# On the stability of the periodic waves for the Benney and Zakharov systems

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July 12, 2022
Mathematical Aspects of the Physics
with non-self-Adjoint Operators
BIRS Workshop

The study of orbital stability of solitary waves for nonlinear dispersive equations goes back to Benjamin and Bona in the early 1970. In the late 1980, Grillakis, Shatah and Strauss developed a general theory for orbital stability of nonlinear systems with symmetries. Many later works rely on the GSS approach in the sense that they establish orbital stability based on conservation laws. This almost always requires a $C^1$ dependence on the wave speed parameters, which is not always easy to establish and has to be assumed in some cases.

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We study two different PDE systems and analyze the spectral stability of their periodic traveling wave solutions. We construct the periodic traveling waves of dnoidal and snoidal type for the Benney system and the dnoidal solutions for the Zakharov system. We then study the corresponding linearized problems and use the index counting theory to analyze the spectral stability of these periodic traveling waves.

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We use the original presentation by Kevrekidis, Kapitula, Sandstede, but the same result appears by Pelinovsky, while the most general version can be found in a recent paper by Lin, Zeng. Consider the Hamiltonian eigenvalue problem

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where $\mathcal{J}, \mathcal{H}$ map real-valued elements into real-valued elements. Introduce the Morse index of a self-adjoint, bounded from below operator $S$, by setting $n(S) = \#\{\lambda \in \sigma(S) : \lambda < 0\}$, counted with multiplicities. Let $k_r := \#\{\lambda \in \sigma_{pt.}(\mathcal{J}\mathcal{L}) : \lambda > 0\}$ represents the number of positive real eigenvalues of $\mathcal{J}\mathcal{L}$, counted with multiplicities, $k_c := \#\{\lambda \in \sigma_{pt.}(\mathcal{J}\mathcal{L}) : \Re\lambda > 0, \Im\lambda > 0\}$ - the number of quadruplets of complex eigenvalues of $\mathcal{J}\mathcal{L}$ with non-zero real and imaginary parts, whereas $k_i^- = \#\{i\lambda, \lambda > 0 : \mathcal{J}\mathcal{L}f = i\lambda f, \langle \mathcal{L}f, f\rangle < 0\}$ is the number of pairs of purely imaginary eigenvalues of negative Krein signature.

Consider the generalized kernel of $\mathcal{J}\mathcal{H}$, and introduce a symmetric matrix $D$ by

$$D := \{\{D_{ij}\}_{i,j=1}^{N} \ : \ D_{ij} = \langle \mathcal{L}\eta_i, \eta_j \rangle\}.$$

Consider the generalized kernel of $\mathcal{JH}$, and introduce a symmetric matrix $D$ by

$$D := \{\{D_{ij}\}_{i,j=1}^{N} \ : \ D_{ij} = \langle \mathcal{L}\eta_i, \eta_j \rangle\}.$$

We have the following formula for the Hamiltonian index,

$$k_{Ham} := k_r + 2k_c + 2k_i^- = n(\mathcal{L}) - n(D). \tag{2}$$

Clearly, spectral stability for (1) follows from $k_{Ham} = 0$, but such a condition is not necessary for spectral stability. For example, one might encounter a situation where $k_{Ham} = 2$, but with $k_i^- = 1$, which is an example of spectrally stable configuration with a non-zero $K_{Ham}$.. On the other hand, it is clear that if $k_{Ham}$ is an odd integer, then $k_r \geq 1$, guaranteeing instability.

We start with the following Benney system, where $\beta$ is a real parameter, $u$ is complex valued function, and $v$ is real-valued function.

$$\begin{cases} iu_t + u_{xx} = uv + \beta|u|^2u, \quad -T \leq x \leq T, t \in \mathbf{R}^1_+ \\ v_t = (|u|^2)_x, \end{cases} \tag{3}$$

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The Cauchy problem on the whole line for the Benney system was studied in 1998 by Bekiranov, Ogawa, Ponce and later by Corcho.

The existence and nonlinear stability of solitary waves was studied by Laurencot in 1995 and independently by by Guo, Chen in 1998.

We consider such model on a periodic domain, that is, we impose periodic boundary conditions. We study the spectral stability of periodic traveling waves of dnoidal and snoidal type, which we construct first. We are interested in the stability of these periodic traveling wave solutions with respect to perturbations that are periodic of the same period as the corresponding wave solutions. The main results are that, for all natural values of the parameters, the periodic dnoidal waves are spectrally stable with respect to perturbations of the same period. For another natural set of parameters, we construct the snoidal waves, which exhibit instabilities, in the same setup.

