mathbb{C}$ extending a such that

$$\tilde{a}(\cdot; h) \in \mathrm{Hol}(\mathbb{R}^{2n} + W), \quad 0 < h \leq 1,$$

  and

$$\exists C > 0 : |\tilde{a}(Z; h)| \leq C m(\mathrm{Re}\, Z), \quad Z \in \mathbb{R}^{2n} + W.$$

- Regarding the symbols p_0 and p_1, we assume that:
  - $\exists$an order function m on $\mathbb{R}^{2n}$ with $m \geq 1$ and $m \in S(m)$ such that $p_0, p_1 \in S_{\mathrm{Hol}}(m)$,
  - p_0 is independent of h,
  - $\mathrm{Re}\, p_0 \geq 0$ with $(\mathrm{Re}\, p_0)^{-1}(0) = \{0\}$,
  - $\mathrm{Im}\, p_0(0) = \nabla(\mathrm{Im}\, p_0)(0) = 0$,
  - $\exists C, c > 0$: $\mathrm{Re}\, p_0(X) \geq c m(X)$ whenever $|X| \geq C$.

L^p-Bounds for Low-Lying EF's in the Analytic Case

Theorem (White '21)

Let $P(h) = Op_h^w(p_0 + hp_1)$, where p_0 and p_1 are as above, and suppose that $u(h) \in L^2(\mathbb{R}^n)$, $0 < h \le 1$, is such that

$$\begin{cases} P(h)u(h) = 0, \\ \|u(h)\|_{L^2} = 1, \end{cases} \qquad 0 < h \le 1.$$

If the quadratic approximation q to p_0 at $0 \in \mathbb{R}^{2n}$ is elliptic along its singular space S, then there exists $0 < h_0 \le 1$ such that for every $1 \le p \le \infty$ there is $C > 0$ such that

$$\|u(h)\|_{L^p} \le C h^{\frac{n}{2p} - \frac{n}{4}}, \quad 0 < h \le h_0. \tag{1}$$

- Partially extends the work of Krupchyk-Uhlmann '18, which established the bounds (1) for $2 \le p \le \infty$ when $p_0, p_1 \in C^\infty(\mathbb{R}^{2n})$ and $\mathrm{Re}\, q > 0$.
- The bounds (1) are saturated by the eigenfunctions of $P(h) = -h^2\Delta + |x|^2$, e.g. $(P(h) - nh)u(h) = 0$ for $u(h) = h^{-n/4} e^{-x^2/2h}$, $\|u(h)\|_{L^p} = Ch^{\frac{n}{2p} - \frac{n}{4}}$.

Schrödinger Operators with Holomorphic Potentials

- Let

$$P(h) := -h^2\Delta + V(x) \ \text{ on } \ \mathbb{R}^n,$$

where $V \in C^\omega(\mathbb{R}^n)$ satisfies
  - $\mathrm{Re}\, V \geq 0$ with $(Re\, V)^{-1}(0) = \{0\}$,
  - $(\mathrm{Im}\, V)(0) = \nabla(\mathrm{Im}\, V)(0) = 0$,
  - $\det V''(0) \neq 0$,
  - $\exists s \geq 0$ such that

  $$\mathrm{Re}\, V(x) \geq \frac{1}{C}\,|x|^s\,, \quad |x| \geq C,$$

  for some $C > 0$, and
  - $\exists$a holomorphic extension $\tilde{V} \in \mathrm{Hol}(\mathbb{R}^n + i(-\epsilon, \epsilon)^n)$ of V such that

  $$\left|\tilde{V}(z)\right| \leq C\langle \mathrm{Re}\, z\rangle^s.$$

  for some $C > 0$.
- *Hitrik-Bellis '18*: $q(x,\xi) = |\xi|^2 + \frac{1}{2}V''(0)x \cdot x$ has trivial singular space.
- Any low-lying EF $u(h)$ of $P(h)$ satisfies $\|u(h)\|_{L^p} \leq Ch^{\frac{n}{2p}-\frac{n}{4}}$, $1 \leq p \leq \infty$.

