New Directions in the Fractalization, Quantization, and Revival of Dispersive Systems

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Dispersion of surface waves on a pond

Peter J. Olver

Introduction to Partial Differential Equations

Undergraduate Texts, Springer, 2014

—, Dispersive quantization, *Amer. Math. Monthly* **117** (2010) 599–610.

Gong Chen & —, Dispersion of discontinuous periodic waves, *Proc. Roy. Soc. London A* **469** (2012), 20120407.

Gong Chen & —, Numerical simulation of nonlinear dispersive quantization, *Discrete Cont. Dyn. Syst. A* **34** (2013), 991–1008.

Dispersion

Definition. A linear partial differential equation is called dispersive if the different Fourier modes travel unaltered but at different speeds.

Substituting

$$u(t,x) = e^{\,\mathrm{i}\,(k\,x-\omega\,t)}$$

produces the dispersion relation

$$\omega = \omega(k), \qquad\qquad \omega, k \in \mathbb{R}$$

relating frequency ω and wave number k.

Phase velocity: $\quad c_p = \dfrac{\omega(k)}{k}$

Group velocity: $\quad c_g = \dfrac{d\omega}{dk} \qquad$ (stationary phase)

A Simple Linear Dispersive Wave Equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3}$$

$\Longrightarrow$ linearized Korteweg–deVries equation

Dispersion relation: $\omega = k^3$

Phase velocity: $c_p = \dfrac{\omega}{k} = k^2$

Group velocity: $c_g = \dfrac{d\omega}{dk} = 3\,k^2$

Thus, wave packets (and energy) move *faster* (to the right) than the individual waves.

Linear Dispersion on the Line

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} \qquad u(0,x) = f(x)$$

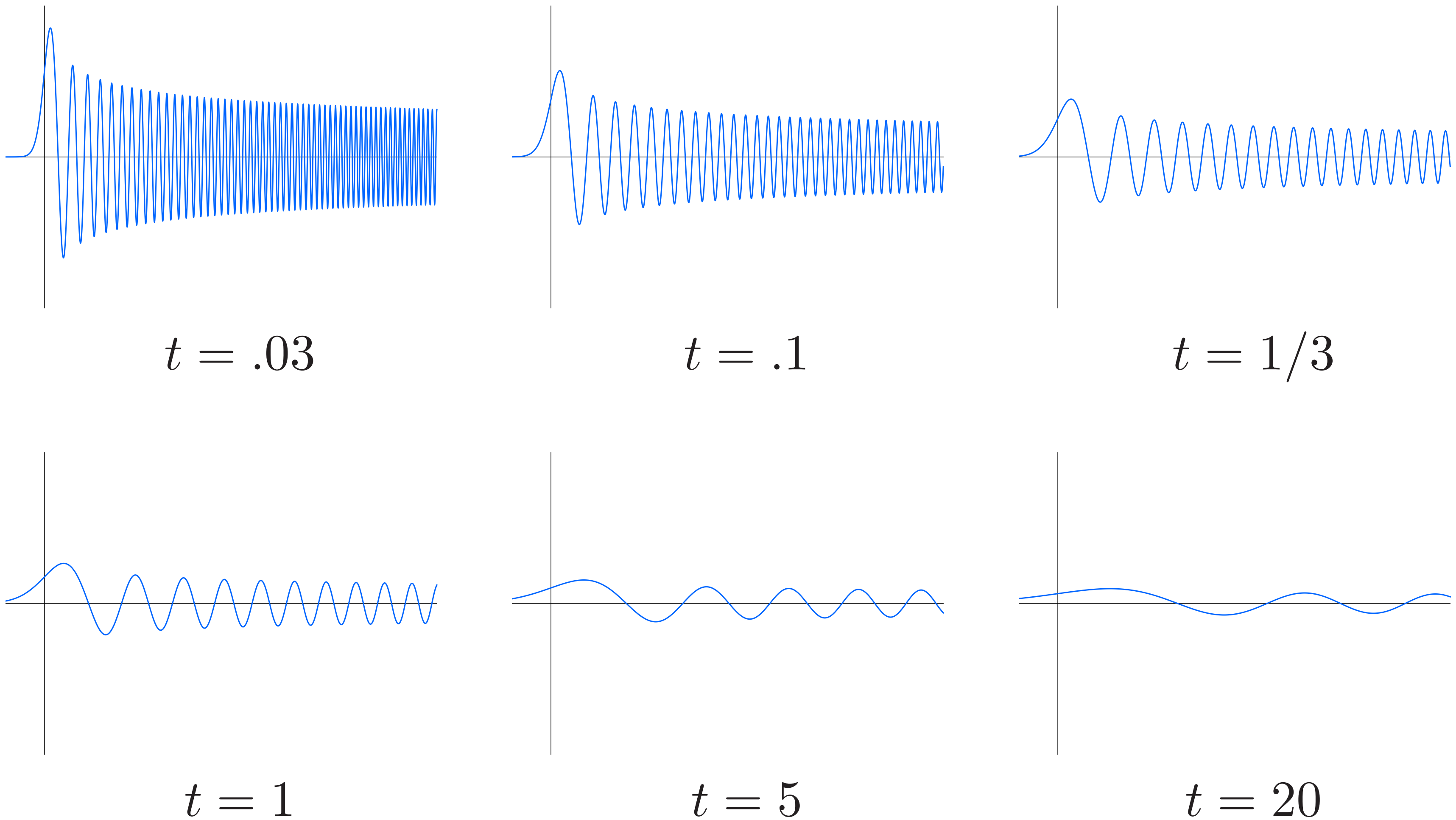
Fourier transform solution:

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(k)\, e^{\,\mathrm{i}\,(k\,x - k^3\,t)}\, dk$$

Fundamental solution $\qquad u(0,x) = \delta(x)$

$$u(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\,\mathrm{i}\,(k\,x - k^3\,t)}\, dk = \frac{1}{\sqrt[3]{3\,t}}\ \mathrm{Ai}\left(-\,\frac{x}{\sqrt[3]{3\,t}} \right)$$

Fundamental solution to linearized KdV


$t = .03$
$t = .1$
$t = 1/3$
$t = 1$
$t = 5$
$t = 20$

Linear Dispersion on the Line

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} \qquad u(0,x) = f(x)$$

Superposition solution formula:

$$u(t,x) = \frac{1}{\sqrt[3]{3\,t}} \int_{-\infty}^{\infty} f(\xi)\ \mathrm{Ai}\left(\frac{\xi - x}{\sqrt[3]{3\,t}}\right) d\xi$$

Linear Dispersion on the Line

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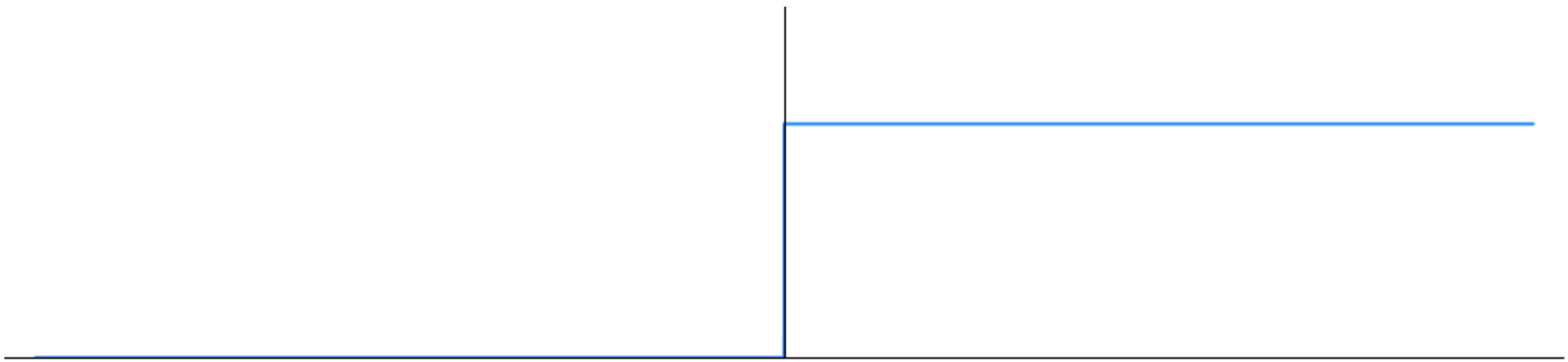
Step function initial data: $\quad u(0,x) = \sigma(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$

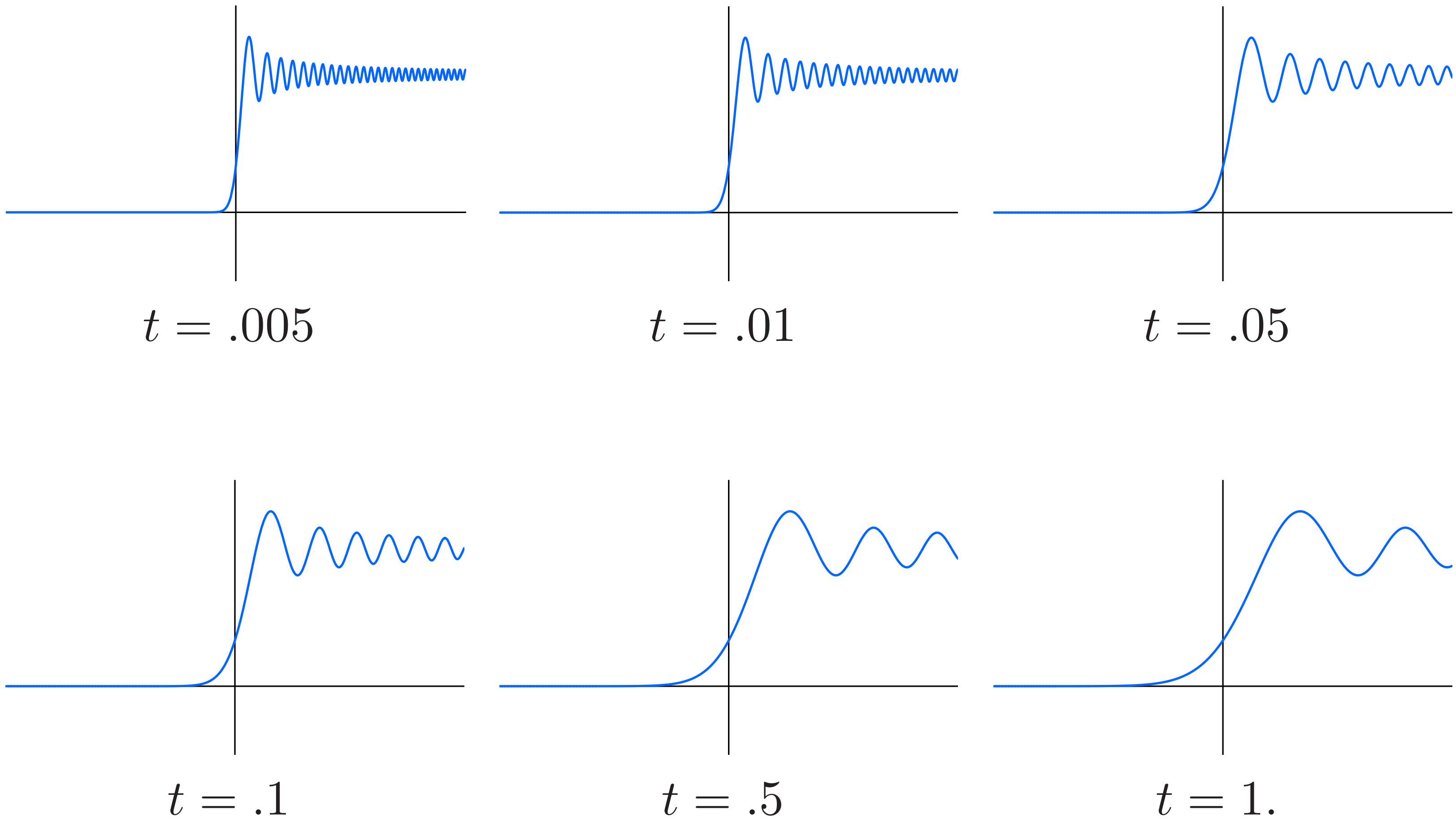
$$u(t,x) = \frac{1}{3} \ - \ H\left(-\frac{x}{\sqrt[3]{3\,t}}\right)$$

$$H(z) = \frac{z\,\Gamma\left(\frac{1}{3}\right)\,{}_1F_2\left(\frac{1}{3};\,\frac{2}{3},\,\frac{4}{3};\,\frac{1}{9}\,z^3\right)}{3^{5/3}\,\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)} - \frac{z^2\,\Gamma\left(\frac{2}{3}\right)\,{}_1F_2\left(\frac{2}{3};\,\frac{4}{3},\,\frac{5}{3};\,\frac{1}{9}\,z^3\right)}{3^{7/3}\,\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{3}\right)}$$

$\Longrightarrow$ MATHEMATICA — via Meijer G functions

Step solution to linearized KdV




$t = .005$
$t = .01$
$t = .05$
$t = .1$
$t = .5$
$t = 1.$

Periodic Linear Dispersion

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3}$$

$$u(t,-\pi) = u(t,\pi) \qquad \frac{\partial u}{\partial x}(t,-\pi) = \frac{\partial u}{\partial x}(t,\pi) \qquad \frac{\partial^2 u}{\partial x^2}(t,-\pi) = \frac{\partial^2 u}{\partial x^2}(t,\pi)$$

Step function initial data:

$$u(0,x) = \sigma(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

Periodic Linear Dispersion

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3}$$

$$u(t,-\pi) = u(t,\pi) \qquad \frac{\partial u}{\partial x}(t,-\pi) = \frac{\partial u}{\partial x}(t,\pi) \qquad \frac{\partial^2 u}{\partial x^2}(t,-\pi) = \frac{\partial^2 u}{\partial x^2}(t,\pi)$$

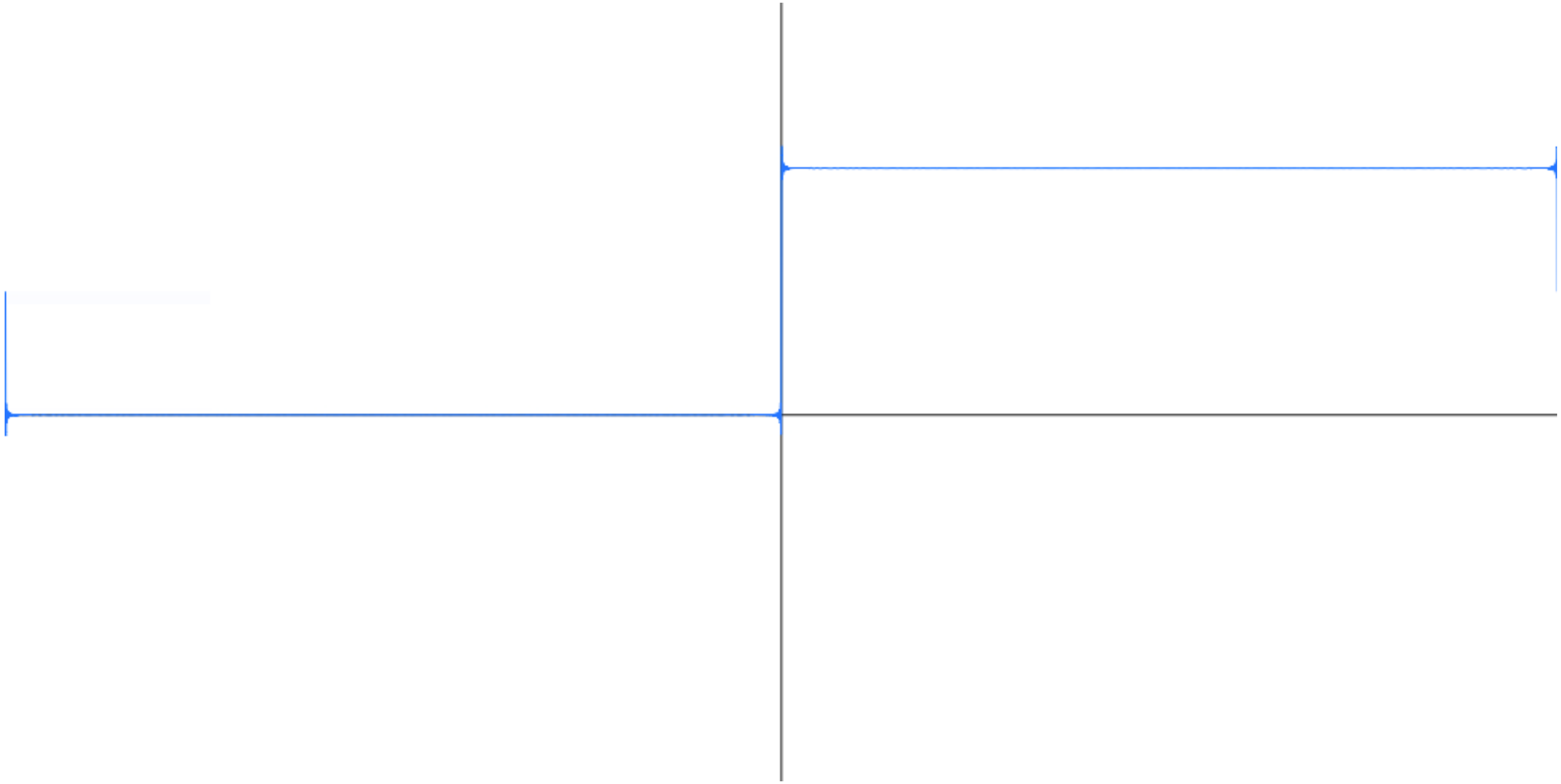
Step function initial data:

$$u(0,x) = \sigma(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

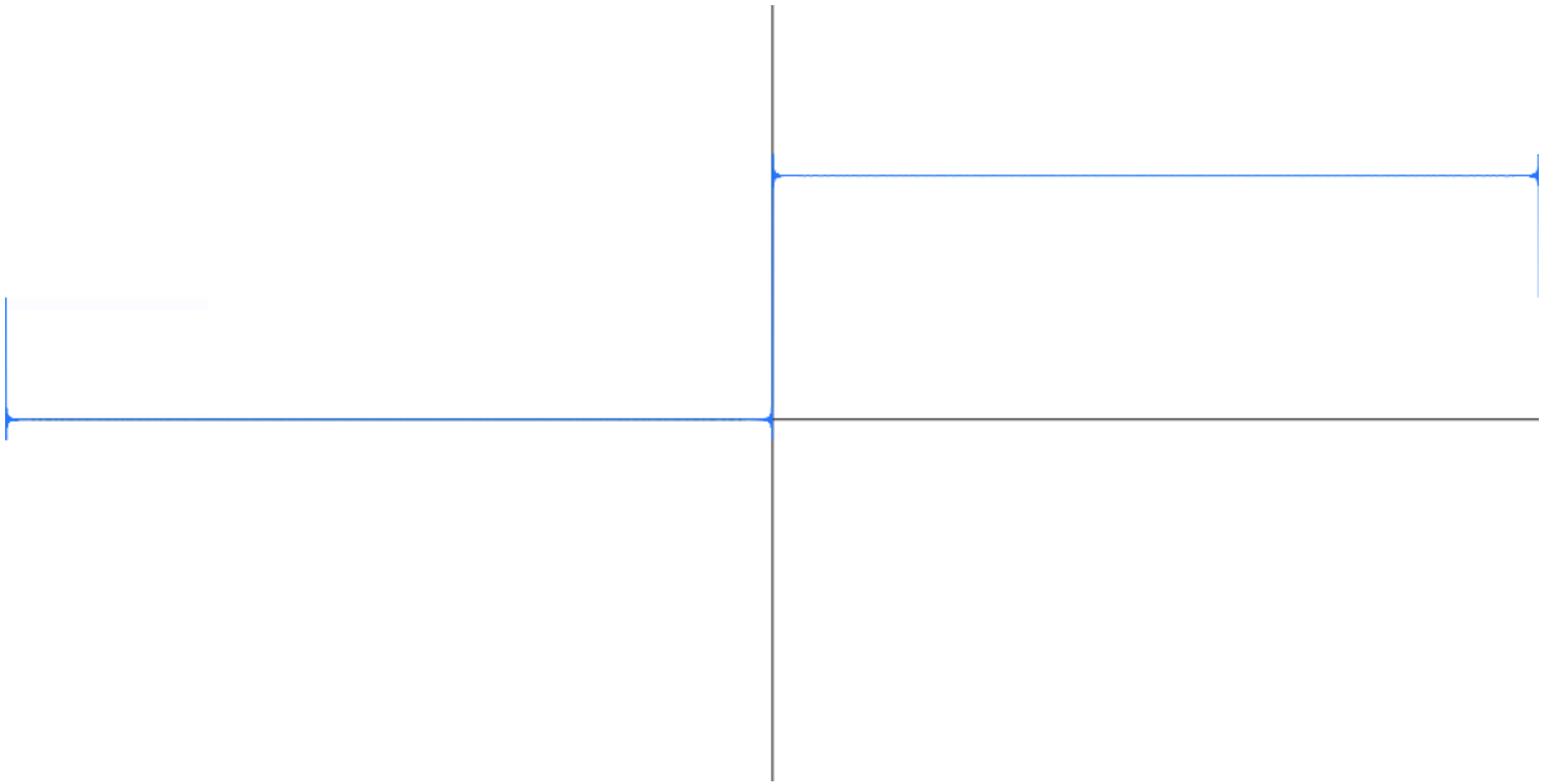
Fourier series solution formula:

$$u^\star(t,x) \ \sim \ \frac{1}{2} + \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{\sin(\,(2\,j+1)\,x - (2\,j+1)^3\,t\,)}{2\,j+1}\,.$$

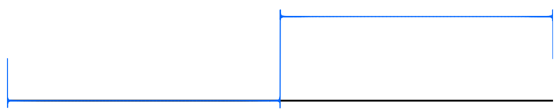
Periodic linearized KdV with $\Delta t = .01$



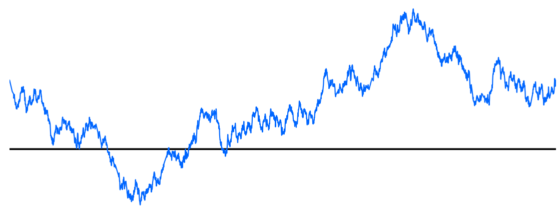
Periodic linearized KdV with $\Delta t = \pi/300$



