The diamagnetic inequality for the Dirichlet-to-Neumann operator

Tom ter Elst

University of Auckland

Joint work with El Maati Ouhabaz

13-7-2022

The diamagnetic inequality for Laplacian

Let $\vec{a} = (a_1, \ldots, a_d)$ with $a_k \in L_{2,\mathrm{loc}}(\mathbb{R}^d)$ for all $k \in \{1, \ldots, d\}$.
Set $H(\vec{a}) = (\nabla - i\vec{a})^*(\nabla - i\vec{a})$.
Then

$$|e^{-tH(\vec{a})} f| \leq e^{t\Delta} |f|$$

for all $t > 0$ and $f \in L_2(\mathbb{R}^d)$.

The same result holds in presence of a real-valued potential V, i.e., with operators $H(\vec{a}) + V$ and $-\Delta + V$.

The setting (1)

Let $\Omega \subset \mathbb{R}^d$ bounded open with Lipschitz boundary Γ.
Let $c_{kl}, b_k, c_k, a_0 \in L_\infty(\Omega, \mathbb{R})$ for all $k, l \in \{1, \ldots, d\}$.
Ellipticity condition: there exists a $\mu > 0$ such that

$$\mathrm{Re} \sum_{k,l=1}^{d} c_{kl}(x)\, \xi_k\, \overline{\xi_l} \geq \mu\, |\xi|^2$$

for all $\xi \in \mathbb{C}^d$ and almost every $x \in \Omega$.

The setting (2)

Consider form $\mathfrak{a}\colon W^{1,2}(\Omega) \times W^{1,2}(\Omega) \to \mathbb{C}$

$$\mathfrak{a}(u,v) = \sum_{k,l=1}^{d} \int_\Omega c_{kl}\,(\partial_l u)\,\overline{\partial_k v} + \sum_{k=1}^{d} \int_\Omega \Big(b_k\, u\, \overline{\partial_k v} \Big) + c_k\,(\partial_k u)\,\overline{v} + \int_\Omega a_0\, u\, \overline{v}.$$

Define $\mathcal{A}\colon W^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ by

$$\langle \mathcal{A}u, v\rangle_{W^{-1,2}(\Omega)\times W_0^{1,2}(\Omega)} = \mathfrak{a}(u,v).$$

Let $\psi \in L_2(\Gamma)$ and $u \in W^{1,2}(\Omega)$ with $\mathcal{A}u \in L_2(\Omega)$.
Definition: u has **weak conormal derivative** ψ if

$$\mathfrak{a}(u,v) - (\mathcal{A}u, v)_{L_2(\Omega)} = (\psi, \operatorname{Tr} v)_{L_2(\Gamma)} \quad \text{for all } v \in W^{1,2}(\Omega).$$

Notation $\partial_\nu^{\mathfrak{a}} u = \psi$.

The Dirichlet-to-Neumann operator $\mathcal{N}$

Assumption: 0 is not a Dirichlet eigenvalue.

Definition: A function $u \in W^{1,2}(\Omega)$ is called $\mathcal{A}$-**harmonic** if

$$\mathfrak{a}(u, v) = 0 \quad \text{for all } v \in W^{1,2}(\Omega).$$

For all $\varphi \in H^{1/2}(\Omega)$ there is a unique $\mathcal{A}$-harmonic $u \in W^{1,2}(\Omega)$ such that $\operatorname{Tr} u = \varphi$.

IF u has a weak conormal derivative, then we say

$$\varphi \in D(\mathcal{N}) \text{ and } \mathcal{N}\varphi = \partial_\nu^{\mathfrak{a}} u.$$

The operator $-\mathcal{N}$ is the generator of a C_0-semigroup.

Two form methods

Define $\mathfrak{b}\colon H^{1/2}(\Gamma) \times H^{1/2}(\Gamma) \to \mathbb{C}$ by

$$\mathfrak{b}(\varphi, \xi) := \mathfrak{a}(u, v),$$

where $u, v \in W^{1,2}(\Omega)$ are $\mathcal{A}$-harmonic with $\operatorname{Tr} u = \varphi$ and $\operatorname{Tr} v = \xi$, respectively.

Then $\mathfrak{b}$ is a densely defined continuous elliptic form and $\mathcal{N}$ is the associated operator.

The operator $\mathcal{N}$ is the operator associated with $\mathfrak{b}$ in the following sense: Let $\varphi, \psi \in L_2(\Gamma)$. Then $\varphi \in D(\mathcal{N})$ and $\mathcal{N}\varphi = \psi$ if and only if $\varphi \in D(\mathfrak{b})$ and

$$\mathfrak{b}(\varphi, \xi) = (\psi, \xi)_{L_2(\Gamma)} \quad \text{for all } \xi \in D(\mathfrak{b}).$$