The problem has been studied in this context by Angulo, Corcho and Hakkaev. The authors proved, via the Fourier restriction method, that the problem is locally well-posed for $(u_0, v_0) \in H^r[-T,T] \times H^s[-T,T]$, if $\max(0, r-1) \leq s \leq \min(r, 2r-1)$. In particular, it is well posed in $H^{\frac{1}{2}}([-T,T]) \times L^2[-T,T]$ and also $H^1([-T,T]) \times L^2[-T,T]$. It is ill-posed (non-uniformly continuous dependence on initial conditions) in $H^r \times H^s$, whenever $r < 0$.

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$$M(u) = \int_{-T}^{T} |u(t,x)|^2 dx$$

$$E(u,v) = \int_{-T}^{T} \left[ v(t,x)|u(t,x)|^2 + |u_x(t,x)|^2 + \frac{\beta}{2}|u(t,x)|^4 \right] dx$$

$$P(u,v) = \int_{-T}^{T} \left[ |v(t,x)|^2 + 2\Im(u(t,x)\overline{u}_x(t,x)) \right] dx.$$

Consider periodic waves of the form

$$u(t,x) = e^{i\omega t} e^{i\frac{c}{2}(x-ct)} \varphi(x-ct), \quad v(t,x) = \psi(x-ct),$$

for the Benney system, which satisfy

$$\begin{cases} \varphi'' - \left(\omega - \frac{c^2}{4}\right)\varphi = \varphi\psi + \beta\varphi^3 \\ -c\psi' = 2\varphi\varphi' \end{cases} \tag{4}$$

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Integrating second equation in (4), we get $\psi = -\frac{1}{c}\varphi^2 + \gamma$, where $\gamma$ is a constant of integration. Substituting $\psi$ in the first equation, we get

$$\varphi'' - \sigma\varphi = \left(\beta - \frac{1}{c}\right)\varphi^3, \tag{5}$$

where we have introduced the important parameter $\sigma = \omega - \frac{c^2}{4} + \gamma$.

Integrating, we get

$$\varphi'^2 = \frac{1}{2}\left(\beta - \frac{1}{c}\right)\varphi^4 + \sigma\varphi^2 + a =: U(\varphi), \tag{6}$$

with a constant of integration $a$.

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$$\varphi'^2 = \frac{1}{2}\left(\beta - \frac{1}{c}\right)\varphi^4 + \sigma\varphi^2 + a =: U(\varphi), \tag{6}$$

with a constant of integration $a$. It is well known that $\varphi$ is a periodic function provided that the energy level set $H(x; y) = a$ of the Hamiltonian system $dH = 0$ with

$$H(x; y) = y^2 - \sigma x^2 + \frac{1}{2}\left(\frac{1}{c} - \beta\right)x^4$$

contains an oval (a simple closed real curve free of critical points). Depending on the properties of the bi-quadratic polynomial $U(\varphi)$, we distinguish two cases, which give rise to different explicit solutions, both in term of the Jacobi elliptic functions.

## Proposition

*Let $(c, \beta, \sigma)$ are three real parameters and $\kappa \in (0,1)$. Then, we can identify the following families of solutions of (6). If $c \neq 0$ and $\beta < \frac{1}{c}, \sigma > 0$, then $\varphi$ is a family of dnoidal solutions given by*

$$\varphi(x) = \varphi_0 dn(\alpha x, \kappa)$$

*with parameters $\varphi_0^2 = \frac{2\sigma}{(2-\kappa^2)(\frac{1}{c}-\beta)}$, $\alpha^2 = \frac{\sigma}{2-\kappa^2}$ and a fundamental period $2T = \frac{2K(\kappa)}{\alpha} = \frac{2K(\kappa)\sqrt{2-\kappa^2}}{\sqrt{\sigma}}$.*
*If $\beta > \frac{1}{c}, \sigma < 0$, we obtain the snoidal family*

$$\varphi(x) = \varphi_0 sn(\alpha x, \kappa),$$

*where $\varphi_0^2 = \frac{2\sigma\kappa^2}{(\frac{1}{c}-\beta)(1+\kappa^2)}$, $\alpha^2 = -\frac{\sigma}{1+\kappa^2}$, and fundamental period given by $2T = 4K(\kappa)\frac{\sqrt{1+\kappa^2}}{\sqrt{-\sigma}}$.*