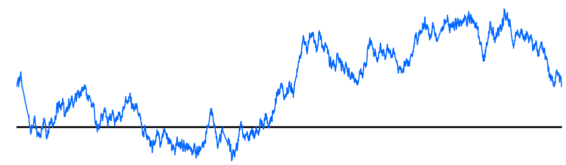
Periodic linearized KdV — irrational times



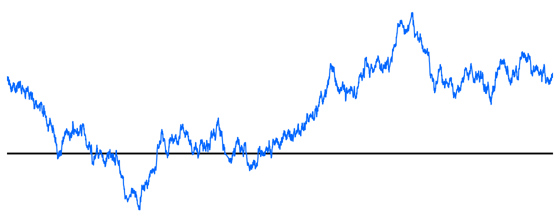
$t = 0.$



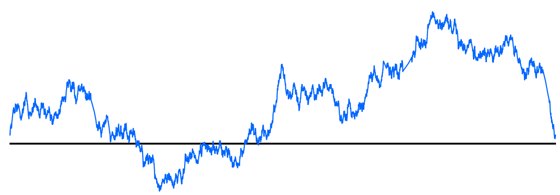
$t = .1$



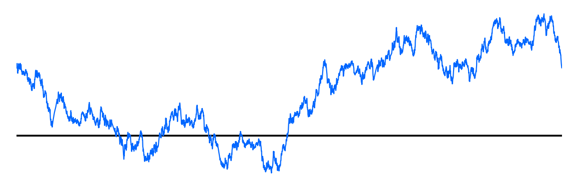
$t = .2$



$t = .3$

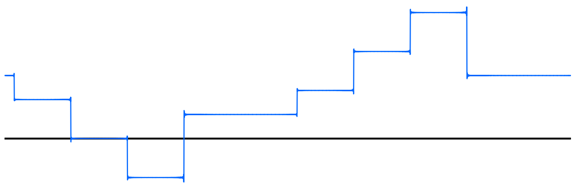


$t = .4$

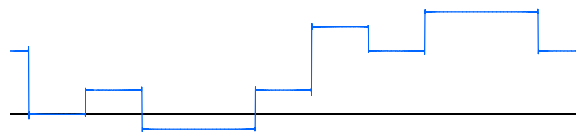


$t = .5$

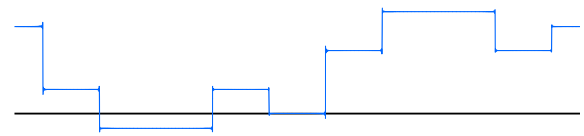
Periodic linearized KdV — rational times



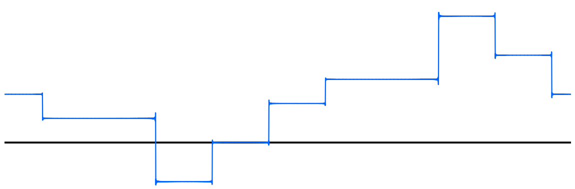
$t = \frac{1}{30}\pi$



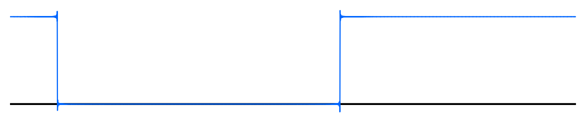
$t = \frac{1}{15}\pi$



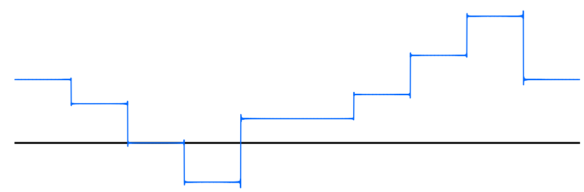
$t = \frac{1}{10}\pi$



$t = \frac{2}{15}\pi$

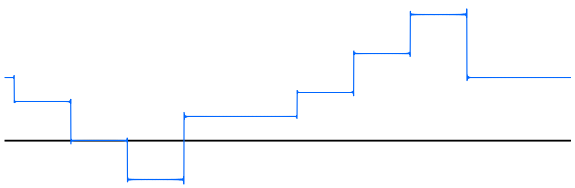


$t = \frac{1}{6}\pi$

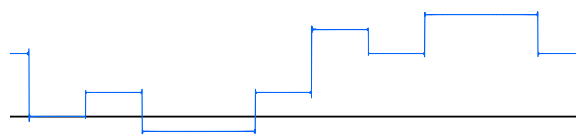


$t = \frac{1}{5}\pi$

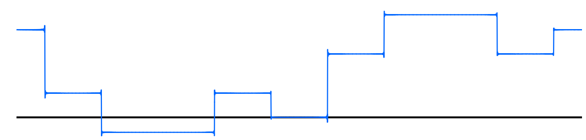
Periodic linearized KdV — rational times



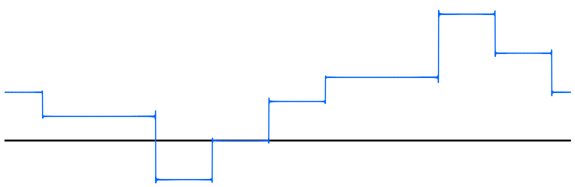
$t = \frac{1}{30}\pi$



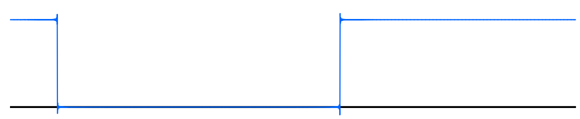
$t = \frac{1}{15}\pi$



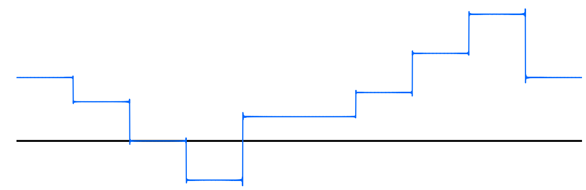
$t = \frac{1}{10}\pi$



$t = \frac{2}{15}\pi$

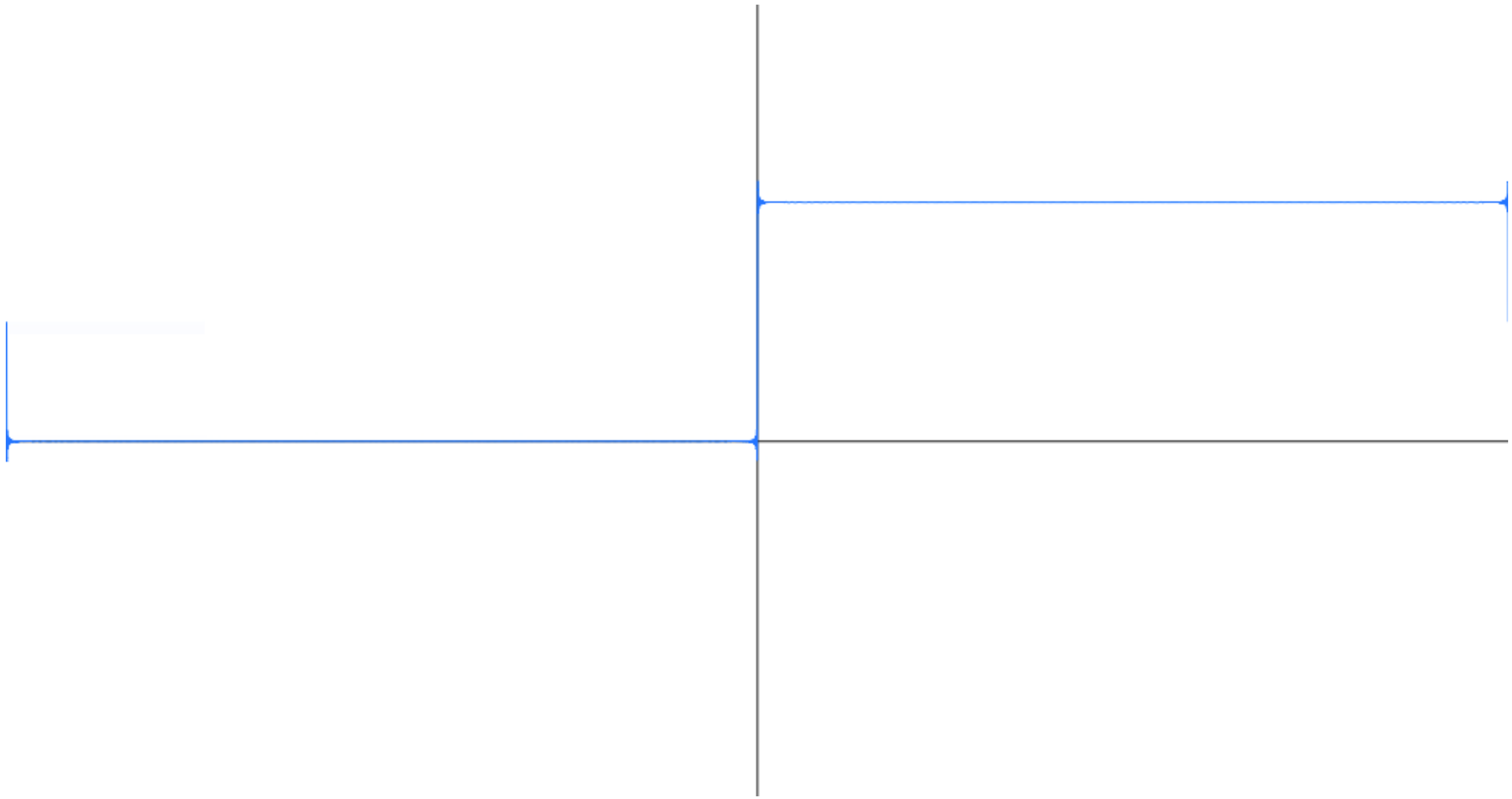


$t = \frac{1}{6}\pi$



$t = \frac{1}{5}\pi$

Periodic linearized KdV with $\Delta t = .0001$



Theorem. At rational time $t = 2\pi\, p/q$, the solution $u^\star(t,x)$ is constant on every subinterval $2\pi\, j/q < x < 2\pi\,(j+1)/q$. At irrational time $u^\star(t,x)$ is a non-differentiable continuous fractal function.

Lemma.

$$f(x) \;\sim \sum_{k=-\infty}^{\infty} c_k\, e^{\,\mathrm{i}\,k\,x}$$

is piecewise constant on intervals $2\pi\, j/q < x < 2\pi\,(j+1)/q$ if and only if

$$\widehat{c}_k = \widehat{c}_l, \quad k \equiv l \not\equiv 0 \mod q, \qquad \widehat{c}_k = 0, \quad 0 \neq k \equiv 0 \mod q.$$

where

$$\widehat{c}_k = \frac{2\pi\, k\, c_k}{\mathrm{i}\, q\,(e^{-2\,\mathrm{i}\,\pi\, k/q} - 1)} \qquad k \not\equiv 0 \mod q.$$

$\Longrightarrow$ DFT

The Fourier coefficients of the solution $u^\star(t,x)$ at rational time $t = 2\pi\, p/q$ are

$$c_k = b_k\, e^{-2\pi\, \mathrm{i}\, k^3 p/q} \qquad (*)$$

where, for the step function initial data,

$$b_k = \begin{cases} -\,\mathrm{i}\,/(\pi\, k), & k \text{ odd}, \\ 1/2, & k = 0, \\ 0, & 0 \neq k \;\; \text{even}. \end{cases}$$

Crucial observation: $\boxed{\text{if } k \equiv l \mod q \text{ then } k^3 \equiv l^3 \mod q}$

which implies

$$e^{-2\pi\, \mathrm{i}\, k^3 p/q} = e^{-2\pi\, \mathrm{i}\, l^3 p/q}$$

and hence the Fourier coefficients $(*)$ satisfy the condition in the Lemma. *Q.E.D.*

Revival

Fundamental Solution: $\quad F(0,x) = \delta(x).$

Theorem. At rational time $t = 2\pi p/q$, the fundamental solution $F(t,x)$ is a linear combination of finitely many periodically extended delta functions, based at $2\pi j/q$ for integers $-\frac{1}{2}q < j \leq \frac{1}{2}q$.

Revival

Fundamental Solution: $F(0,x) = \delta(x)$.

Theorem. At rational time $t = 2\pi p/q$, the fundamental solution $F(t,x)$ is a linear combination of finitely many periodically extended delta functions, based at $2\pi j/q$ for integers $-\frac{1}{2}q < j \leq \frac{1}{2}q$.

Corollary. At rational time, any solution profile $u(2\pi p/q, x)$ to the periodic initial-boundary value problem is a linear combination of $\leq q$ translates of the initial data, namely $f(x + 2\pi j/q)$, and hence its value depends on only finitely many values of the initial data.

★★ The same quantization/fractalization phenomenon appears in any linearly dispersive equation with "integral polynomial" dispersion relation:

$$\omega(k) = \sum_{m=0}^{n} c_m k^m$$

where

$$c_m = \alpha\, n_m \qquad n_m \in \mathbb{Z}$$

Linear Free-Space Schrödinger Equation

$$\mathrm{i}\,\frac{\partial u}{\partial t} = -\,\frac{\partial^2 u}{\partial x^2}$$

Dispersion relation: $\omega = k^2$

Phase velocity: $c_p = \dfrac{\omega}{k} = k$

Group velocity: $c_g = \dfrac{d\omega}{dk} = 2\,k$

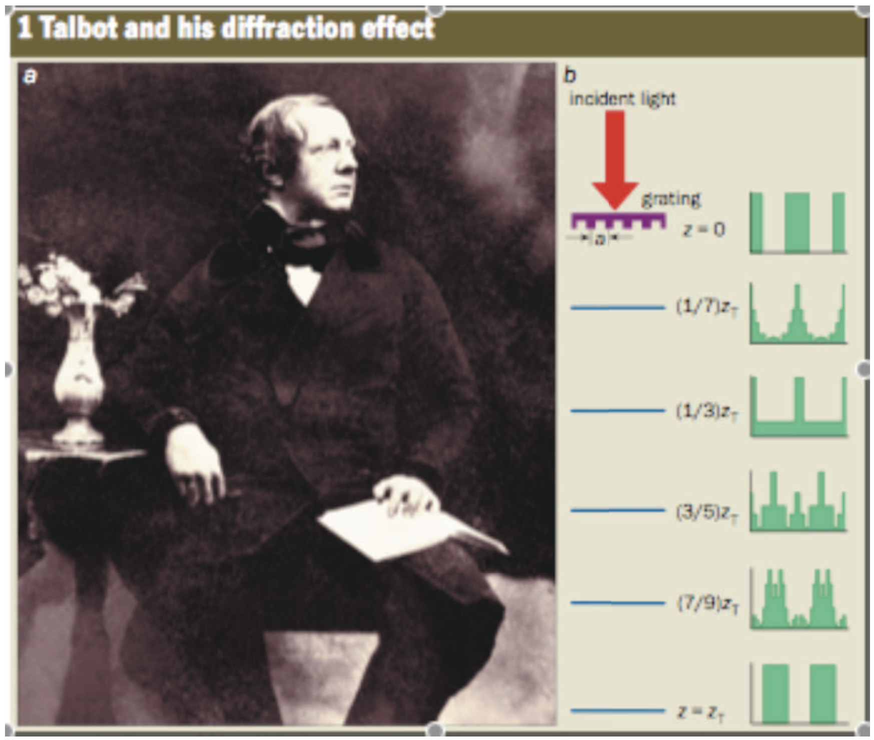
The Talbot Effect

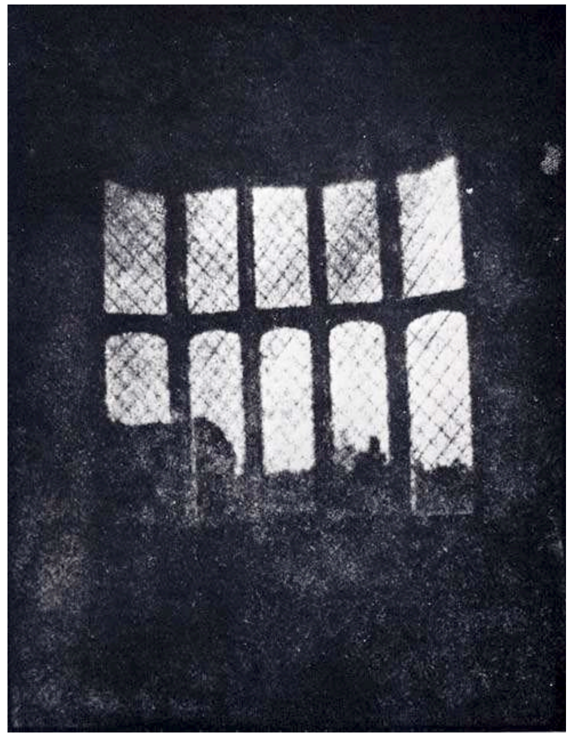
$$\mathrm{i}\,\frac{\partial u}{\partial t} = -\,\frac{\partial^2 u}{\partial x^2}$$

$$u(t,-\pi) = u(t,\pi) \qquad \frac{\partial u}{\partial x}(t,-\pi) = \frac{\partial u}{\partial x}(t,\pi)$$

- Michael Berry et. al.
- Oskolkov
- Kapitanski, Rodnianski
  "Does a quantum particle know the time?"
- Michael Taylor
- Bernd Thaller, *Visual Quantum Mechanics*

William Henry Fox Talbot (1800–1877)





⋆ Talbot's 1835 image of a latticed window in Lacock Abbey

$\Longrightarrow$ oldest photographic negative in existence.

A Talbot Experiment

Fresnel diffraction by periodic gratings (1836):

"It was very curious to observe that though the grating was greatly out of the focus of the lens . . . the appearance of the bands was perfectly distinct and well defined . . . the experiments are communicated in the hope that they may prove interesting to the cultivators of optical science."

— *Fox Talbot*

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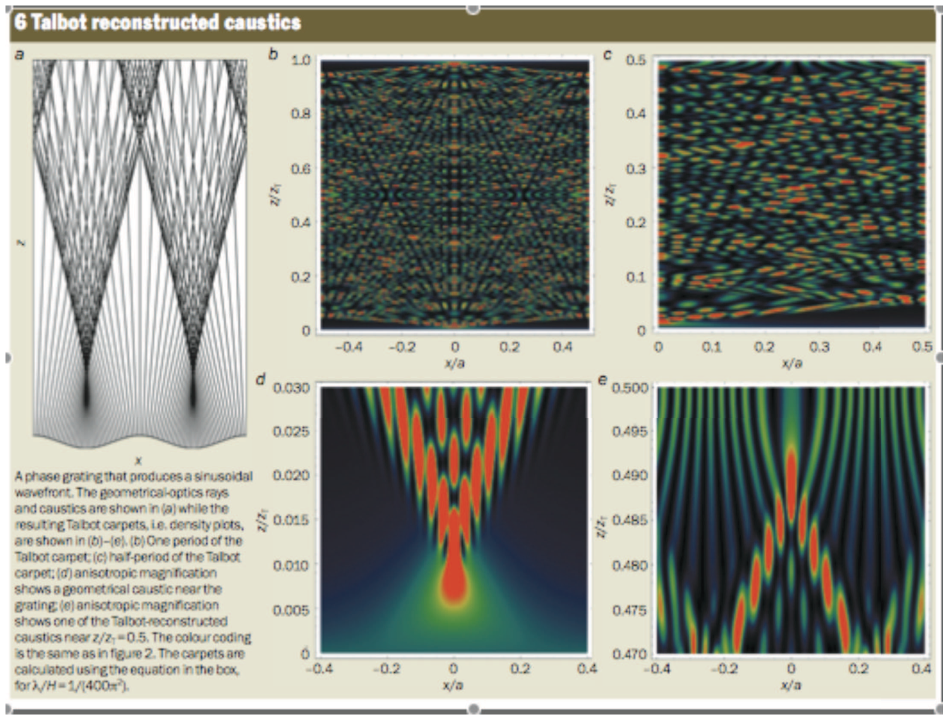
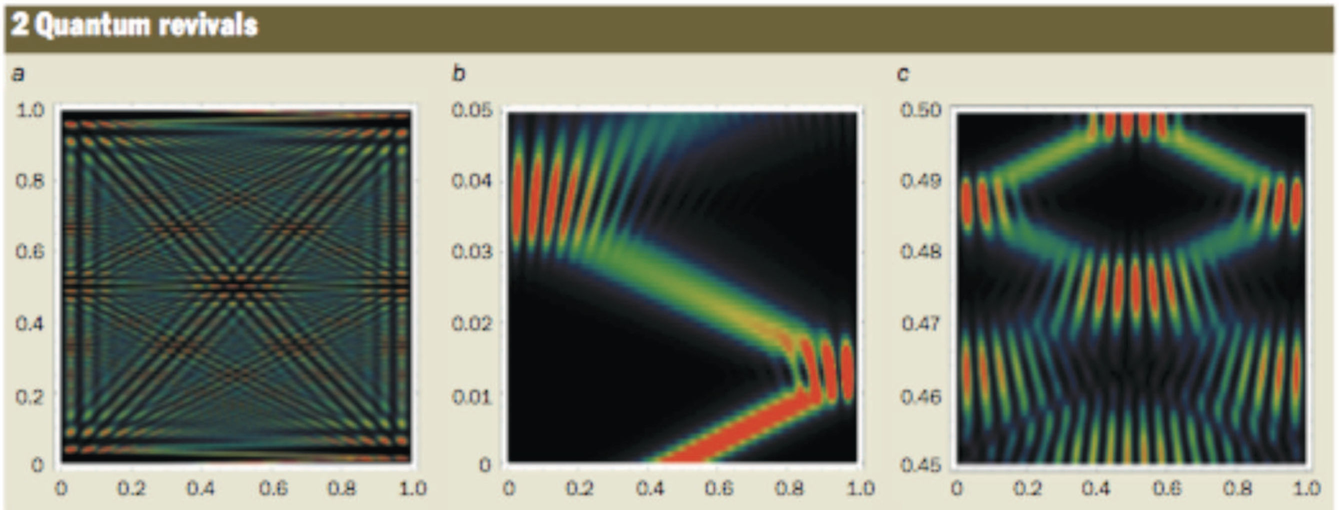
$\Longrightarrow$ Lord Rayleigh calculates the Talbot distance (1881)

The Quantized/Fractal Talbot Effect



- Optical experiments — Berry & Klein
- Diffraction of matter waves (helium atoms) — Nowak et. al.