Second form method

Let V and H be Hilbert spaces.
Let $\mathfrak{a} \colon V \times V \to \mathbb{C}$ be a continuous sesquilinear form.
Let $j \colon V \to H$ be a continuous operator with dense range.
Suppose $\mathfrak{a}$ is j-elliptic, that is, there are $\mu > 0$ and $\omega \in \mathbb{R}$ such that

$$\operatorname{Re} \mathfrak{a}(u,u) + \omega \, \|j(u)\|_H^2 \geq \mu \, \|u\|_V^2 \quad \text{for all } u \in V.$$

The operator A **associated with** $(\mathfrak{a}, j)$ is defined as follows:
Let $x, f \in H$. Then $x \in D(A)$ and $Ax = f$ if and only if there exists a $u \in V$ such that $j(u) = x$ and

$$\mathfrak{a}(u,v) = (f, j(v))_H \quad \text{for all } v \in V.$$

Theorem (Arendt–tE). The operator A is well defined and $-A$ is the generator of a holomorphic C_0-semigroup in H.

In our case, if $\mathfrak{a}$ is Tr-elliptic, then the operator $\mathcal{N}$ is the operator associated with $(\mathfrak{a}, \operatorname{Tr})$.

The magnetic Dirichlet-to-Neumann operator $\mathcal{N}(\vec{a})$

Let $\vec{a} := (a_1, \ldots, a_d)$ with $a_k \in L_\infty(\Omega, \mathbb{R})$ for all $k \in \{1, \ldots, d\}$. Set

$$D_k := \partial_k - i a_k$$

Consider form $\mathfrak{a}(\vec{a})\colon W^{1,2}(\Omega) \times W^{1,2}(\Omega) \to \mathbb{C}$

$$\mathfrak{a}(\vec{a})(u,v) = \sum_{k,l=1}^{d} \int_\Omega c_{kl}\,(D_l u)\,\overline{D_k v} + \sum_{k=1}^{d} \int_\Omega \Big(b_k\,u\,\overline{D_k v}\Big) + c_k\,(D_k u)\,\overline{v} + \int_\Omega a_0\,u\,\overline{v}.$$

Assumption: 0 is not a Dirichlet eigenvalue.

Define similarly that an element of $W^{1,2}(\Omega)$ is $\mathcal{A}(\vec{a})$-harmonic and the magnetic Dirichlet-to-Neumann operator $\mathcal{N}(\vec{a})$.

Formally, if $u \in D(\mathcal{N}(\vec{a}))$ is $\mathcal{A}(\vec{a})$-harmonic with trace $\operatorname{Tr} u = \varphi$, then

$$\mathcal{N}(\vec{a})\varphi = \partial_\nu^{\mathfrak{a}(\vec{a})}\, u = \sum_{k,l=1}^{d} \nu_k \operatorname{Tr}(c_{kl}\,\partial_l u) - i \sum_{k,l=1}^{d} \nu_k \operatorname{Tr}(c_{kl}\,a_l\,u) + \sum_{k=1}^{d} \nu_k \operatorname{Tr}(b_k\,u)$$

The diamagnetic inequality

Let $T_{\vec{a}} = (T_{\vec{a}}(t))_{t>0}$ and $T = (T(t))_{t>0}$ be the semigroups generated by $-\mathcal{N}(\vec{a})$ and $-\mathcal{N}$ on $L_2(\Gamma)$, respectively.

Theorem (tE–Ouhabaz). Suppose $\mathfrak{a}$ is accretive and there exist $\mu, \omega > 0$ such that

$$\operatorname{Re} \mathfrak{a}(u,u) + \omega \, \|\operatorname{Tr} u\|_{L_2(\Gamma)}^2 \geq \mu \, \|u\|_{W^{1,2}(\Omega)}^2 \quad \text{for all } u \in W^{1,2}(\Omega).$$

Then

$$|T_{\vec{a}}(t)\varphi| \leq T(t)|\varphi|$$

for all $t > 0$ and $\varphi \in L_2(\Gamma)$.

Kernel bounds

Suppose Ω is of class $C^{1+\kappa}$ for some $\kappa > 0$.
Suppose also that $c_{kl} = c_{lk} \in C^{\kappa}(\Omega, \mathbb{R})$, $b_k = c_k = 0$ and $a_k \in L_\infty(\Omega, \mathbb{R})$ for all $k, l \in \{1, \ldots, d\}$.
Suppose $a_0 \geq 0$ a.e. on Ω.
Then $T_{\vec{a}}$ has a kernel $K_{\vec{a}}$ and there exists a constant $c > 0$ such that

$$|K_{\vec{a}}(t, z, w)| \leq \frac{c\,(t \wedge 1)^{-(d-1)}\, e^{-\lambda_1 t}}{\Big(1 + \dfrac{|z - w|}{t}\Big)^d}$$

for all $z, w \in \Gamma$ and $t > 0$, where λ_1 is the first eigenvalue of the operator $\mathcal{N}(\vec{a})$.