Table of Contents

Semiclassical FBI Transforms on $\mathbb{R}^n$

Definition (Fourier-Bros-Iagolnitzer (FBI) transforms)

An **FBI phase function** is a holomorphic quadratic form $\varphi = \varphi(z, y)$ on $\mathbb{C}^{2n} = \mathbb{C}^n_z \times \mathbb{C}^n_y$ such that

$$\det \varphi''_{zy} \neq 0, \quad \mathrm{Im}\, \varphi''_{yy} > 0.$$

The **semiclassical FBI transform** associated to an FBI phase function φ is the linear transformation $\mathcal{T}_\varphi : \mathcal{S}'(\mathbb{R}^n) \to \mathrm{Hol}(\mathbb{C}^n)$ given by

$$\mathcal{T}_\varphi u(z) = c_\varphi h^{-\frac{3n}{4}} \int_{\mathbb{R}^n} e^{\frac{i}{h}\varphi(z,y)} u(y)\, dy, \quad u \in \mathcal{S}'(\mathbb{R}^n),$$

where $c_\varphi = 2^{-n/2}\pi^{-3n/4}(\det \mathrm{Im}\, \partial^2_{yy}\varphi)^{-1/4} \left|\det \partial^2_{yz}\varphi\right|$.

- $\mathcal{T}_\varphi$ is unitary $L^2(\mathbb{R}^n) \to L^2(\mathbb{C}^n, e^{-2\Phi(z)/h} L(dz)) \cap \mathrm{Hol}(\mathbb{C}^n)$, where $L(dz)$ is the Lebesgue measure on $\mathbb{C}^n$, and $\Phi(z) = \max_{y \in \mathbb{R}^n}(-\mathrm{Im}\, \varphi(z, y))$, $z \in \mathbb{C}^n$.
- The weight Φ is a strictly plurisubharmonic quadratic form, i.e. $\partial^2_{\bar{z}z}\Phi > 0$.

Bounding $\|u\|_{L^p}$

- Since $\mathrm{Op}_h^w(p_0 + hp_1)u = 0$ where $(\mathrm{Re}\, p_0)^{-1}(0) = \{0\}$ and $\mathrm{Re}\, p_0$ is elliptic at infinity, we have

$$\int_{|z|\geq\delta} |\mathcal{T}_\varphi u(z)|\, e^{-\frac{\Phi(z)}{h}}\, L(dz) = \mathcal{O}(h^\infty) \text{ for any } \delta > 0.$$

- Because $\mathcal{T}_\varphi$ is unitary, we have

$$u(x) = \mathcal{T}_\varphi^* \mathcal{T}_\varphi u(x) = c_\varphi h^{-\frac{3n}{4}} \int_{\mathbb{C}^n} e^{-\frac{i}{h}\overline{\varphi(z,x)}} \mathcal{T}_\varphi u(z) e^{-\frac{2}{h}\Phi(z)}\, L(dz), \quad x \in \mathbb{R}^n.$$

  By the triangle inequality,

$$|u(x)| \leq c_\varphi h^{-\frac{3n}{4}} \int_{\mathbb{C}^n} e^{-\frac{c}{h}|x-x(z)|^2}\, |\mathcal{T}_\varphi u(z)|\, e^{-\frac{\Phi(z)}{h}}\, L(dz), \quad x \in \mathbb{R}^n,$$

  where $x(z) \in \mathbb{R}^n$ is an $\mathbb{R}$-linear function of $z \in \mathbb{C}^n$.

- Thus, for any $1 \leq p \leq \infty$, $\delta > 0$, and $N \geq 0$, there is $C > 0$ such that

$$\begin{aligned}\|u\|_{L^p} &\leq c_\varphi h^{-\frac{3n}{4}} \int_{\mathbb{C}^n} \|e^{-\frac{c}{h}|(\cdot)-x(z)|^2}\|_{L^p}\, |\mathcal{T}_\varphi u(z)|\, e^{-\frac{\Phi(z)}{h}}\, L(dz) \\ &\leq C h^{\frac{n}{2p}-\frac{3n}{4}} \int_{|z|<\delta} |\mathcal{T}_\varphi u(z)|\, e^{-\frac{\Phi(z)}{h}}\, L(dz) + Ch^N.\end{aligned}$$