Quantum Revival



- Electrons in potassium ions — Yeazell & Stroud
- Vibrations of bromine molecules —
  Vrakking, Villeneuve, Stolow

Periodic Linear Schrödinger Equation

$$\mathrm{i}\,\frac{\partial u}{\partial t} = -\,\frac{\partial^2 u}{\partial x^2}$$

$$u(t,-\pi) = u(t,\pi) \qquad \frac{\partial u}{\partial x}(t,-\pi) = \frac{\partial u}{\partial x}(t,\pi)$$

Integrated fundamental solution:

$$u(t,x) = \frac{1}{2\pi} \sum_{0 \neq k = -\infty}^{\infty} \frac{e^{\,\mathrm{i}\,(k\,x - k^2 t)}}{k}\,.$$

For $x/t \in \mathbb{Q}$, this is known as a Gauss sum (or, more generally, k^n, a Weyl sum), of great importance in number theory

$\Longrightarrow$ Hardy, Littlewood, Weil, I. Vinogradov, etc.

Periodic Linear Schrödinger Equation

$$\mathrm{i}\,\frac{\partial u}{\partial t} = -\,\frac{\partial^2 u}{\partial x^2}$$

$$u(t,-\pi) = u(t,\pi) \qquad \frac{\partial u}{\partial x}\,(t,-\pi) = \frac{\partial u}{\partial x}\,(t,\pi)$$

Integrated fundamental solution:

$$u(t,x) = \frac{1}{2\,\pi} \sum_{0\neq k\,=\,-\infty}^{\infty} \frac{e^{\,\mathrm{i}\,(k\,x - k^2 t)}}{k}\,.$$

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★ ★ The Riemann Hypothesis!

Periodic Linear Dispersion

$$\frac{\partial u}{\partial t} = L(D_x)\, u, \qquad u(t, x + 2\pi) = u(t, x)$$

Dispersion relation:

$$u(t,x) = e^{\,\mathrm{i}\,(k\,x - \omega\,t)} \quad \Longrightarrow \quad \omega(k) = -\,\mathrm{i}\,L(-\,\mathrm{i}\,k) \qquad \text{assumed real}$$

Riemann problem: step function initial data

$$u(0,x) = \sigma(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

Solution:

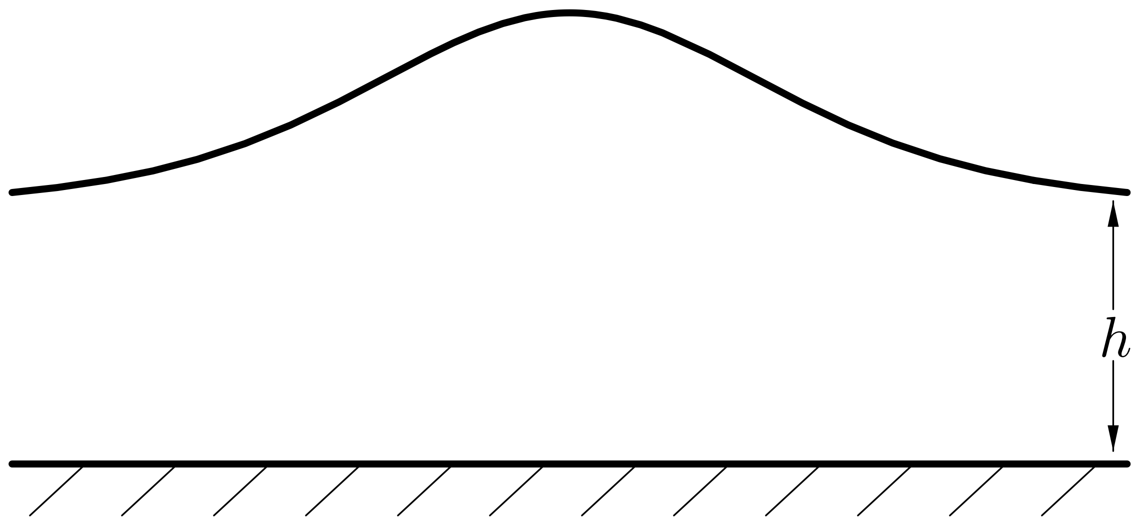
$$u(t,x) \;\sim\; \frac{1}{2} \;+\; \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{\sin[\,(2\,j+1)\,x - \omega(2\,j+1)\,t\,]}{2\,j+1}\,.$$

★ ★ $\omega(-\,k) = -\,\omega(k)$ odd

Polynomial dispersion, rational $t \;\Longrightarrow\;$ Weyl exponential sums

Water Waves

2D Water Waves



$y = h + \eta(t, x)$ surface elevation

$\phi(t, x, y)$ velocity potential

2D Water Waves

- Incompressible, irrotational fluid.
- No surface tension

$$\left.\begin{aligned} \phi_t + \tfrac{1}{2}\,\phi_x^2 + \tfrac{1}{2}\,\phi_y^2 + g\,\eta = 0 \\ \eta_t = \phi_y - \eta_x \phi_x \end{aligned}\right\} \qquad y = h + \eta(t,x)$$

$$\phi_{xx} + \phi_{yy} = 0 \qquad 0 < y < h + \eta(t,x)$$

$$\phi_y = 0 \qquad y = 0$$

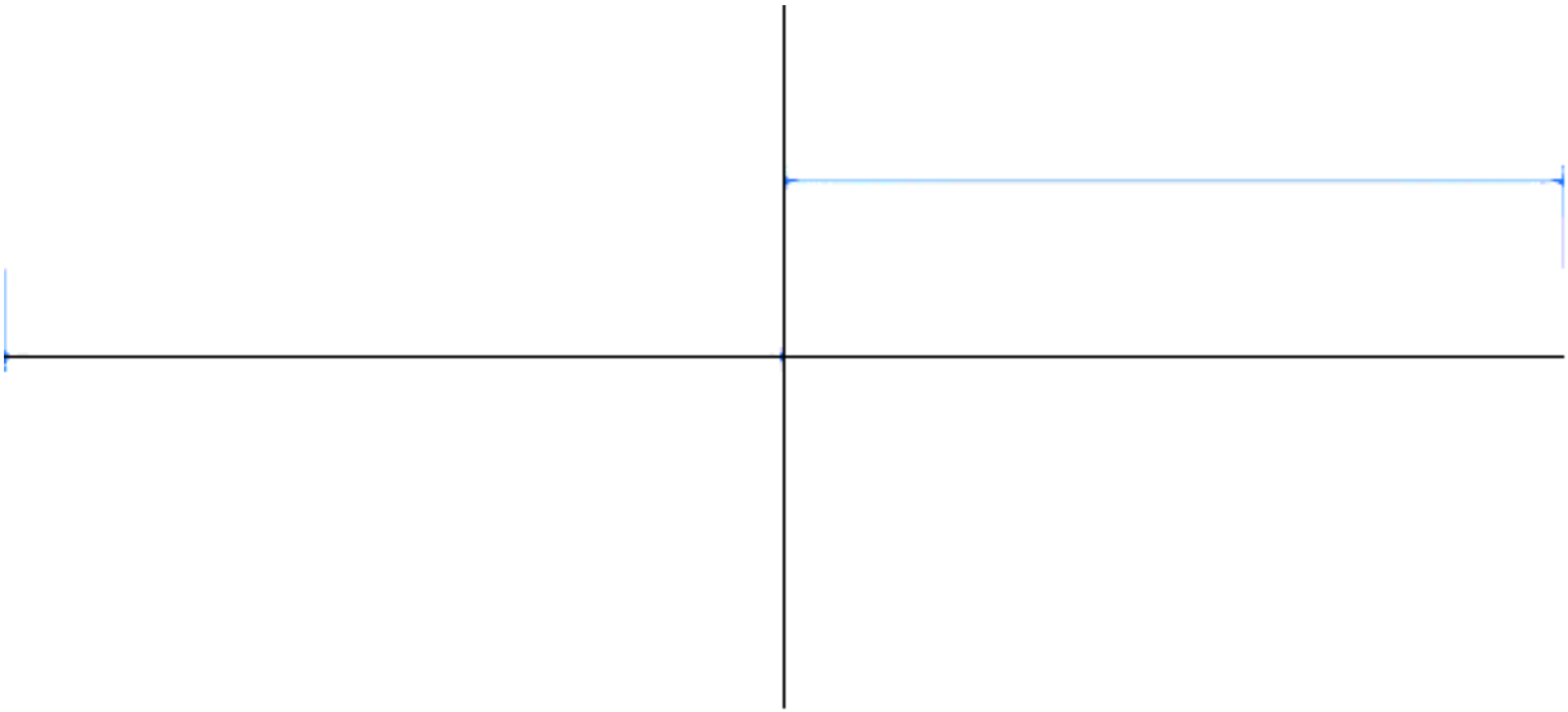
- Wave speed (maximum group velocity): $c = \sqrt{g\,h}$
- Dispersion relation: $\sqrt{g\,k\,\tanh(h\,k)} = c\,k - \frac{1}{6}\,c\,h^2 k^3 + \cdots$

Shallow Water Dispersion Relations

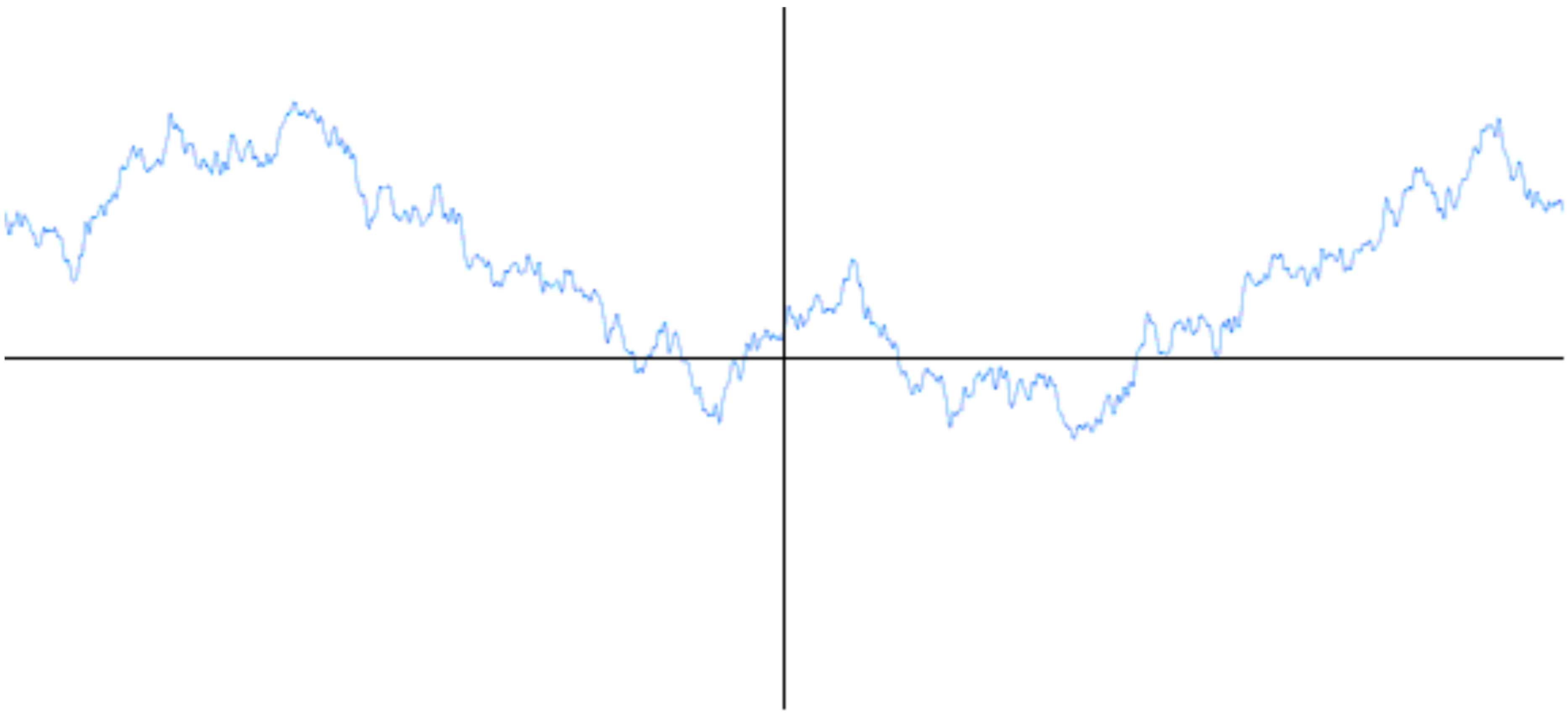
Water waves	$\pm\sqrt{k\ \tanh k}$
Boussinesq system	$\pm\dfrac{k}{\sqrt{1+\frac{1}{3}k^2}}$
Boussinesq equation	$\pm k\sqrt{1+\frac{1}{3}k^2}$
Korteweg–deVries	$k-\frac{1}{6}k^3$
BBM	$\dfrac{k}{1+\frac{1}{6}k^2}$

Water waves

$$\omega = \sqrt{k \tanh k}\, \operatorname{sign} k$$

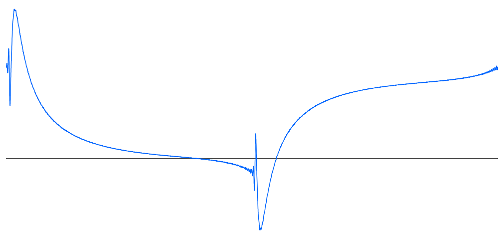


Water waves: t > 1000 $\omega = \sqrt{k \tanh k}\, \operatorname{sign} k$

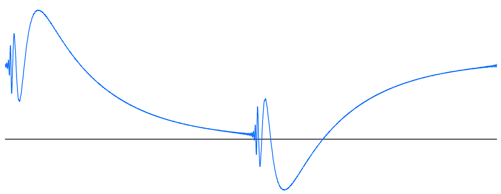


Water waves

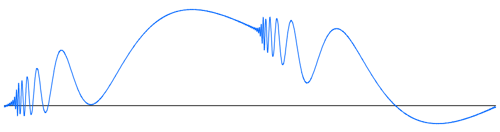
$$\omega = \sqrt{k \ \tanh k} \ \operatorname{sign} k$$



$t = 1$



$t = 2$



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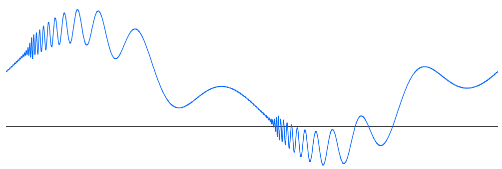
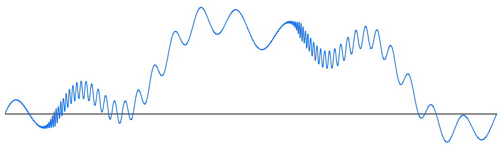
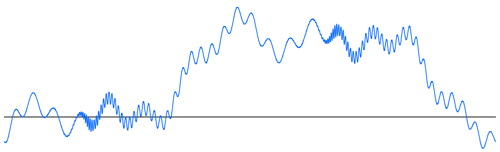


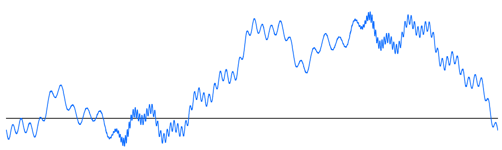
$t = 10$



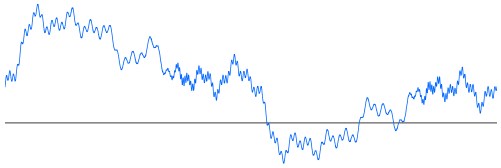
$t = 20$



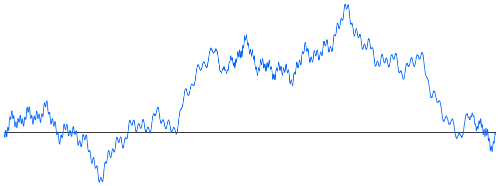
$t = 35$



$t = 50$



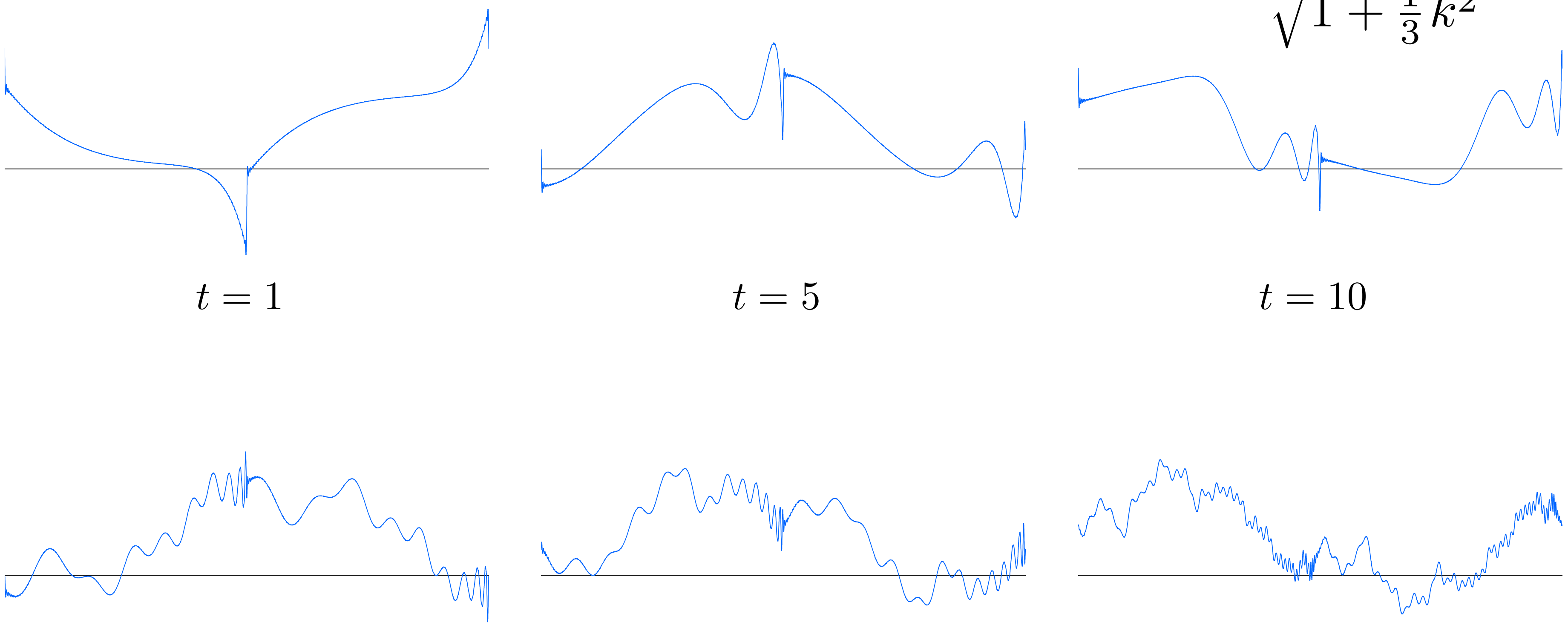
$t = 75$



$t = 100$

BBM equation

$$\omega = \frac{k}{\sqrt{1 + \frac{1}{3}k^2}}$$

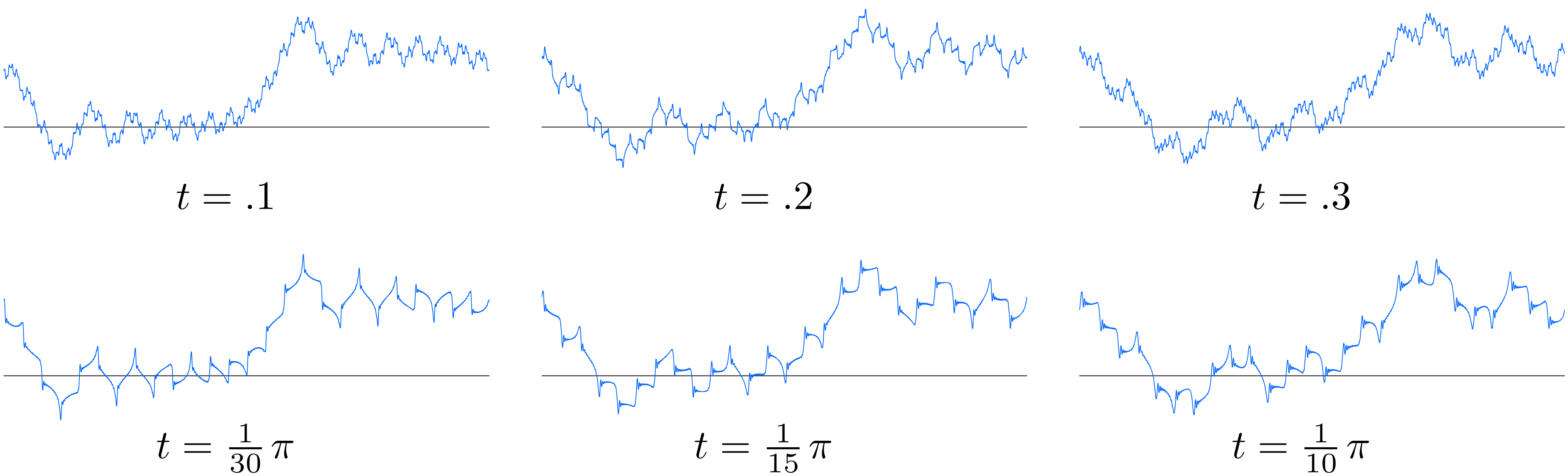


$t = 1$ $t = 5$ $t = 10$

$t = 50$ $t = 100$ $t = 1000$

Boussinesq equation

$$\omega = k\sqrt{1 + \tfrac{1}{3}k^2}$$



Dispersion Asymptotics

⋆ The qualitative behavior of the solution to the periodic problem depends crucially on the asymptotic behavior of the dispersion relation $\omega(k)$ for large wave number $k \to \pm\infty$.

$$\omega(k) \;\sim\; k^{\alpha}$$

- $\alpha = 0$ — large scale oscillations
- $0 < \alpha < 1$ — dispersive oscillations
- $\alpha = 1$ — traveling waves
- $1 < \alpha < 2$ — oscillatory becoming fractal
- $\alpha \geq 2$ — fractal/quantized