Hölder continuous kernel bounds

Same assumptions. In addition suppose that $d \geq 3$.
Then for all $\varepsilon, \tau' \in (0,1)$, $\tau > 0$ there exist $c, \nu > 0$ such that

$$|K_{\vec{a}}(t,z,w) - K_{\vec{a}}(t,z',w')|$$

$$\leq c\,(t \wedge 1)^{-(d-1)} \left(\frac{|z-z'| + |w-w'|}{t + |z-w|}\right)^{\nu} \frac{1}{\left(1 + \frac{|z-w|}{t}\right)^{d-\varepsilon}} \,(1+t)^{\nu}\, e^{-\lambda_1 t}$$

for all $z, w, z', w' \in \Gamma$ and $t > 0$ with $|z-z'| + |w-w'| \leq \tau\, t + \tau'\, |z-w|$.

Sketch of proof

The diamagnetic inequality is obtained by proving the invariance of the closed convex set

$$\{(\varphi, \psi) \in L_2(\Gamma) \times L_2(\Gamma) : |\varphi| \leq \psi\}$$

for the semigroup

$$\begin{pmatrix} T_{\vec{a}}(t) & 0 \\ 0 & T(t) \end{pmatrix}_{t>0}.$$

Invariance of closed convex sets

Let V and $\widetilde{H}$ be Hilbert spaces with V densely and continuously embedded in $\widetilde{H}$.
Let $\mathfrak{a}\colon V \times V \to \mathbb{C}$ be a continuous accretive sesquilinear form.
Suppose $\mathfrak{a}$ is elliptic, that is i-elliptic, where i is the inclusion map.
Let $\widetilde{S}$ be the associated semigroup.
Let $\widetilde{C} \subset \widetilde{H}$ be a non-empty closed convex set and let $\widetilde{P}\colon \widetilde{H} \to \widetilde{C}$ be the projection.

Theorem (Ouhabaz). The following are equivalent.

- $\widetilde{C}$ is invariant under $\widetilde{S}$, that is $\widetilde{S}_t \widetilde{C} \subset \widetilde{C}$ for all $t > 0$.
- $\widetilde{P} V \subset V$ and $\operatorname{Re} \mathfrak{a}(\widetilde{P}u, u - \widetilde{P}u) \geq 0$ for all $u \in V$.
- $\widetilde{P} V \subset V$ and $\operatorname{Re} \mathfrak{a}(u, u - \widetilde{P}u) \geq 0$ for all $u \in V$.

Invariance of closed convex sets

Let V and H be Hilbert spaces.
Let $\mathfrak{a}\colon V \times V \to \mathbb{C}$ be a continuous sesquilinear form.
Let $j\colon V \to H$ be a continuous operator with dense range.
Suppose $\mathfrak{a}$ is j-elliptic and accretive.
Let S be the semigroup associated with $(\mathfrak{a}, j)$.
Let $C \subset H$ be a non-empty closed convex set and let $P\colon H \to C$ be the projection.

Theorem (Arendt–tE). The following are equivalent.

- C is invariant under S, that is $S_t C \subset C$ for all $t > 0$.
- For all $u \in V$ there exists a $w \in V$ such that $P(j(u)) = j(w)$ and $\operatorname{Re} \mathfrak{a}(w, u - w) \geq 0$.
- For all $u \in V$ there exists a $w \in V$ such that $P(j(u)) = j(w)$ and $\operatorname{Re} \mathfrak{a}(u, u - w) \geq 0$.

Invariance of closed convex sets

Proposition. Let $C \subset H$ be a non-empty closed convex set and let $P \colon H \to C$ be the projection.
Let $\widetilde{C} \subset \widetilde{H}$ be a non-empty closed convex set and let $\widetilde{P} \colon \widetilde{H} \to \widetilde{C}$ be the projection.
Suppose $\mathfrak{a}$ is j-elliptic and accretive.
Suppose $\widetilde{C}$ is invariant under the semigroup $\widetilde{S}$ and

$$P \circ j = j \circ \widetilde{P} \quad \text{on } V.$$

Then C is invariant under the semigroup S.

Our situation

$V = W^{1,2}(\Omega)$.
$H = L_2(\Gamma)$.
$\widetilde{H} = L_2(\Omega)$.
$j = \mathrm{Tr} : W^{1,2}(\Omega) \to L_2(\Gamma)$.
S semigroup generated by Dirichlet-to-Neumann operator.
$\widetilde{S}$ semigroup on $L_2(\Omega)$ with Neumann boundary conditions.

We need in addition to prove a diamagnetic inequality for differential operators in divergence form with lower-order terms and Neumann boundary conditions on Ω.

The latter was done by Hundertmark and Simon for the Laplacian.

References

A.F.M. ter Elst and E.M. Ouhabaz
The diamagnetic inequality for the Dirichlet-to-Neumann operator.
Bull. Lond. Math. Soc. (2022).
In press.

W. Arendt and A.F.M. ter Elst
Sectorial forms and degenerate differential operators.
J. Operator Theory **67** (2012), 33–72.

D. Hundertmark and B. Simon
A diamagnetic inequality for semigroup differences.
J. Reine Angew. Math. **571** (2004), 107–130.

E.M. Ouhabaz
Invariance of closed convex sets and domination criteria for semigroups.
Potential Anal. **5** (1996), 611–625.