Bounding $\|u\|_{L^p}$ Cont'd

- After making a prudent choice of the FBI phase φ, we can show that there exists a strictly plurisubharmonic function $\Phi^* \in C^\omega(\mathrm{neigh}(0;\mathbb{C}^n);\mathbb{R})$ and $\delta > 0$ and $c > 0$ such that

$$\|\mathcal{T}_\varphi u\|^2_{H_{\Phi^*}(\{|z|<\delta\})} := \int_{|z|<\delta} |\mathcal{T}_\varphi u(z)|^2 e^{-2\Phi^*(z)/h} L(dz) = \mathcal{O}(1), \quad h \to 0^+,$$

and

$$\Phi(z) - \Phi^*(z) \geq c\,|z|^2, \quad |z| < \delta.$$

- Consequently, if $N \gg 1$ is taken sufficiently large, we have

$$\begin{aligned}
\|u\|_{L^p} &\leq Ch^{\frac{n}{2p}-\frac{3n}{4}} \int_{|z|<\delta} |\mathcal{T}_\varphi u(z)|\, e^{-\Phi(z)/h} L(dz) + Ch^N \\
&\leq Ch^{\frac{n}{2p}-\frac{3n}{4}} \int_{|z|<\delta} |\mathcal{T}_\varphi u(z)|\, e^{-\Phi^*(z)/h} e^{-(\Phi(z)-\Phi^*(z))/h} L(dz) + Ch^N \\
&\leq Ch^{\frac{n}{2p}-\frac{3n}{4}} \int_{|z|<\delta} |\mathcal{T}_\varphi u(z)|\, e^{-\Phi^*(z)/h} e^{-c|z|^2/h} L(dz) + Ch^N \\
&\leq C\|\mathcal{T}_\varphi u\|_{H_{\Phi^*}(\{|z|<\delta\})} h^{\frac{n}{2p}-\frac{n}{4}} + Ch^N = \mathcal{O}(1) h^{\frac{n}{2p}-\frac{n}{4}}.
\end{aligned}$$

Constructing the Weight Φ^*

- Let $\kappa_\varphi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be the complex linear canonical transformation given implicitly by $\kappa_\varphi : (y, -\partial_y \varphi(z,y)) \to (z, \partial_z \varphi(z,y)), \quad (z,y) \in \mathbb{C}^{2n}$.
- Let $\Lambda_\Phi = \mathrm{graph}\left(\frac{2}{i}\partial_z \Phi\right) = \kappa_\varphi(\mathbb{R}^{2n}) \subset \mathbb{C}^{2n}$ and let $p_0 = p_0 \circ \kappa_\varphi^{-1} \in \mathrm{Hol}(\Lambda_\Phi + W)$, where W is a small open neighborhood of 0 in $\mathbb{C}^{2n}$.
- Write $\mathbb{C}^{2n} = \mathbb{C}^n_z \times \mathbb{C}^n_\zeta$, let $H_{p_0} = \partial_\zeta p_0 \cdot \partial_z - \partial_z p_0 \cdot \partial_\zeta$ be the complex Hamilton vector field of p_0, and let

$$\kappa_t(Z) = \exp\left(1\widehat{H_{tp_0}}\right), \quad Z \in \Lambda_\Phi + W, \quad t \in \mathbb{C},$$

  where $\widehat{H_{tp_0}} = H_{tp_0} + \overline{H_{tp_0}}$, be the complex time Hamilton flow of p_0.
- If U is a small open neighborhood of 0 in $\mathbb{C}^n$, then

$$\kappa_t(\Lambda_\Phi) \cap U \times U = \Lambda_{\Phi_t} := \mathrm{graph}\left(\frac{2}{i}\partial_z \Phi_t\right), \quad |t| \ll 1,$$

  where $(\Phi_t)_{|t| \ll 1}$ is a family of strictly plurisubharmonic functions defined in a neighborhood of $0 \in \mathbb{C}^n$, depending analytically on t.
- We can find $t_0 \in \mathbb{C}$ with $0 < |t_0| \ll 1$ so that $\Phi^* := \Phi_{t_0}$ has the desired properties.

Thank you for your attention!