Linearized Benjamin Ono equation

➡ *waves on fluid interfaces*

$$u_t = \mathcal{H}[u_{xx}], \qquad \omega_{BO}(k) = k^2 \operatorname{sign} k.$$

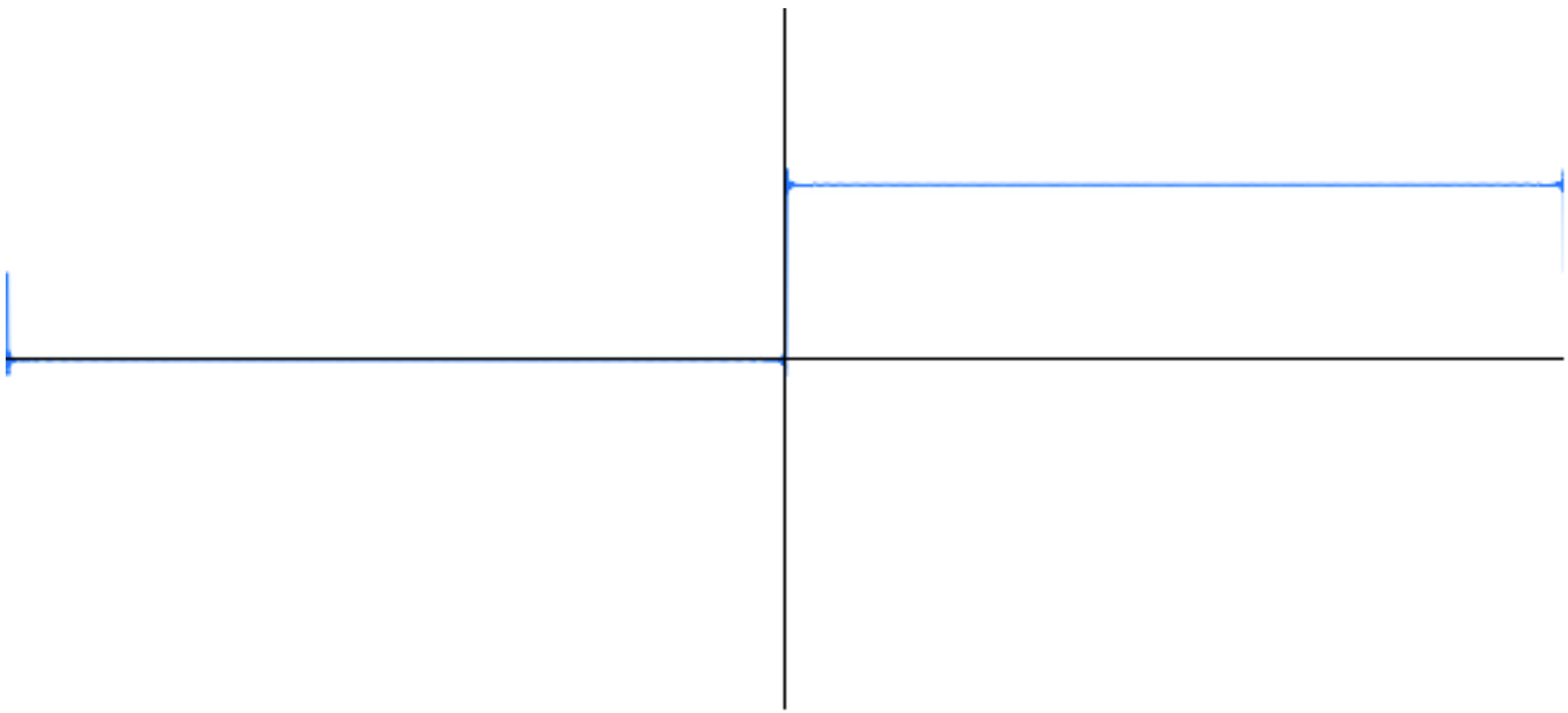
Hilbert transform

$$\mathcal{H}[f](x) = H * f\,(x) = \frac{1}{\pi} \,⨍_{-\infty}^{\infty} \frac{f(y)}{x-y}\, dy$$

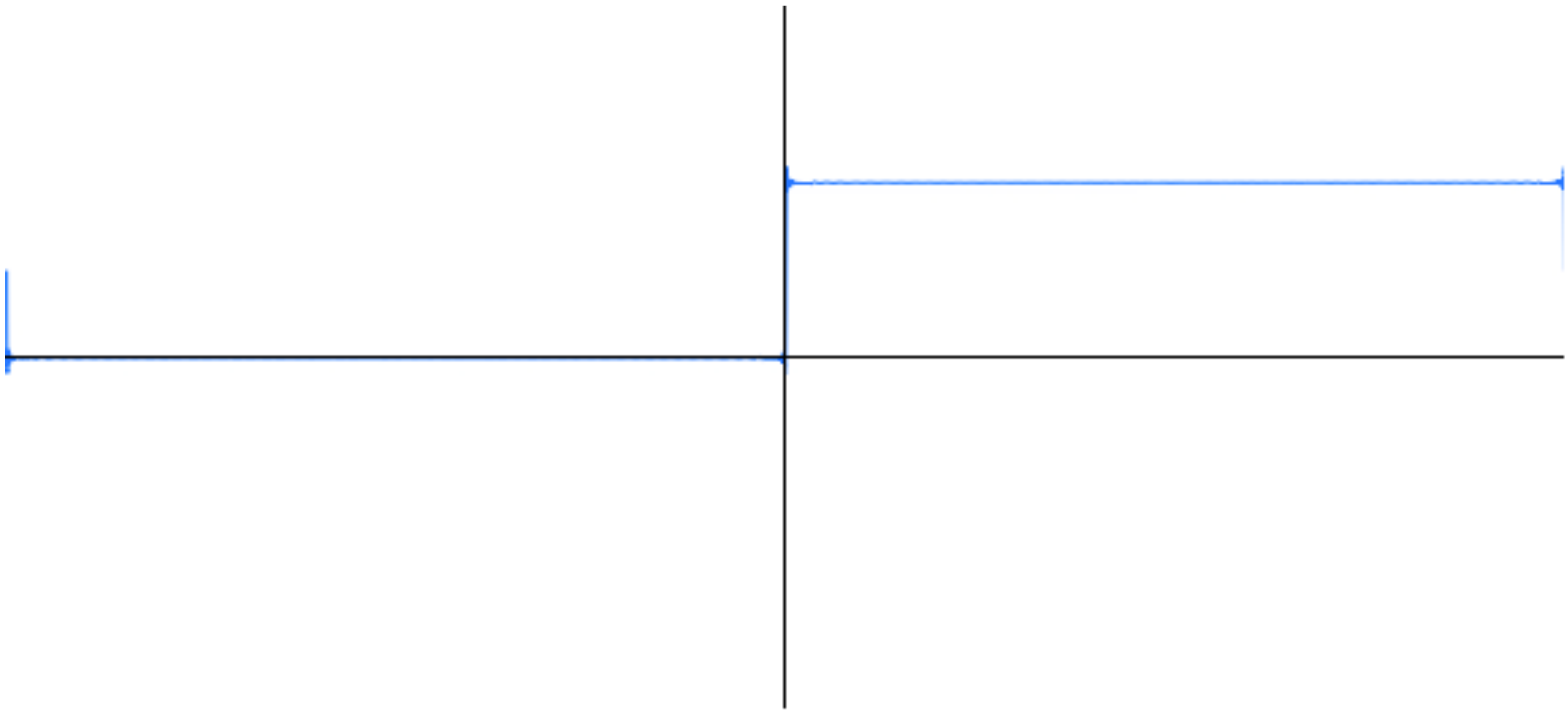
periodic Hilbert transform

$$\mathcal{H}[f](x) = \frac{1}{\pi} \sum_{k=-\infty}^{\infty} ⨍_{-\pi}^{\pi} \frac{f(y)}{x - y + 2\pi k}\, dy = \frac{1}{2\pi} ⨍_{-\pi}^{\pi} \cot\left[\,\tfrac{1}{2}(x-y)\,\right] f(y)\, dy$$

Benjamin–Ono equation: irrational times $\omega = k^2 \operatorname{sign} k$

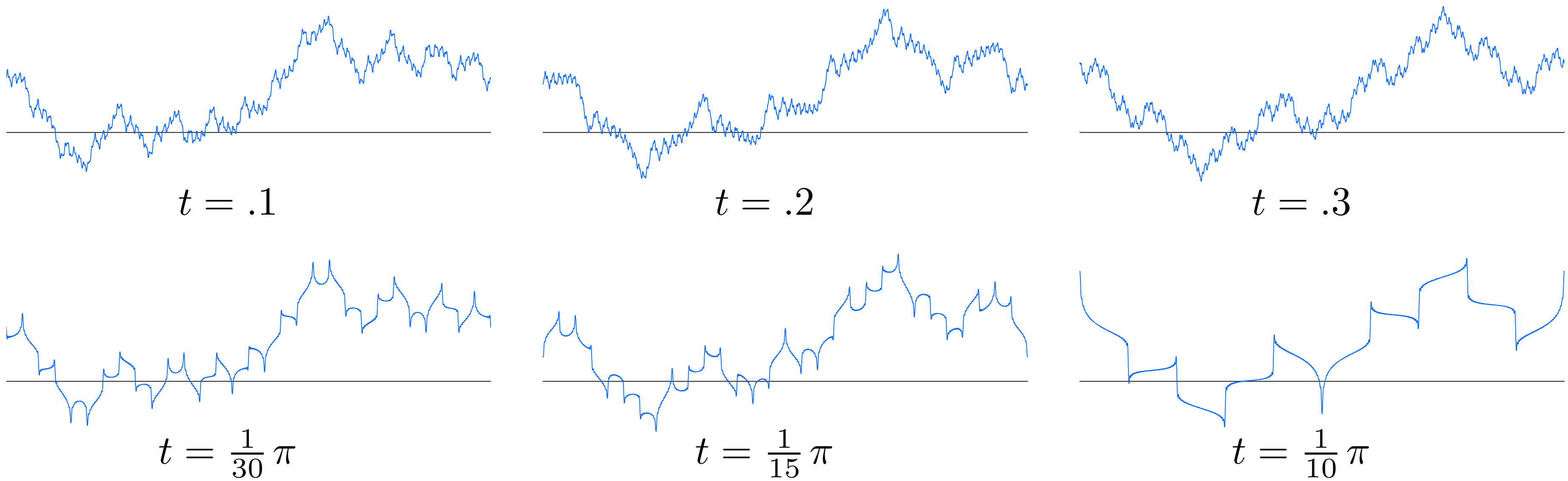


Benjamin–Ono equation: rational times $\omega = k^2 \operatorname{sign} k$



Benjamin–Ono equation

$$\omega = |\, k \,|^2 \ \mathrm{sign}\, k$$



Generalized Revival

Theorem At a rational time $t = \pi\, p/q$, the solution to the periodic initial-boundary value problem for the linearized Benjamin–Ono equation on the interval $-\pi < x < \pi$ is a linear combination of

- translates $f(x + \pi\, j/q)$ of the initial condition $u(0,x) = f(x)$, and
- translates $g(x + \pi\, j/q)$ of its periodic Hilbert transform: $g(x) = \mathcal{H}[f](x)$,

for $j = 0, \ldots, 2\,q - 1$.

➢ L. Boulton, PJO, B. Pelloni, D. Smith

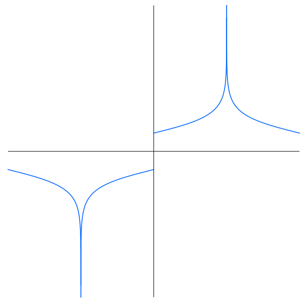
Trigonometric hypergeometric functions

$$S_j^k(x) = S_{j,1}^k(x) = \sum_{n=0}^{\infty} \frac{\sin(nk+j)x}{nk+j}.$$

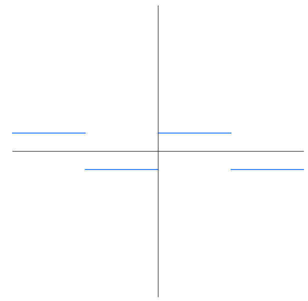
$$S_j^k(x) = \frac{1}{k}\sum_{l=1}^{k}\left[\sin\left(\frac{2\pi j\,l}{k}\right)\log\left|2\sin\left(\frac{x}{2}+\frac{\pi l}{k}\right)\right|\right.$$
$$\left. + \cos\left(\frac{2\pi j\,l}{k}\right)\frac{\operatorname{sign}(x+2\pi l/k)\,\pi - (x+2\pi l/k)}{2}\right].$$

$$\frac{dS_j^k}{dx} = \frac{\pi}{k}\sum_{l=0}^{k-1}\cos\left(\frac{2\pi\,l\,j}{k}\right)\delta_{[-\pi,\pi]}\left(x+\frac{2\pi\,l}{k}\right) + \frac{1}{2k}\sum_{l=1}^{k-1}\sin\left(\frac{2\pi\,l\,j}{k}\right)\cot\left(\tfrac{1}{2}x+\frac{\pi\,l}{k}\right)$$

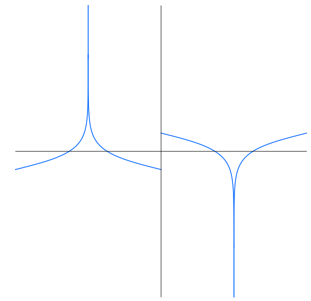
- Produces the periodic fundamental solution
- The cotangent is the Hilbert transform of the delta function



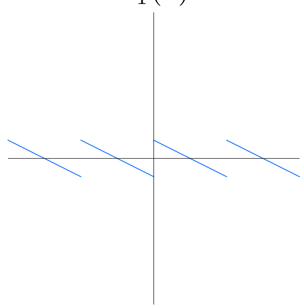
$S_1^4(x)$



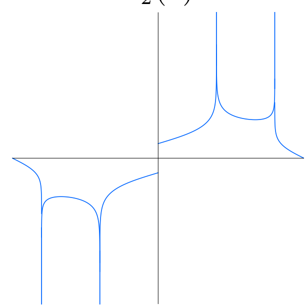
$S_2^4(x)$



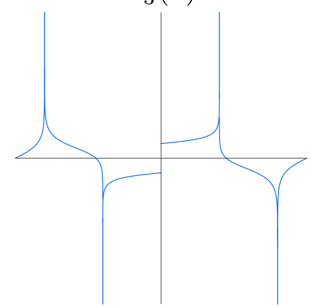
$S_3^4(x)$



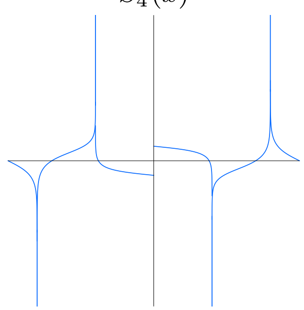
$S_4^4(x)$



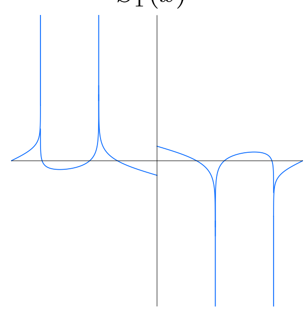
$S_1^5(x)$



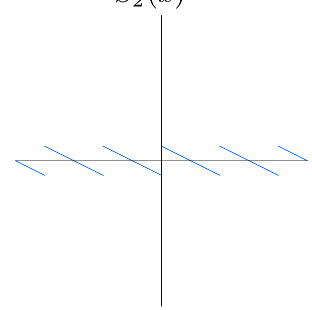
$S_2^5(x)$



$S_3^5(x)$

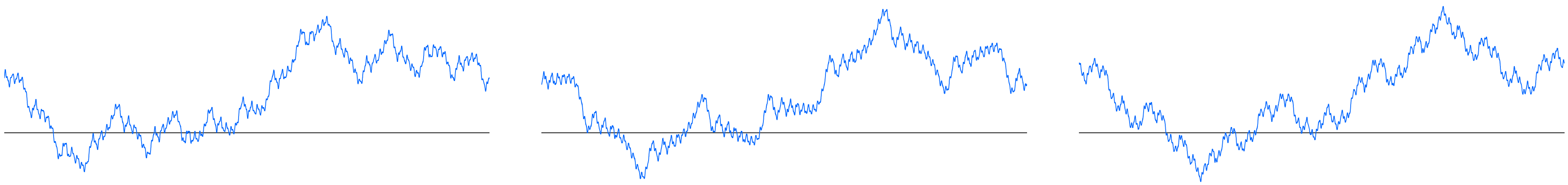


$S_4^5(x)$

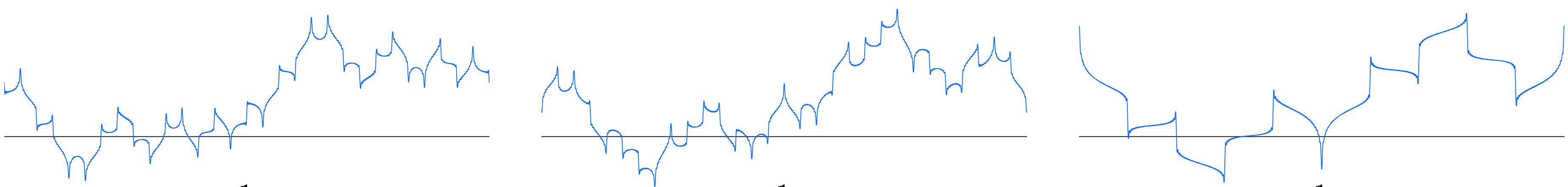


$S_5^5(x)$

Benjamin–Ono equation $\qquad \omega = |\,k\,|^2 \ \mathrm{sign}\, k$



$t = .1$ $\qquad t = .2$ $\qquad t = .3$



$t = \frac{1}{30}\,\pi$ $\qquad t = \frac{1}{15}\,\pi$ $\qquad t = \frac{1}{10}\,\pi$

Linearized Intermediate Long Wave Equation

$$\mathcal{L}[u] = \mathcal{I}_\delta[u_{xx}] - \frac{1}{\delta}\, u_x \qquad\qquad \omega_\delta(k) = k^2 \coth(\delta k) - \frac{k}{\delta}.$$

$$\mathcal{I}_\delta[f](x) = -\frac{1}{2\,\delta} \fint_{-\infty}^{\infty} \coth\left[\frac{\pi}{2\,\delta}(x-y)\right] f(y)\, dy$$

Periodic kernel:

$$\mathcal{I}_\delta[f](x) = -\frac{1}{2\,\delta} \fint_{-\pi}^{\pi} \left[\sum_{n=-\infty}^{\infty} \coth\left(\frac{\pi}{2\,\delta}(x-y) + \frac{\pi^2 n}{\delta}\right)\right] f(y)\, dy$$

$$= \frac{1}{\pi} \fint_{-\pi}^{\pi} \left[\mathrm{i}\,\frac{\zeta(-\mathrm{i}\,\delta)}{\delta}(x-y) - \zeta(x-y) + \frac{\pi^2 n}{\delta}\right] f(y)\, dy$$

Weierstrass zeta function

$$\zeta(z) = \frac{\eta_1}{\omega_1}\, z + \frac{\pi}{2\,\omega_1} \sum_{n=-\infty}^{\infty} \cot\left(\frac{\pi}{2\,\omega_1}\, z + \frac{\pi\,\omega_3\, n}{\omega_1}\right), \quad \eta_1 = \zeta(\omega_1), \quad \omega_1 = -\mathrm{i}\,\delta, \quad \omega_3 = \pi$$

Linearized Smith Equation

$$u_t = \mathcal{S}_\delta[u_x]$$

$$\omega_S(k) = k\,\sqrt{\frac{1}{\delta} + k^2}.$$

$$\mathcal{S}_\delta[f] = -\frac{\mathrm{i}}{\pi\sqrt{\delta}} \int_{-\infty}^{\infty} \frac{K_1\big(\,|\,x-y\,|/\sqrt{\delta}\,\big)}{|\,x-y\,|}\, f(y)\, dy.$$

$K_1(x)$ denotes the modified Bessel function of the second kind

Periodic kernel:

$$\mathcal{S}_\delta[f] = -\frac{\mathrm{i}}{\pi\sqrt{\delta}} \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{K_1\big(\,|\,x-y+2\,n\,\pi\,|/\sqrt{\delta}\,\big)}{|\,x-y+2\,n\,\pi\,|}\, f(y)\, dy.$$

What about nonlinear equations?

Periodic Korteweg–deVries equation

$$\frac{\partial u}{\partial t} = \alpha\,\frac{\partial^3 u}{\partial x^3} + \beta\,u\,\frac{\partial u}{\partial x} \qquad u(t, x + 2\ell) = u(t, x)$$

Zabusky–Kruskal (1965)

$$\alpha = 1, \qquad \beta = .000484, \qquad \ell = 1, \qquad u(0, x) = \cos \pi x.$$

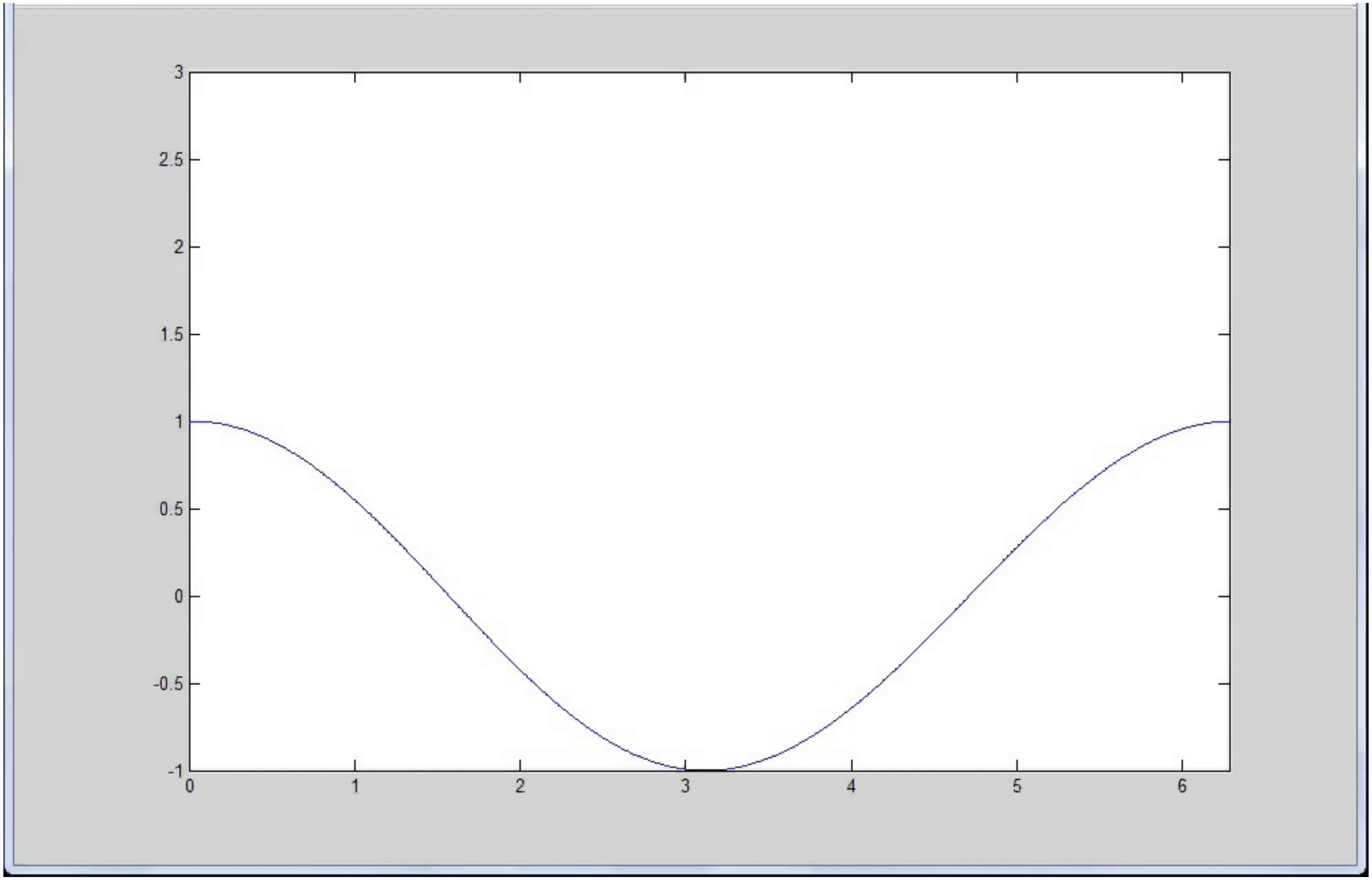
Lax–Levermore (1983) — small dispersion

$$\alpha \longrightarrow 0, \qquad \beta = 1.$$

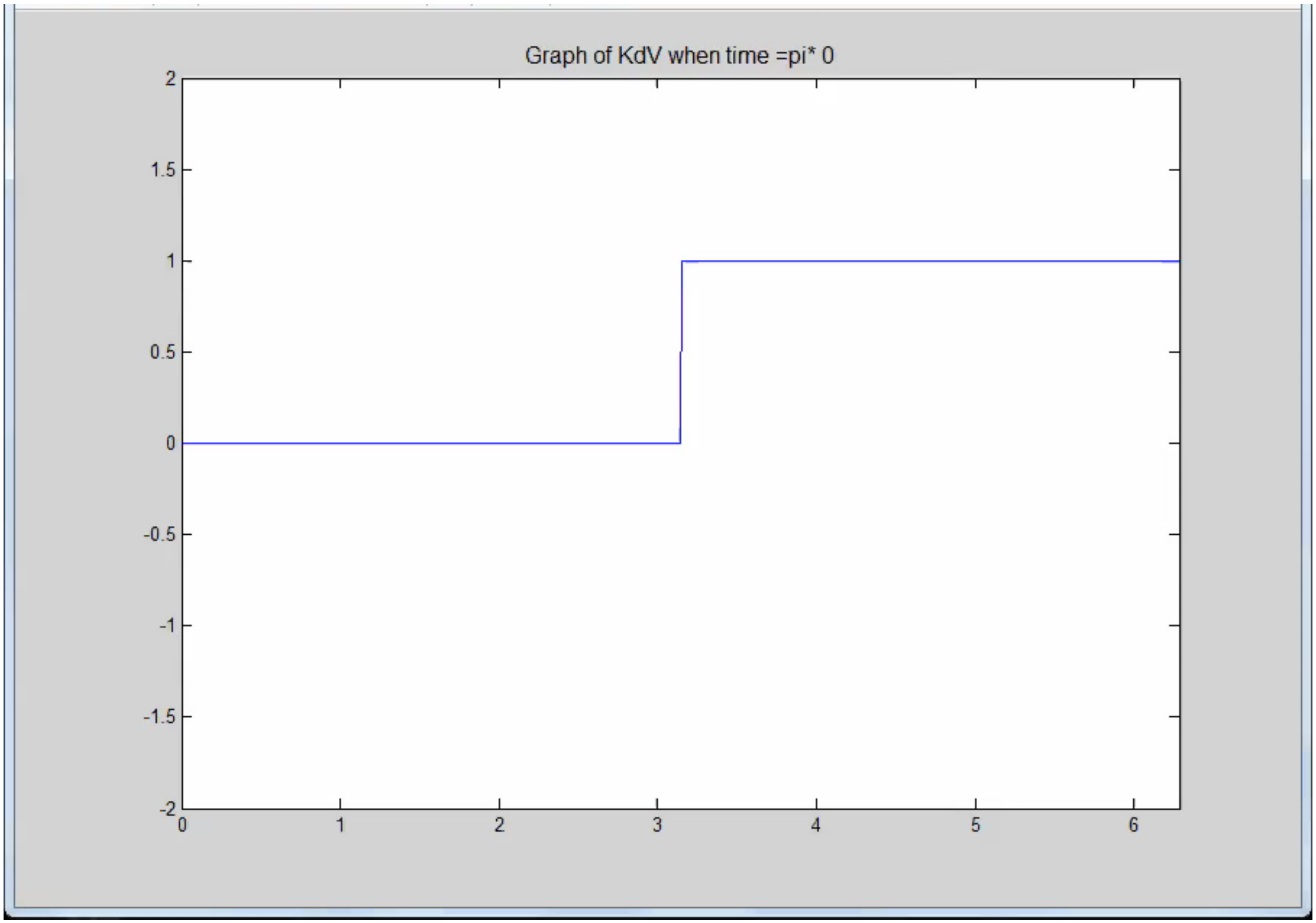
Gong Chen (2011)

$$\alpha = 1, \qquad \beta = .000484, \qquad \ell = 1, \qquad u(0, x) = \sigma(x).$$

Zabusky & Kruskal — birth of the soliton



Periodic KdV — dispersive quantization



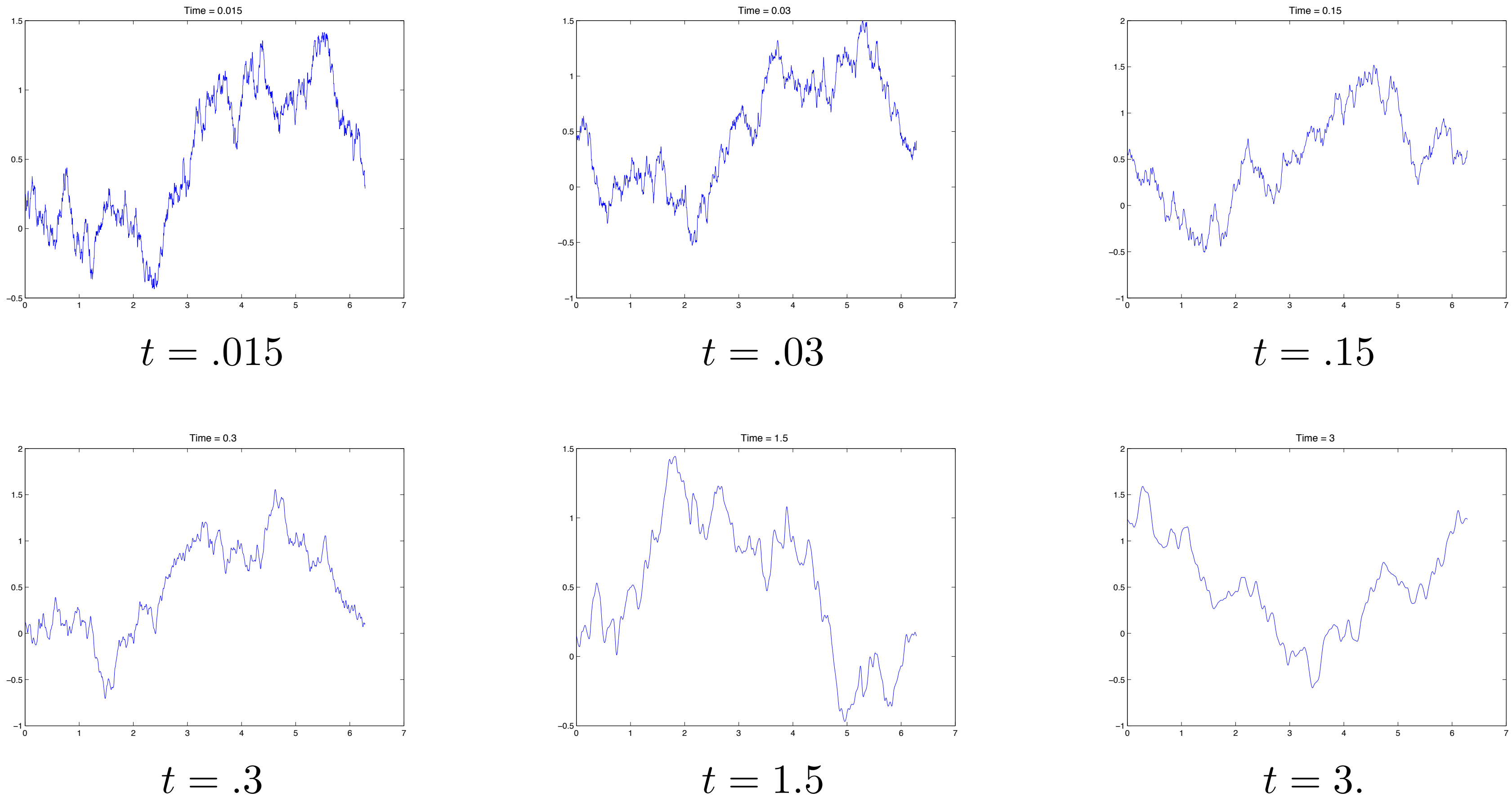


Figure 13. Korteweg–deVries Equation: Irrational Times.

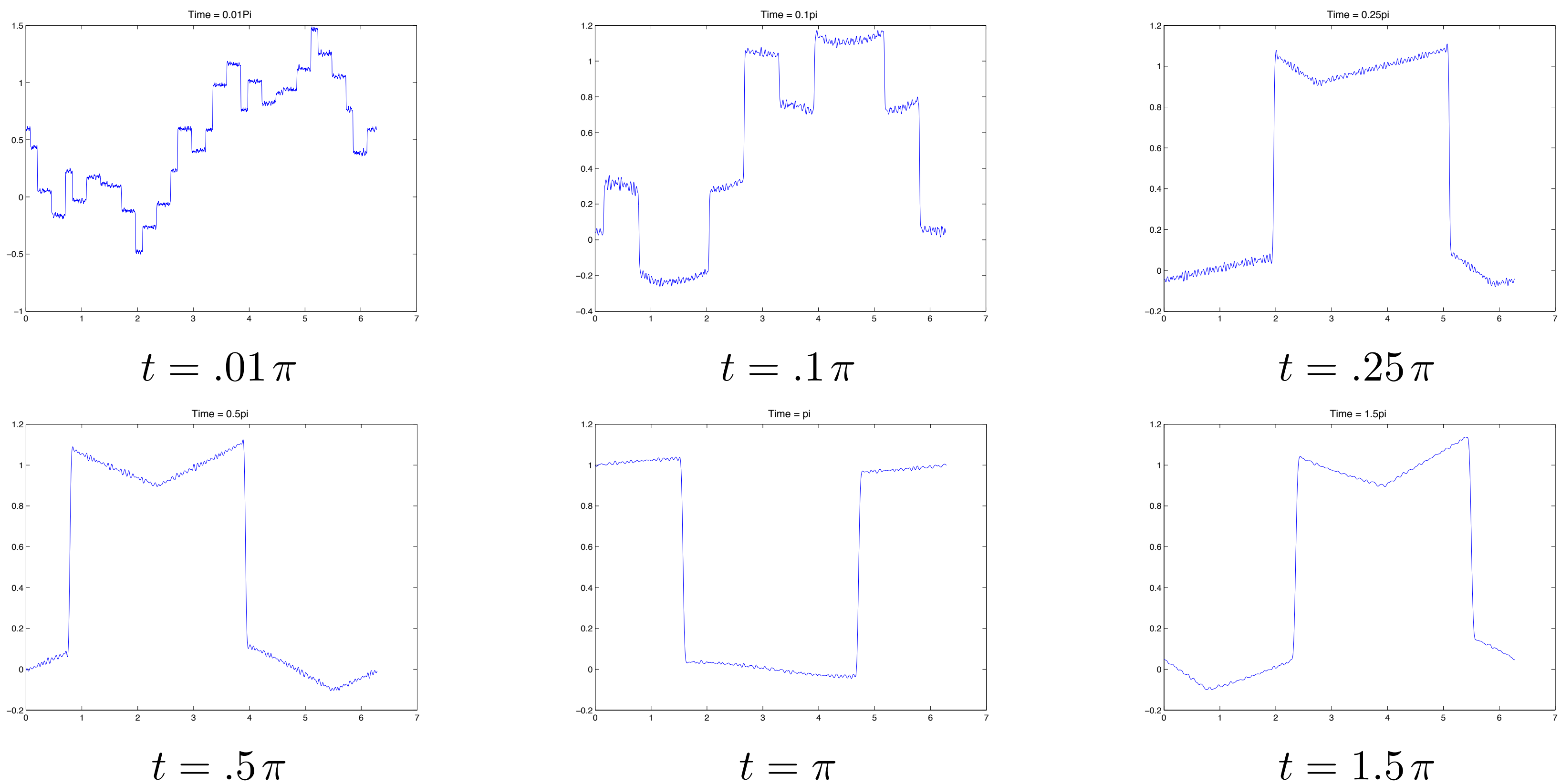
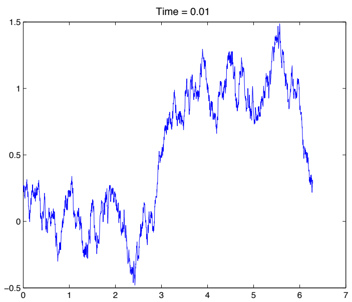
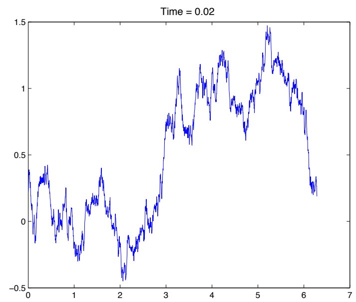


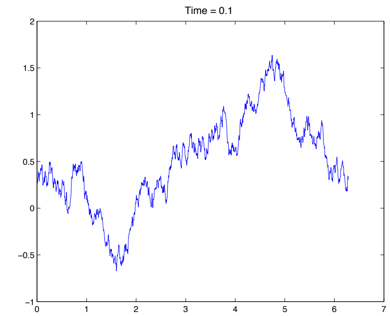
Figure 14. Korteweg–deVries Equation: Rational Times.



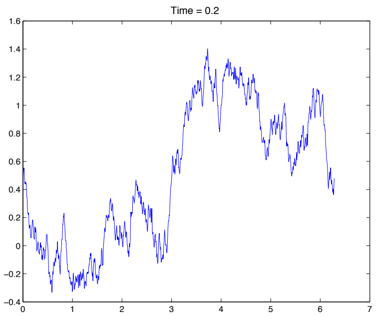
$t = .01$



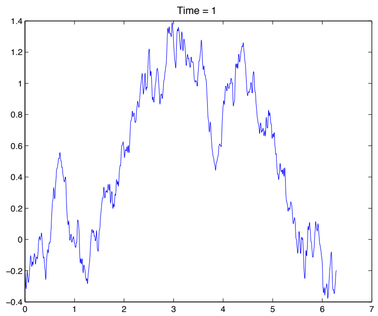
$t = .02$



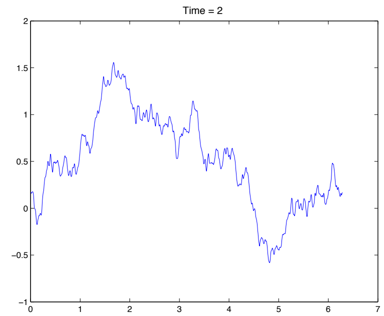
$t = .1$



$t = .2$



$t = 1$



$t = 2$

Figure 15. Quartic Korteweg–deVries Equation: Irrational Times.

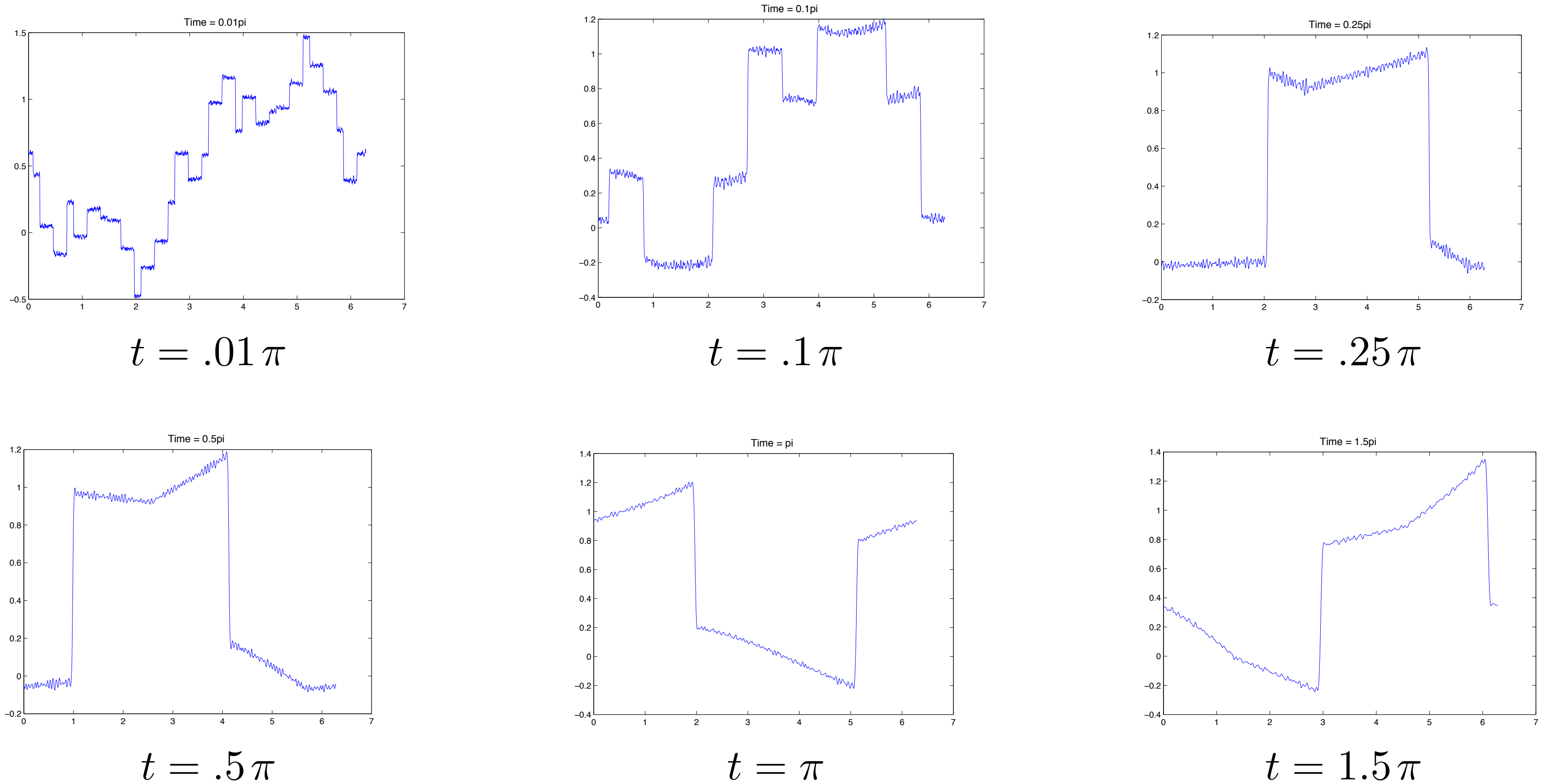


Figure 16. Quartic Korteweg–deVries Equation: Rational Times.

Periodic Nonlinear Schrödinger Equation

$$\mathrm{i}\, u_t + u_{xx} + |\, u\,|^p\, u = 0, \qquad x \in \mathbb{R}/\mathbb{Z}, \qquad u(0,x) = g(x).$$

Theorem. (Erdoğan, Tzirakis)

Suppose $p = 2$ (the integrable case) and $g \in \mathrm{BV}$. Then

(i) $u(t,\cdot)$ is continuous at irrational times $t \notin \mathbb{Q}$

(ii) $u(t,\cdot)$ is bounded with at most countably many discontinuities at rational times $t \in \mathbb{Q}$

(iii) When the initial data is sufficiently "rough", i.e., $g \notin \bigcup_{\epsilon>0} H^{1/2+\epsilon}$ then, at almost all t, the real or imaginary part of the graph of $u(t,\,\cdot\,)$ has fractal (upper Minkowski) dimension $\frac{3}{2}$.

Periodic Linear Dispersive Equations

$\Longrightarrow$ Chousionis, Erdoğan, Tzirakis

Theorem. Suppose $3 \leq k \in \mathbb{Z}$ and

$$\mathrm{i}\, u_t + (-\mathrm{i}\, \partial_x)^k u = 0, \qquad x \in \mathbb{R}/\mathbb{Z}, \qquad u(0,x) = g(x) \in \mathrm{BV}$$

(i) $u(t,\cdot)$ is continuous for almost all t

(ii) When $g \notin \bigcup_{\epsilon>0} H^{1/2+\epsilon}$, then, at almost all t, the real and imaginary parts of the graph of $u(t,\,\cdot\,)$ has fractal dimension $1 + 2^{1-k} \leq D \leq 2 - 2^{1-k}$.

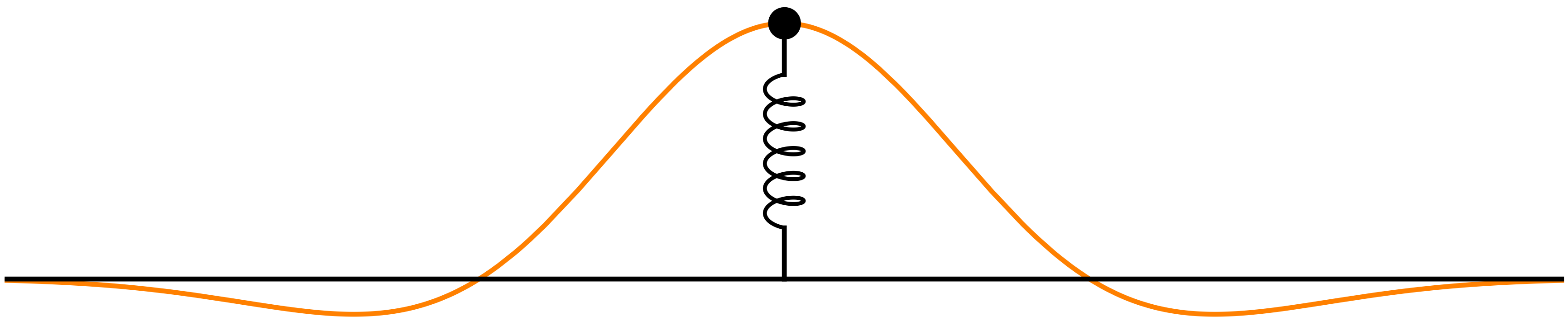
Theorem. For the periodic Korteweg–deVries equation

$$u_t + u_{xxx} + u\, u_x = 0, \qquad x \in \mathbb{R}/\mathbb{Z}, \qquad u(0,x) = g(x) \in \mathrm{BV}$$

(i) $u(t,\cdot)$ is continuous for almost all t

(ii) When $g \notin \bigcup_{\epsilon>0} H^{1/2+\epsilon}$, then, at almost all t, the real and imaginary parts of the graph of $u(t,\,\cdot\,)$ has fractal dimension $\frac{5}{4} \leq D \leq \frac{7}{4}$.

The Lamb Problem

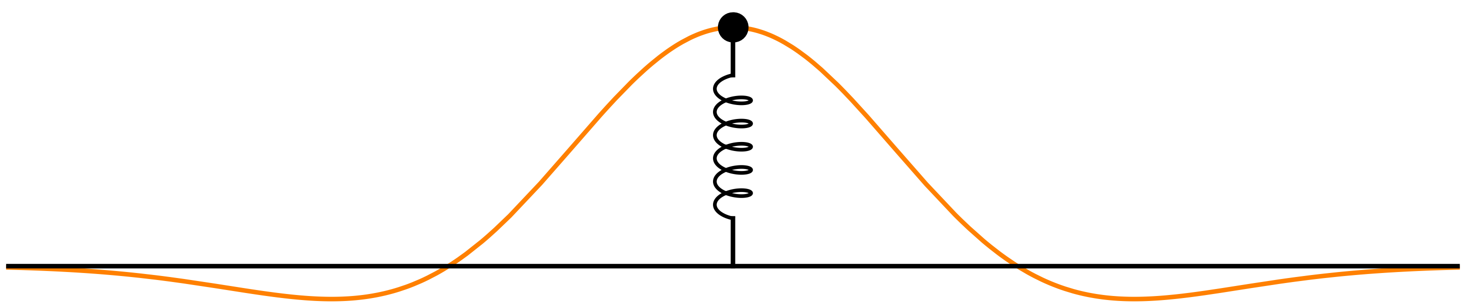


oscillating mass connected to an elastic string

The Lamb Problem

$\Longrightarrow$ Horace Lamb, 1900

Consider an oscillating mass connected to an elastic string.



Starting at rest, the mass is subject to a sudden blow.

$\Longrightarrow$ 1D model of radiation damping
of a vibrating body in a medium

Radiation Damping in Applications

- Vibrations of elastic sphere in a gaseous medium
- Electrical oscillations of a spherical conductor
- Dielectic sphere with large inductance
- Relativistic radiation of energy via gravity waves
- Quantum resonance of nuclei, etc.
- Radiative decay of sine–Gordon breathers

The Lamb Problem

- m — mass
- σ — mass oscillation frequency
- T — string tension
- ρ — string density
- $c = \sqrt{T/\rho}$ — wave speed of string
- $b = m/(2\,\rho)$ — mass damping coefficient
- $\kappa = \sqrt{\dfrac{\sigma^2}{c^2} - \dfrac{\rho^2}{m^2}}$ — damped oscillation frequency

In the linear regime, the string displacement $u(t, x)$ satisfies the usual wave equation

$$u_{tt} = c^2 u_{xx} \qquad x \neq 0$$

Force balance on the mass displacement $h(t) = u(t, 0)$ yields

$$m(h'' + \sigma^2 h) = -T\,[\,u_x\,]_0$$

the right hand side being the jump in u_x at the location of the mass: $x = 0$.

Equivalent model

$$u_{tt} = c^2 u_{xx} - 2\,c\,h'(t)\,\delta(x),$$

where

$$h'' + 2\,\beta\,h' + \sigma^2 h = 0, \qquad h(0) = 0, \qquad \beta = c/(2\,b).$$

$\Longrightarrow$ damped oscillator

Solution:

$$u(t,x) = \begin{cases} C\,e^{(|\,x\,|-c\,t)/(2\,b)} \sin \kappa\,(|\,x\,| - c\,t) & |\,x\,| < c\,t \\ 0 & |\,x\,| > c\,t \end{cases}$$

The Lamb Problem

Periodic Lamb Problem

We can solve the periodic problem by superposition or by Fourier series:

$$u(t,x) = \tfrac{1}{2}\,a_0(t) + \sum_{k=1}^{\infty} a_k(t)\cos k\,x.$$

where

$$a_k'' + \omega(k)^2 a_k = h'(t)/\pi, \qquad a_k(0) = a_k'(0) = 0.$$

$\omega(k)$ — dispersion relation wave equation: $\omega(k) = c\,k$.

Dispersion Asymptotics for the Lamb Problem

In general, if

$$\omega(k) \;\sim\; k^m \qquad \text{as} \qquad k \to \infty,$$

where $m > 0$, then

$$a_k(t) \;\sim\; \omega(k)^{-2} \;\sim\; k^{-2m} \qquad \text{as} \qquad k \to \infty.$$

Thus, the physical water wave dispersion

$$\omega(k) \;\sim\; \sqrt{|\,k\,|}$$

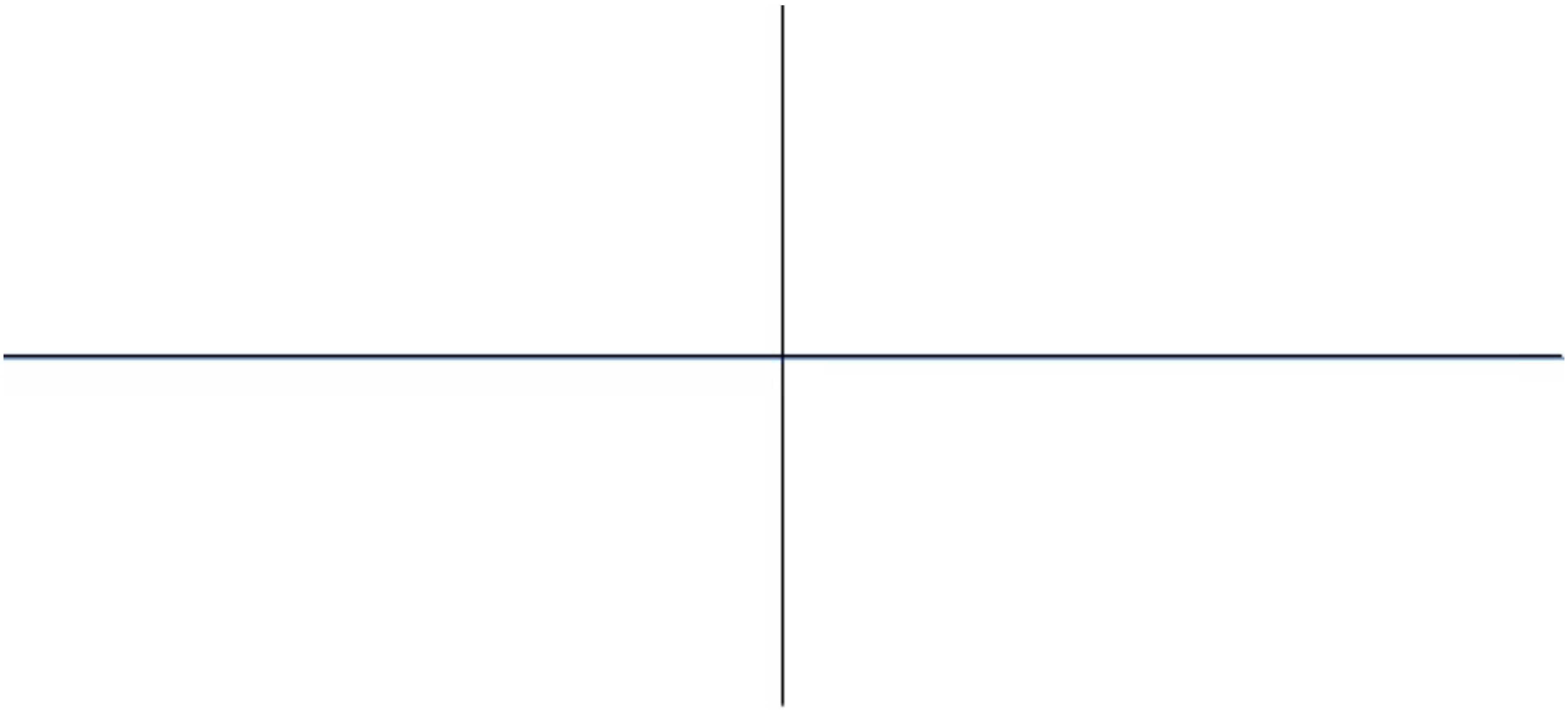
produces slow decay

$$a_k(t) \;\sim\; 1/|\,k\,|$$

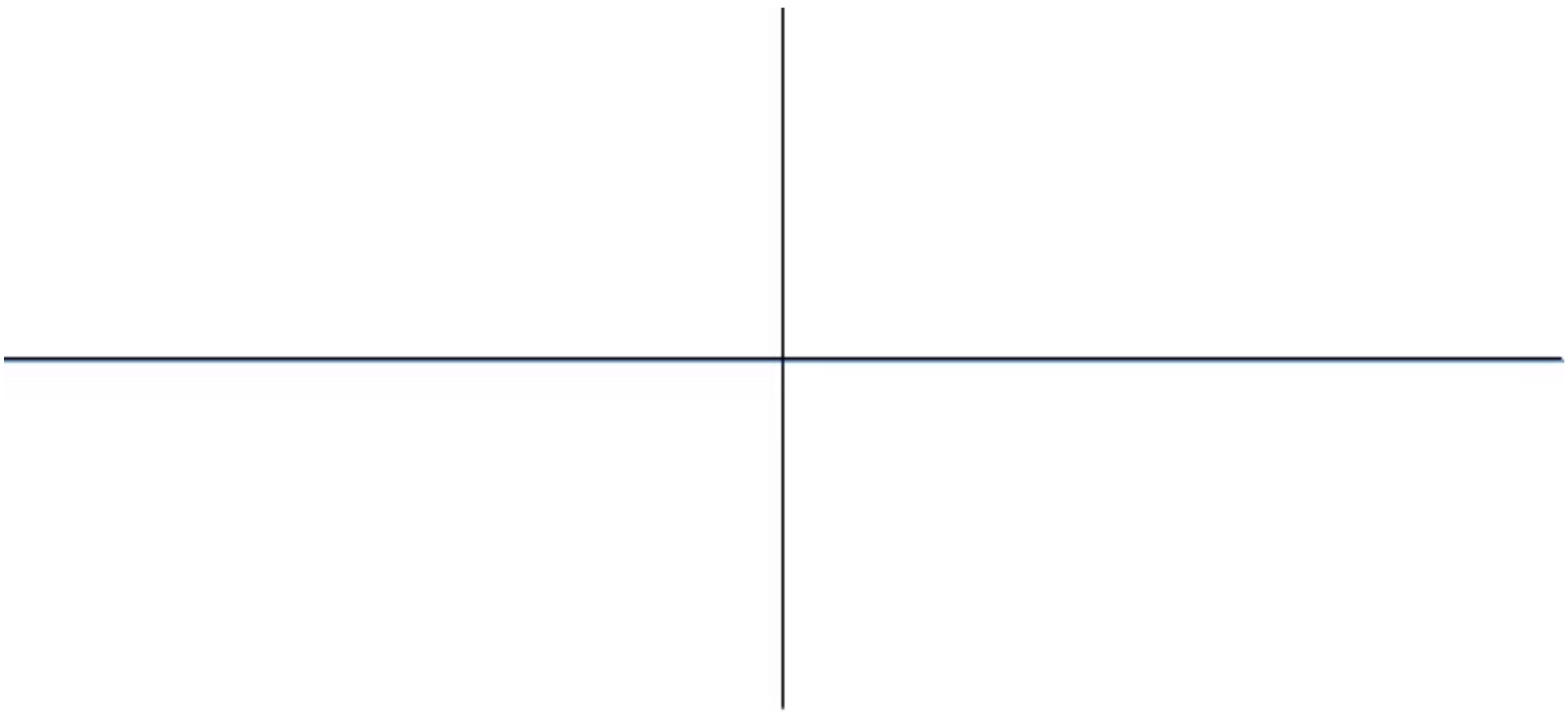
in the dispersive Lamb system indicative of fractalization.

Periodic Lamb

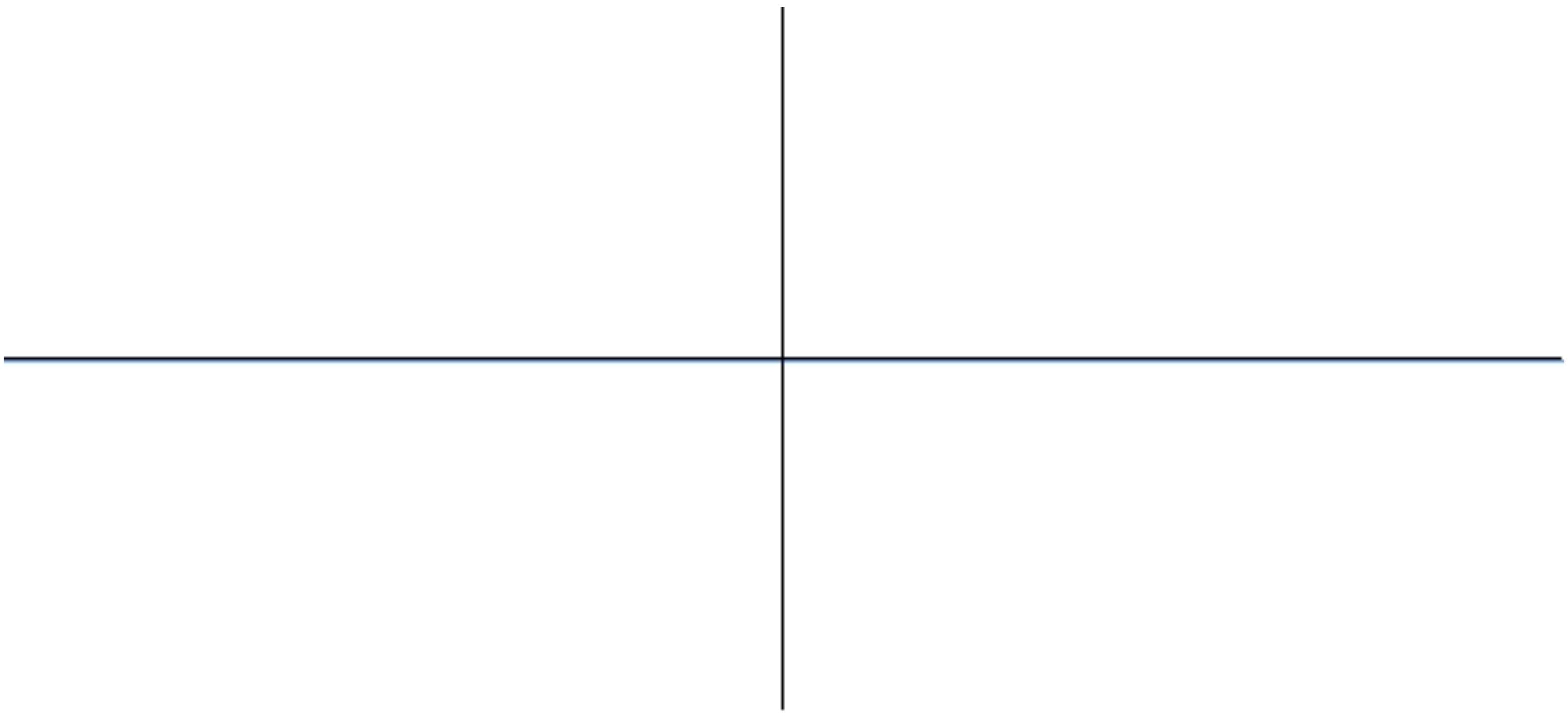
Higher order string model



Square root dispersion

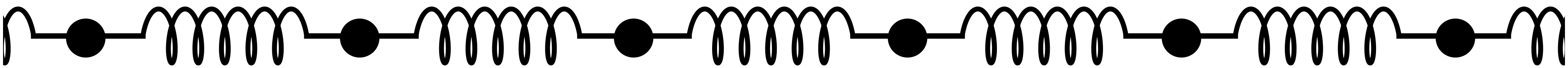


Square root dispersion



The Fermi–Pasta–Ulam–Tsingou Problem

$\Longrightarrow$ Los Alamos Report, 1955



- PJO + Ari Stern

The Fermi–Pasta–Ulam–Tsingou Problem

$\Longrightarrow$ Los Alamos Report, 1955

Our problem turned out to have been felicitously chosen. The results were entirely different qualitatively from what even Fermi, with his great knowledge of wave motions, had expected. . . . To our surprise, the string started playing a game of musical chairs, only between several low notes, and perhaps even more amazingly, after what would have been several hundred ordinary up and down vibrations, it came back almost exactly to its original sinusoidal shape.

— Stanislaw Ulam, *Adventures of a Mathematician*, pp. 226–7

The Fermi–Pasta–Ulam–Tsingou System

$$\mu^{-2}\,\frac{d^2u_n}{dt^2} = F(u_{n+1}-u_n) - F(u_n - u_{n-1})$$

$$= u_{n+1} - 2\,u_n + u_{n-1} + N(u_{n+1}-u_n) - N(u_n - u_{n-1}).$$

Forcing function and potential

$$F(y) = y + N(y) = V'(y), \qquad \text{where} \qquad V(y) = \tfrac{1}{2}\,y^2 + W(y)$$

Classical potentials: $N(y) = \alpha\,y^{\beta}$, $\beta = 2, 3$

Toda lattice: $N(y) = \alpha\,e^{\beta\,y}$

Continuum Limit

Periodic problem: m masses on a circle of unit radius with intermass spacing $h = 2\pi/m$. We suppose $m \longrightarrow \infty$.

Rescale time: $t \longmapsto h\,t$

$$\frac{d^2 u_n}{dt^2} = \frac{c^2}{h^2}\left[\, F(u_{n+1} - u_n) - F(u_n - u_{n-1})\,\right],$$

$$c = \mu\, h \quad \text{— wave speed}$$

Assume the displacements are obtained by sampling a function $u(t,x)$ at the nodes:

$$u_n(t) = u(t, x_n), \qquad \text{where} \qquad x_n = n\,h = 2\pi\, n/m.$$

Taylor expansion:

$$u_{n\pm 1}(t) = u(t, x_n \pm h) = u \pm h\, u_x + \tfrac{1}{2} h^2 u_{xx} \pm \tfrac{1}{6} h^3 u_{xxx} + \ \cdots \ ,$$

Continuum Models

$$u_{tt} = c^2 (\, K[\,u\,] + M[\,u\,] \,)$$

Linear component

$$K[\,u\,] = u_{xx} + \tfrac{1}{12} h^2 u_{xxxx} + \mathrm{O}(h^4)$$

Quadratic nonlinear component:

$$M[\,u\,] = 2\,\alpha\, h\, u_x u_{xx} + \tfrac{1}{6}\,\alpha\, h^3 u_x u_{xxxx} + \tfrac{1}{3}\,\alpha\, h^3 u_{xx} u_{xxx} + \mathrm{O}(h^5)$$

Bidirectional continuum model = potential Boussinesq equation

$$u_{tt} = c^2 (\, u_{xx} + 2\,\alpha\, h\, u_x u_{xx} + \tfrac{1}{12} h^2 u_{xxxx} \,)$$

Unidirectional model = Korteweg–deVries equation:

$$u_t = c\,(\, u_x + \alpha\, h\, u\, u_x + \tfrac{1}{24} h^2 u_{xxx} \,)$$

Linear FPU

Discrete wave equation:

$$\frac{d^2 u_n}{dt^2} = \frac{c^2}{h^2}(u_{n+1} - 2\,u_n + u_{n-1}),$$

Bidirectional continuum model

$$u_{tt} = c^2 u_{xx} + \tfrac{1}{12}\, c^2 h^2 u_{xxxx},$$

★ linearized "bad Boussinesq equation" — ill-posed.

Dispersion relation:

$$\omega^2 = p_4(k) = c^2 k^2 (\,1 - \tfrac{1}{12}\, h^2 k^2\,) < 0 \qquad \text{for} \qquad k \gg 0$$

Regularized Bidirectional Models

Sixth order linearized model:

$$u_{tt} = c^2\left(u_{xx} + \tfrac{1}{12} h^2 u_{xxxx} + \tfrac{1}{360} h^4 u_{xxxxxx} \right),$$

Dispersion relation:

$$\omega^2 = p_6(k) = c^2 k^2 \left(1 - \tfrac{1}{12} h^2 k^2 + \tfrac{1}{360} h^4 k^4 \right) > 0 \quad \text{for all} \quad k \neq 0$$

Alternatively, replacing

$$u_{xx} = c^{-2} u_{tt} + \mathrm{O}(h^2)$$

leads to the *linear Boussinesq equation*

$$u_{tt} = c^2 u_{xx} + \tfrac{1}{12} h^2 u_{xxtt}$$

Dispersion relation:

$$\omega^2 = q(k) = \frac{c^2 k^2}{1 + \tfrac{1}{12} h^2 k^2} > 0 \qquad \text{for all} \qquad k \neq 0$$

FPU Lattice Dispersion Relation

Substituting $u(t,x) = e^{\,\mathrm{i}\,(k\,x-\omega\,t)}$ evaluated at $x = x_n = n\,h$ into the linearized FPU system

$$\frac{d^2u_n}{dt^2} = \frac{c^2}{h^2}(u_{n+1} - 2\,u_n + u_{n-1}),$$

produces

$$-\omega^2 e^{\,\mathrm{i}\,(k\,x_n-\omega\,t)} = \frac{c^2}{h^2}\left(e^{\,\mathrm{i}\,(k\,x_n+k\,h-\omega\,t)} - 2\,e^{\,\mathrm{i}\,(k\,x_n-\omega\,t)} + e^{\,\mathrm{i}\,(k\,x_n-k\,h-\omega\,t)} \right)$$

$$= -\frac{2\,c^2}{h^2}(1 - \cos k\,h)\,e^{\,\mathrm{i}\,(k\,x_n-\omega\,t)}.$$

Discrete FPU dispersion relation:

$$\omega^2 = \frac{2\,c^2}{h^2}(1 - \cos k\,h) = \frac{4\,c^2}{h^2}\sin^2 \tfrac{1}{2}\,k\,h = \frac{c^2 m^2}{\pi^2}\sin^2 \frac{k\,\pi}{m}$$

The Continuum Riemann Problem

Step function initial data:

$$u(0,x) = \sigma(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{\sin(2j+1)x}{2j+1}$$

$$u_t(0,x) = 0$$

Bidirectional solution

$$u(t,x) = \frac{1}{2} + \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{\cos\omega(2j+1)t\ \sin(2j+1)x}{2j+1}.$$

Unidirectional right-moving constituent:

$$u_R(t,x) = \frac{1}{2} + \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{\sin[\,(2j+1)x - \omega(2j+1)t\,]}{2j+1},$$

The Discrete Riemann Problem

$$u_n(0) = \begin{cases} 1, & 0 < n < m, \\ 0, & -m < n < 0, \\ \frac{1}{2}, & n = -m,\ 0,\ m. \end{cases}$$

Discrete Fourier Transform:

$$u(0,x) \ \sim\ \frac{1}{2} + \frac{1}{m} \sum_{j=0}^{[m/2]} \cot \frac{(2j+1)\pi}{2m}\ \sin(2j+1)x.$$

Linear FPU solution:

$$u(t,x) \sim \frac{1}{2} + \frac{1}{m} \sum_{j=0}^{[m/2]} \cot \frac{(2j+1)\pi}{2m}\ \cos\left(\frac{cmt}{\pi} \sin \frac{(2j+1)\pi}{m} \right) \sin(2j+1)x,$$

Right-moving constituent:

$$u_R(t,x) \sim \frac{1}{2} + \frac{1}{2m} \sum_{j=0}^{[m/2]} \cot \frac{(2j+1)\pi}{2m}\ \sin\left((2j+1)x - \frac{cmt}{\pi} \sin \frac{(2j+1)\pi}{m} \right).$$

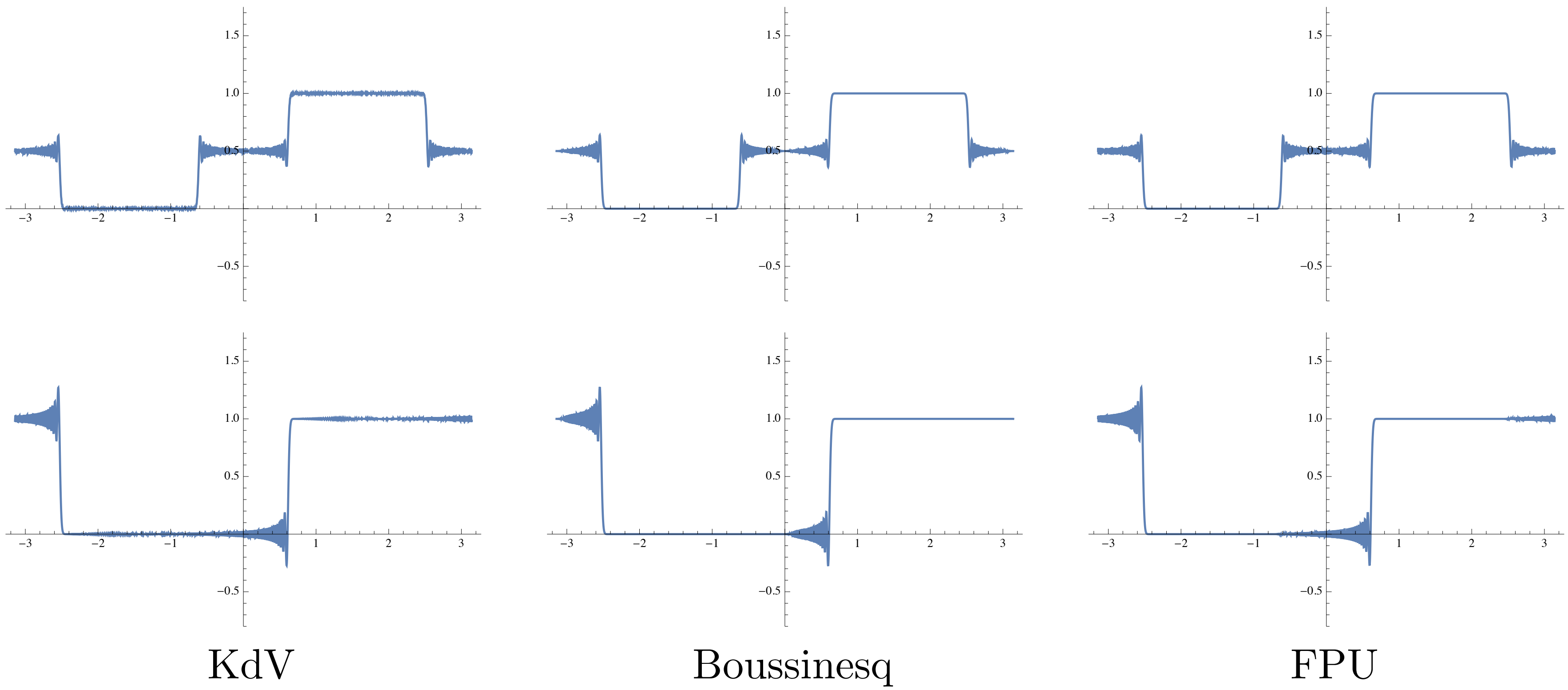


Figure 1. Bi- and uni-directional solution profiles at $t = \frac{1}{5}\pi$.

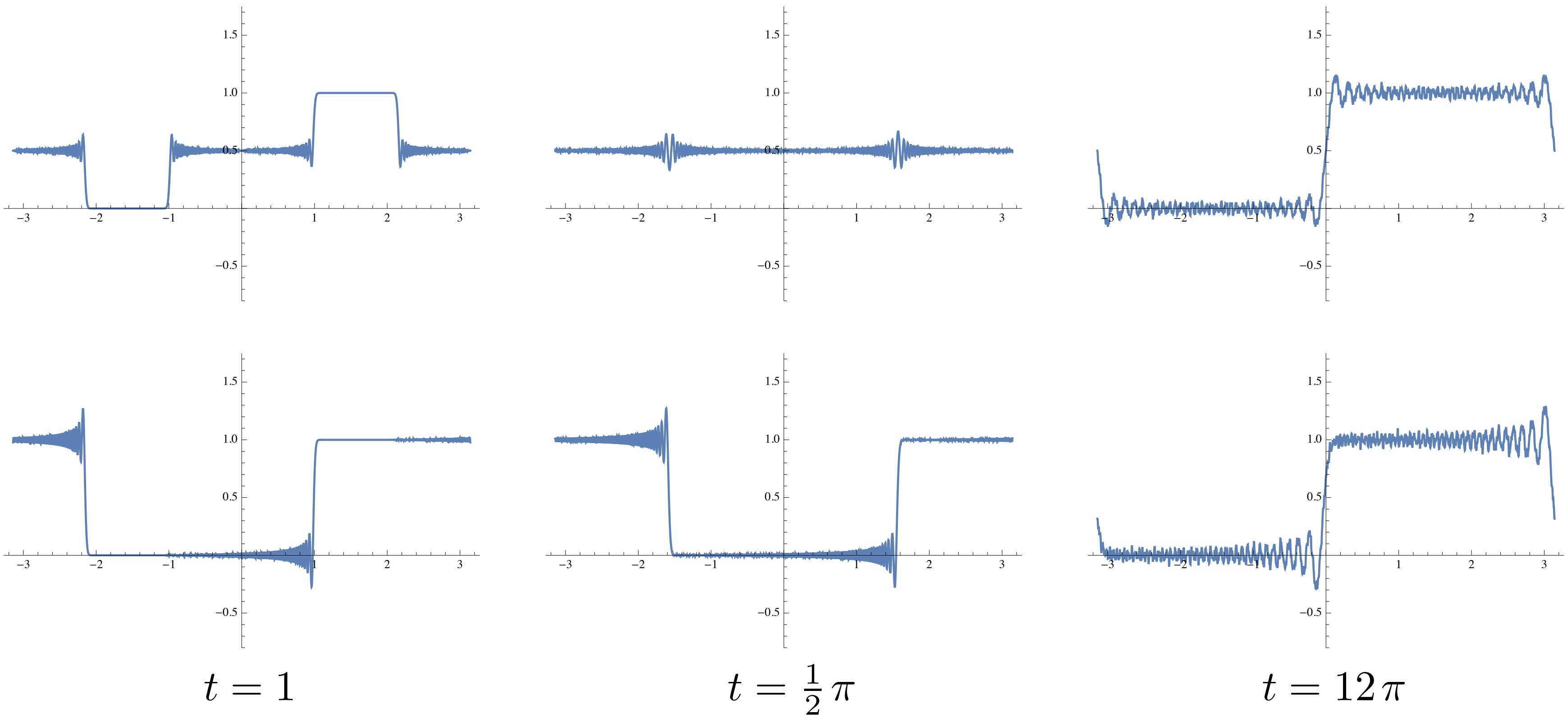
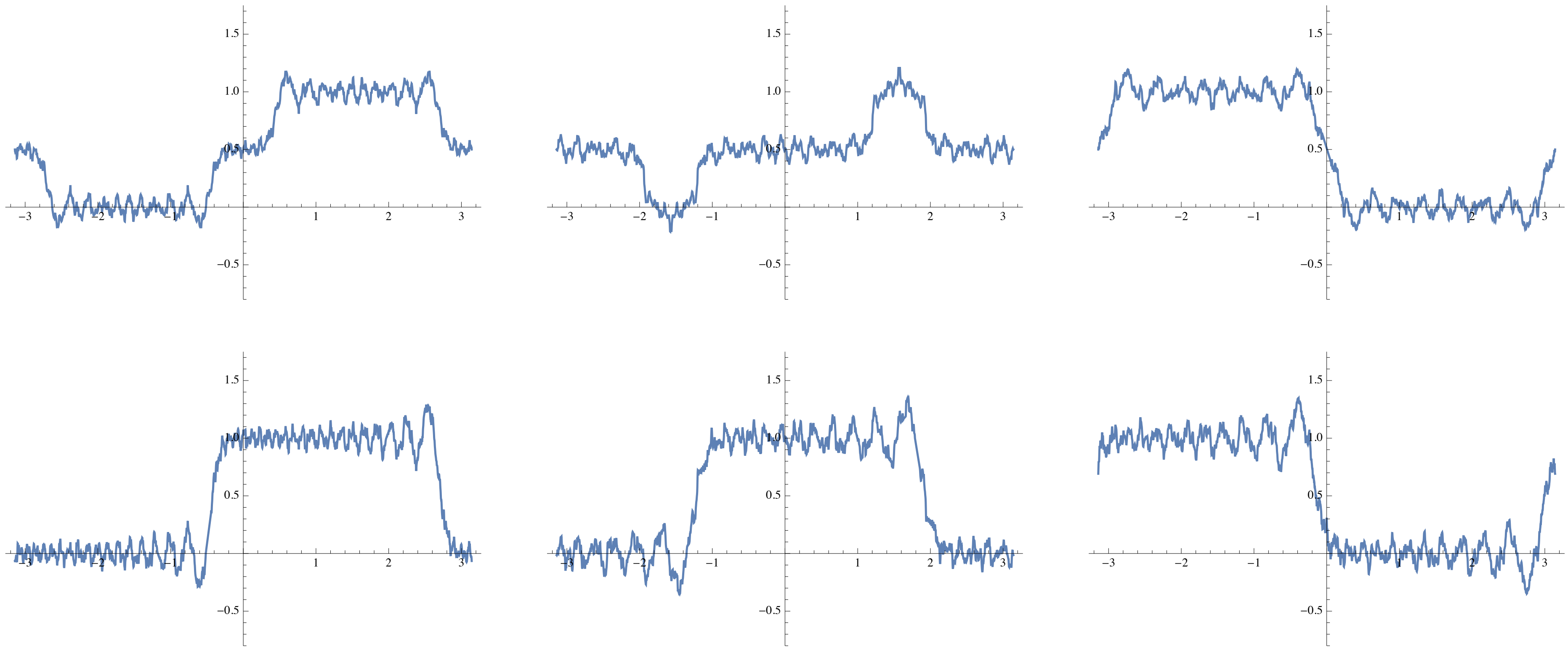
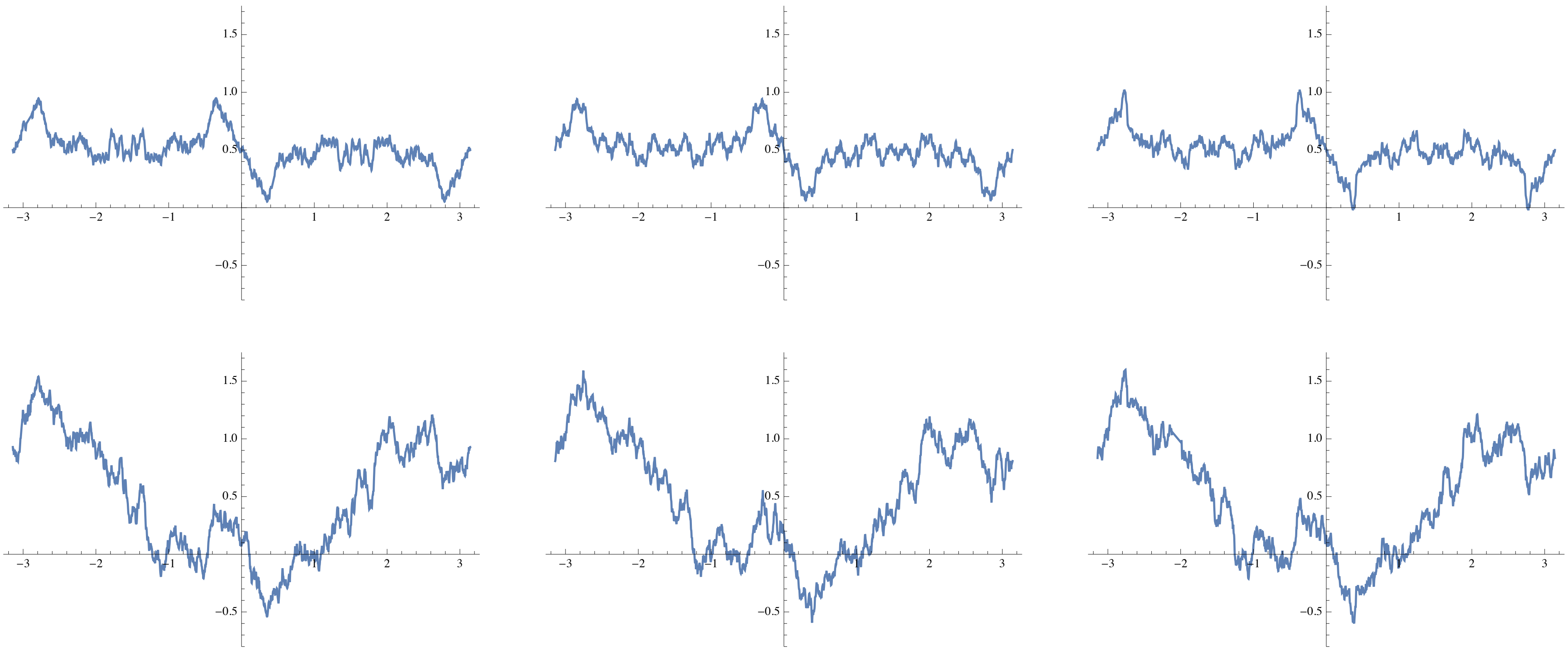


Figure 2. Bi- and unidirectional FPU solution profiles.



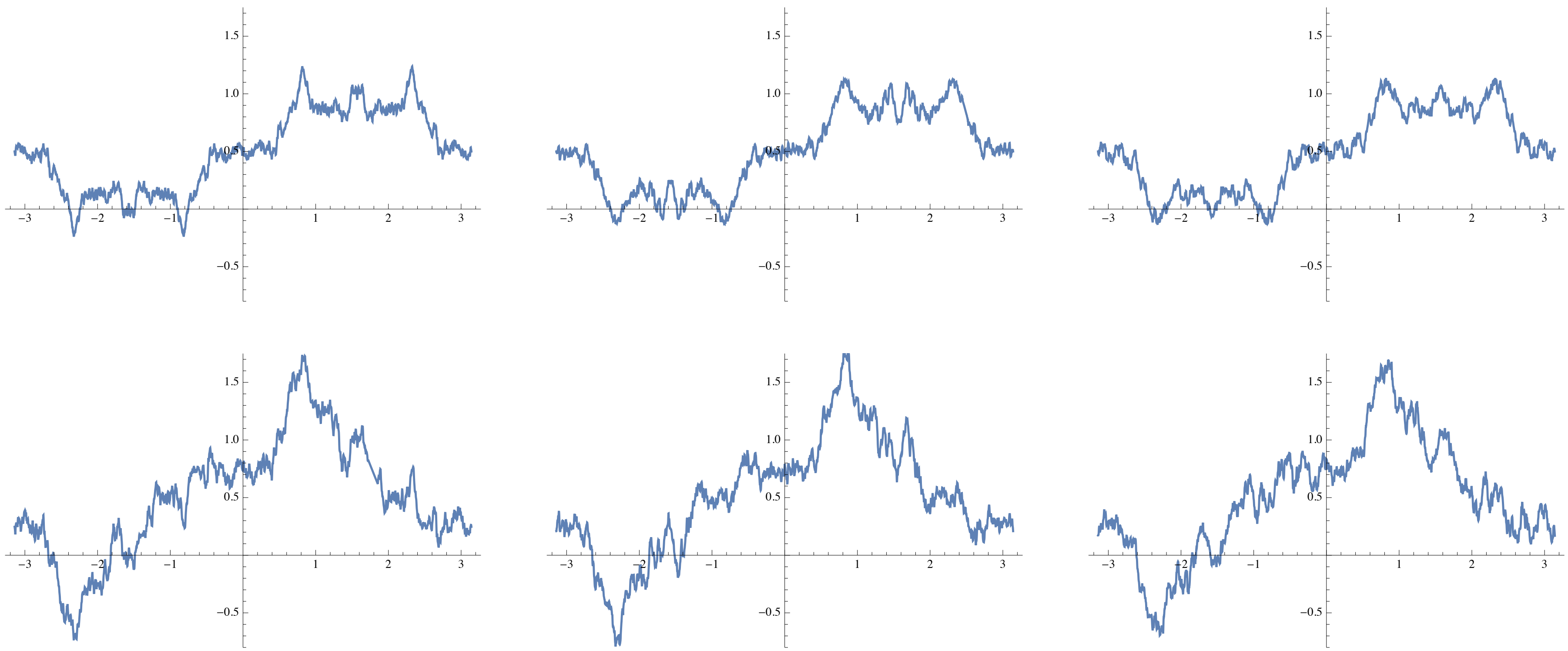
$t = 1/h$ $t = 3/h$ $t = \pi/h$

Figure 3. Bi- and unidirectional FPU solution profiles.



KdV Boussinesq FPU

Figure 4. Bi- and unidirectional solution profiles at $t = 1/h^2$.



KdV Boussinesq FPU

Figure 5. Bi- and unidirectional solution profiles at $t = 400{,}000$.

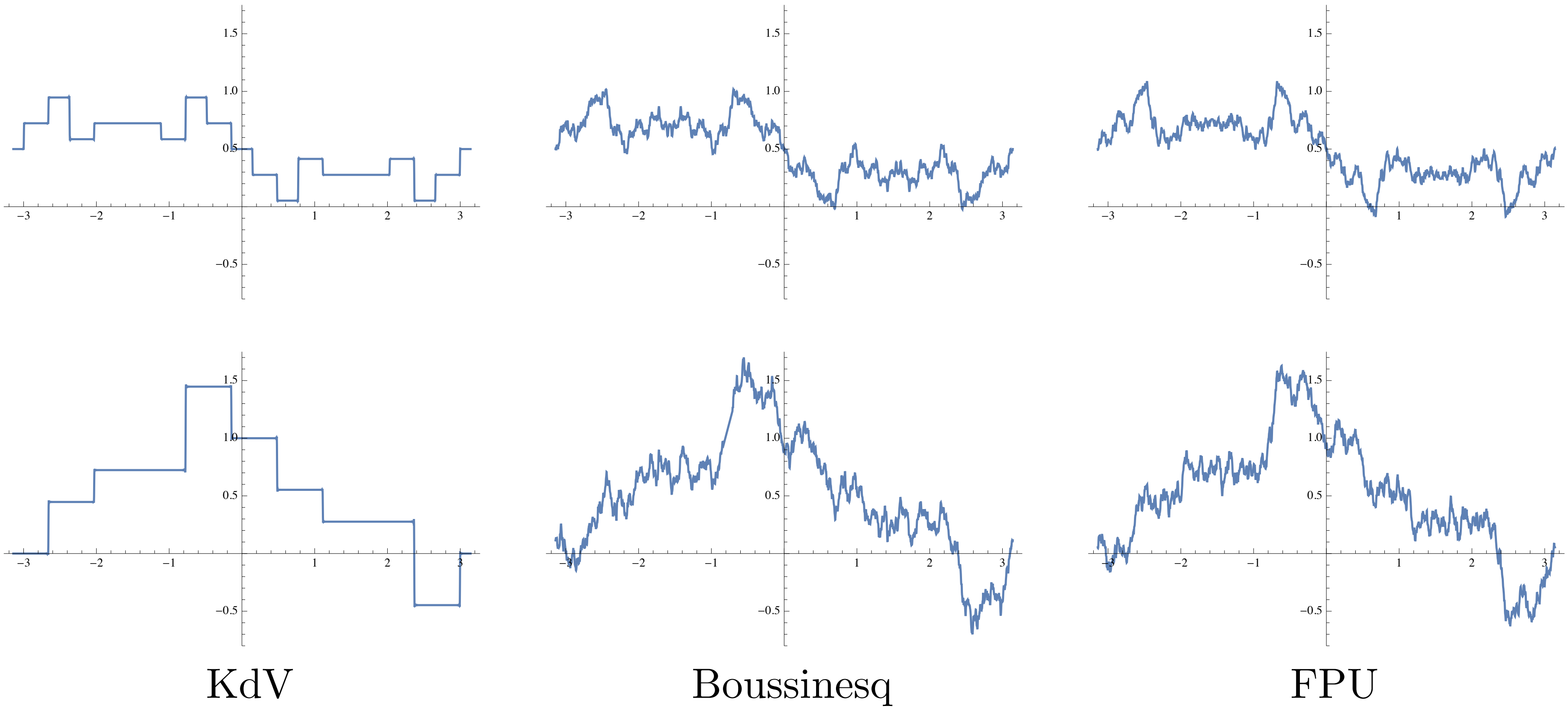


Figure 6. Bi- and unidirectional solution profiles at $t = 24\pi/(5h^2) \approx 400{,}527$.

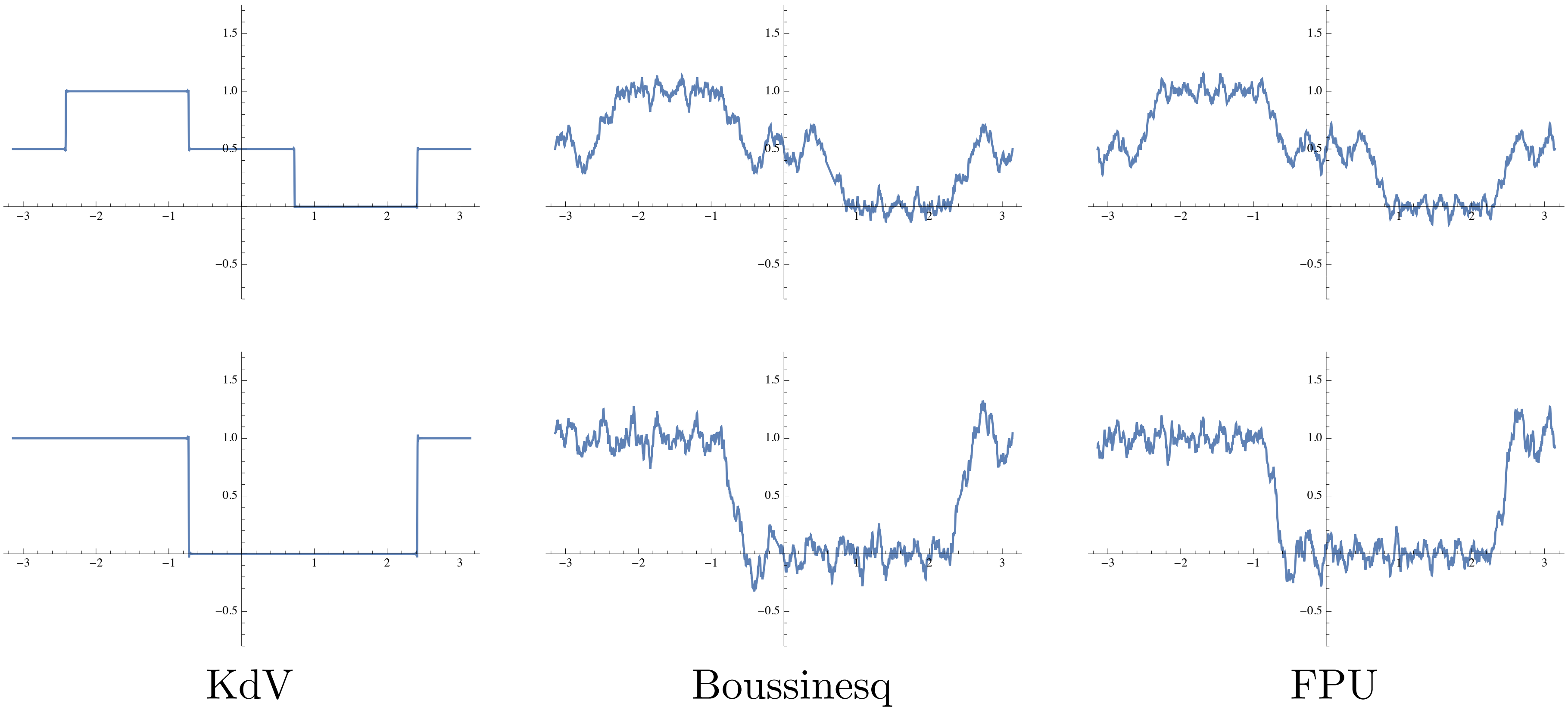


Figure 7. Bi- and unidirectional solution profiles at $t = 24\pi/h^2$.

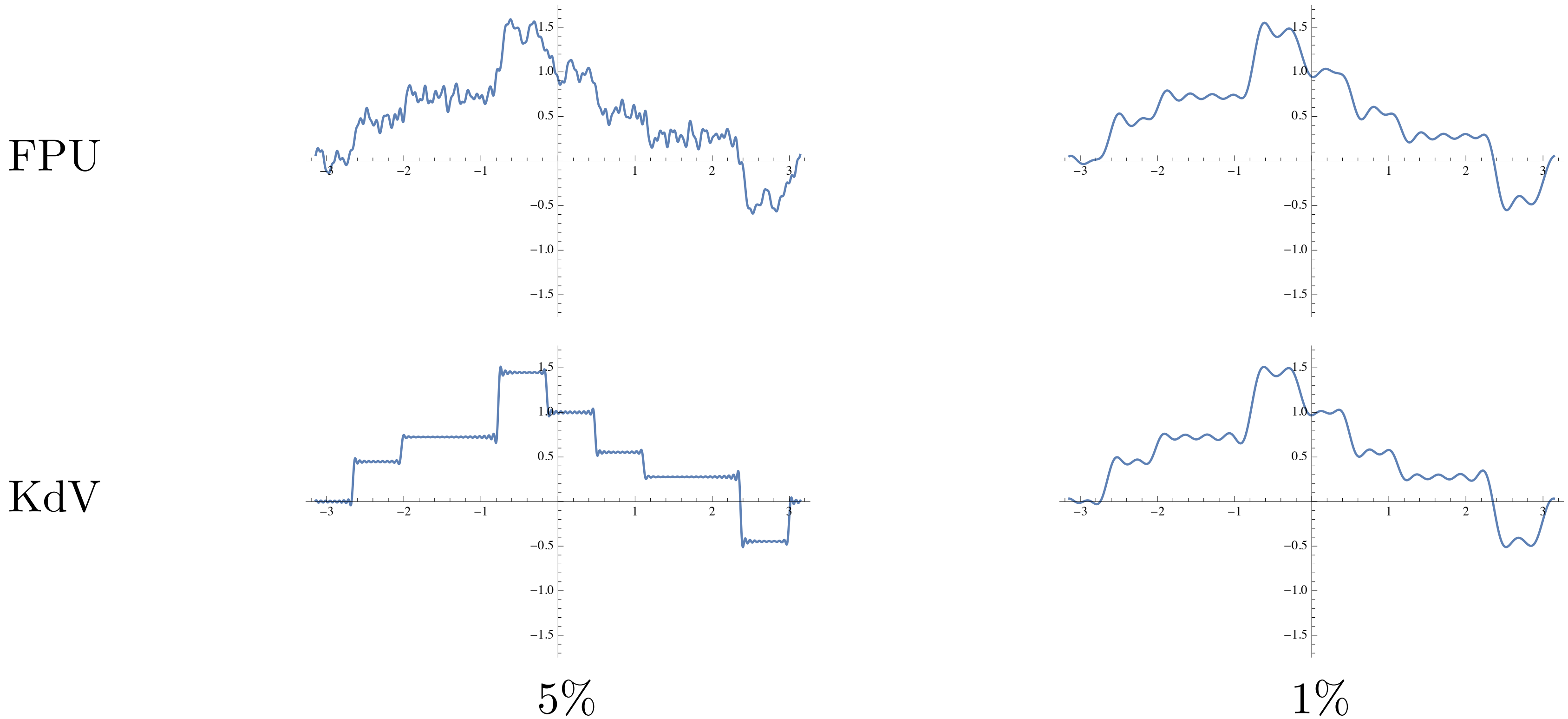


Figure 8. Truncated unidirectional solution profiles at $t = 24\pi/(5h^2) \approx 400{,}527$.

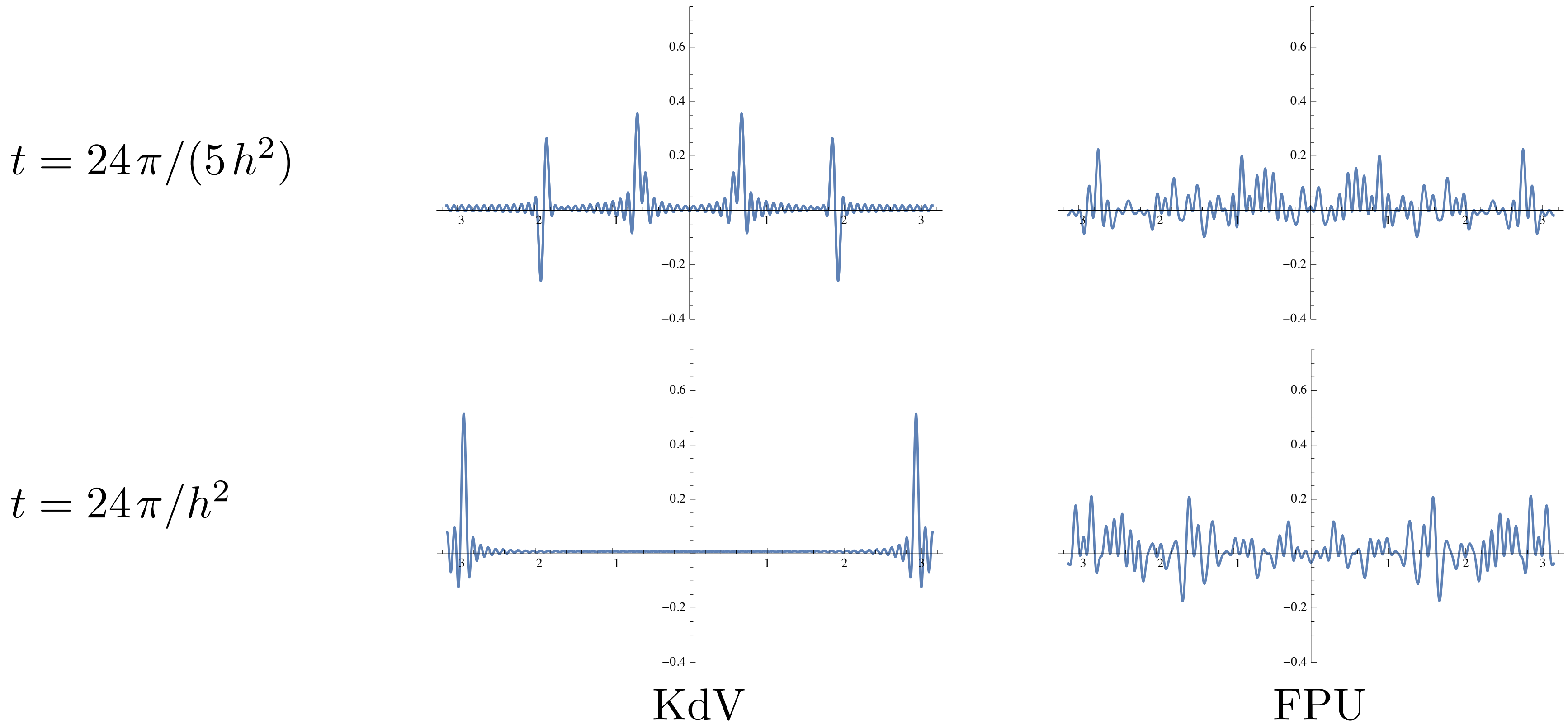


Figure 11. Revival and lack thereof.

Numerical Integration of Linear FPUT

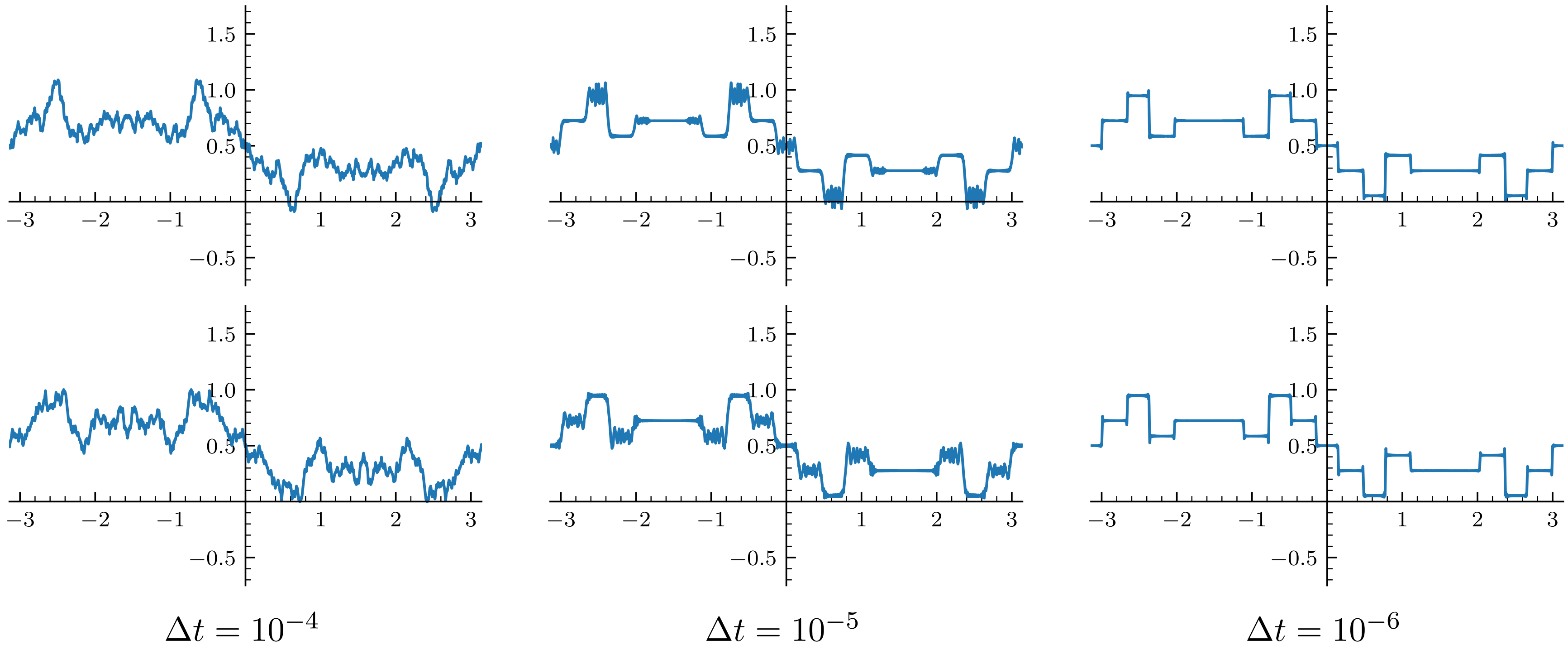


FIGURE 12. Numerical approximation of the bidirectional KdV solution profile with $m = 512$ at $t = 24\pi/(5\,h^2)$, showing the effect of time step size Δt for the Störmer/Verlet method (top) and midpoint method (bottom).

Numerical Integration of Nonlinear FPUT

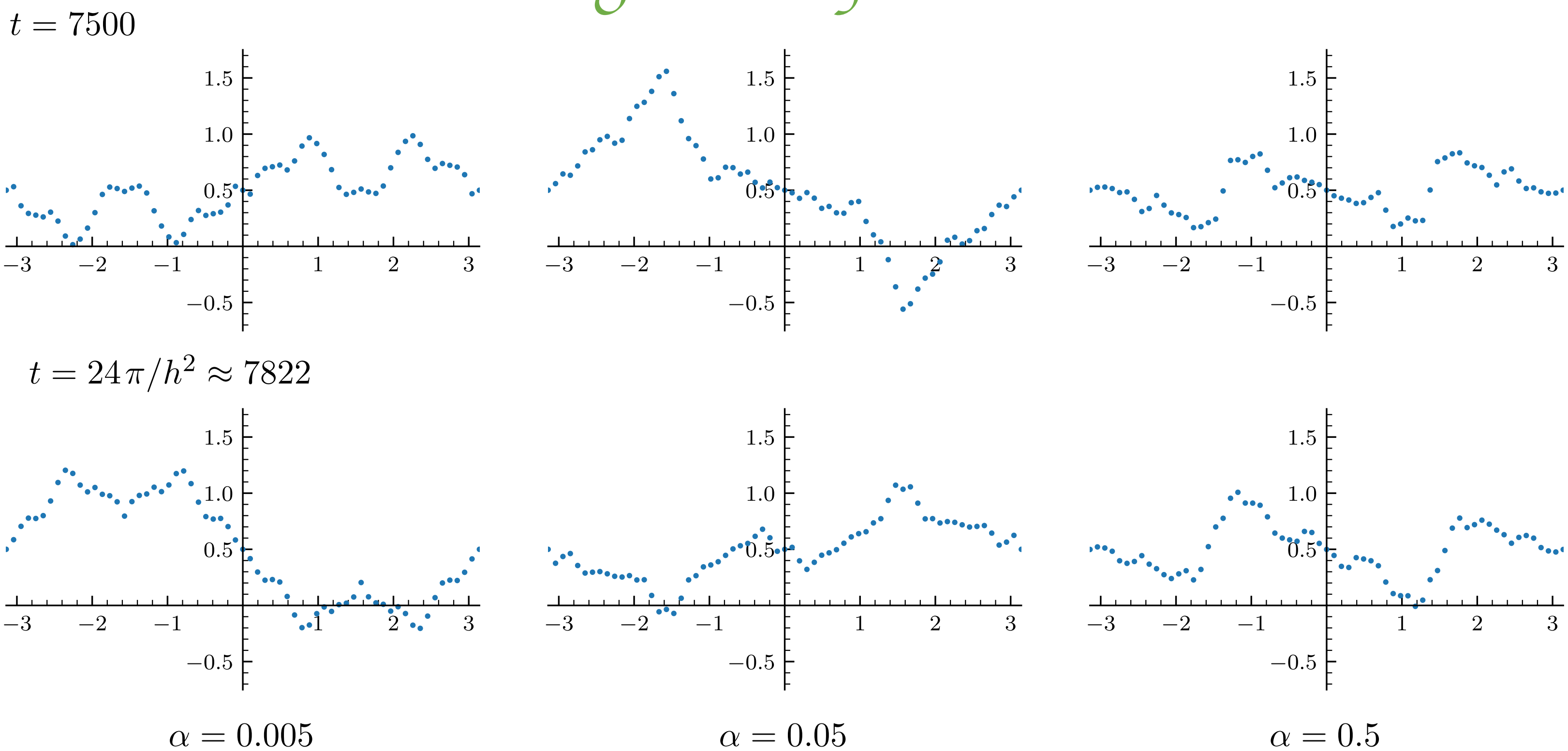


FIGURE 14. Bidirectional solution profiles for the discrete FPUT system with $m = 32$ and quadratic nonlinearity $N(y) = \alpha y^2$.

❖ Blanes–Moan Runge–Kutta–Nyström method with 10^6 time steps.

Runge–Kutta–Nyström (RKN) methods are designed specifically for splittings of the form (4.6), i.e., for second-order Newtonian systems written in first-order form using a velocity variable. Of these, we chose the optimal 14-stage order-6 RKN method of Blanes and Moan, [6], which has the symmetric form

$$\varphi^A_{a_1\Delta t} \circ \varphi^B_{b_1\Delta t} \circ \cdots \circ \varphi^A_{a_7\Delta t} \circ \varphi^B_{b_7\Delta t} \circ \varphi^A_{a_8\Delta t} \circ \varphi^B_{b_7\Delta t} \circ \varphi^A_{a_7\Delta t} \circ \cdots \circ \varphi^B_{b_1\Delta t} \circ \varphi^A_{a_1\Delta t}, \tag{4.8}$$

where the coefficients a_i, b_i are

$$\begin{aligned}
a_1 &= 0.0378593198406116, & b_1 &= 0.09171915262446165,\\
a_2 &= 0.102635633102435, & b_2 &= 0.183983170005006,\\
a_3 &= -0.0258678882665587, & b_3 &= -0.05653436583288827,\\
a_4 &= 0.314241403071447, & b_4 &= 0.004914688774712854,\\
a_5 &= -0.130144459517415, & b_5 &= 0.143761127168358,\\
a_6 &= 0.106417700369543, & b_6 &= 0.328567693746804,\\
a_7 &= -0.00879424312851058, & b_7 &= \tfrac{1}{2} - (b_1 + \cdots + b_6),\\
a_8 &= 1 - 2(a_1 + \cdots + a_7).
\end{aligned} \tag{4.9}$$

Future Directions

- General dispersion behavior explanation/justification
- Stability analysis
- Improved numerical solution techniques
- Other boundary conditions
- Nonlinearly dispersive models: Camassa–Holm, . . .
- Discrete systems: Fermi–Pasta–Ulam, spin chains, . . .
- Higher space dimensions and other domains: tori, spheres, . . .
- Experimental verification in dispersive media?