We take the perturbation in the form

$$u(t,x) = e^{i\omega t} e^{i\frac{c}{2}(x-ct)} (\varphi(x-ct) + U(t,x-ct)) \quad (7)$$

$$v(t,x) = \psi(x-ct) + V(t,x-ct)$$

where $U(t,x)$ is complex valued, $V(t,x)$ is real valued. Plugging in the system, using (4), and ignoring all quadratic and higher order terms yields a linear equation for $(U,V)$. Next, split the real and imaginary parts of complex valued function $U$ as $U = P + iQ$, which recasts the linearized problem as the following system

$$\begin{cases} -Q_t = -P_{xx} + \left(w - \frac{c^2}{4}\right) P + 3\beta\varphi^2 P + \varphi V + \psi P \\ P_t = -Q_{xx} + \left(w - \frac{c^2}{4}\right) Q + \psi Q + \beta\varphi^2 Q \\ V_t - cV_x = 2\partial_x(\varphi P). \end{cases} \quad (8)$$

Let us denote

$$\mathcal{J} := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2\partial_x & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \mathcal{H} := \begin{pmatrix} L_1 & \varphi & 0 \\ \varphi & \frac{c}{2} & 0 \\ 0 & 0 & L_2 \end{pmatrix},$$

where

$$\begin{aligned} L_1 &= -\partial_x^2 + \sigma + \left(3\beta - \tfrac{1}{c}\right)\varphi^2 \\ L_2 &= -\partial_x^2 + \sigma + \left(\beta - \tfrac{1}{c}\right)\varphi^2. \end{aligned}$$

Then the system (8) can be written of the form

$$\vec{Z}_t = \mathcal{J}\mathcal{H}\vec{Z}, \quad \vec{Z} = \begin{pmatrix} P \\ V \\ Q \end{pmatrix}. \tag{9}$$

The standard mapping to a time independent problem $\vec{Z} \to e^{\lambda t}\vec{z}$ transforms the linear differential equation (9) into the eigenvalue problem

$$\mathcal{J}\mathcal{H}\vec{z} = \lambda\vec{z}. \tag{10}$$

By general properties of Hamiltonian systems, and the operators $\mathcal{J}, \mathcal{H}$ in particular, if $\lambda$ is an eigenvalue of (10), then so are, $\bar{\lambda}, -\lambda, -\bar{\lambda}$.

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## Definition

We say that the wave $\varphi$ is spectrally unstable, if the eigenvalue problem (10) has a non-trivial solution $(\vec{u}, \lambda)$, so that $\vec{z} \neq 0, \vec{z} \in H^2[-T, T] \times H^1[-T, T] \times H^2[-T, T]$ and $\lambda : \Re\lambda > 0$.
In the opposite case, that is (10) has no non-trivial solutions, with $\Re\lambda > 0$, we say that the wave is spectrally stable.

# Remark:

The definition of linear stability is closely related to the one for spectral stability. More precisely, $\varphi$ is a linearly stable wave, if the flow of the differential equation (or equivalently the semigroup generated by $\mathcal{J}\mathcal{H}$) has Lyapunov exponent less or equal to zero. Equivalently,

$$\limsup_{t\to\infty} \frac{\ln \|\vec{U}(t)\|}{t} \leq 0, \tag{11}$$

for each initial data $\vec{U}(0) \in H^2[-T,T] \times H^1[-T,T] \times H^2[-T,T]$. It is a standard fact that these two notions coincide in the case of periodic domains, due to the fact that the spectrum of $\mathcal{J}\mathcal{H}$ consists of eigenvalues only. A general justification of (11), which applies to our case, is provided in Theorem 2.2 of Lin, Zeng.

## Theorem

*(Stability of the dnoidal waves)*
*Let* $\omega \in \mathbf{R}^1$ *and* $c \neq 0, \beta < \frac{1}{c}, \sigma > 0$. *Then, the Benney system has a family of dnoidal solutions* $(e^{i\omega t} e^{i\frac{c}{2}(x-ct)}\varphi(x-ct), \psi(x-ct)) = (e^{i\omega t} e^{i\frac{c}{2}(x-ct)}\varphi(x-ct), -\frac{1}{c}\varphi^2(x-ct) + \sigma + \frac{c^2}{4} - \omega)$, *which are spatially periodic, provided*

$$c\frac{K(\kappa)\sqrt{2-\kappa^2}}{\sqrt{\sigma}} \in 2\pi\mathbb{Z}. \tag{12}$$

*Under these assumptions, the periodic dnoidal waves are spectrally stable for all values of the parameters,* $\omega \in \mathbf{R}^1, \sigma > 0, \beta < \frac{1}{c}, \kappa \in (0,1)$, *subject to* (12).

# Remark:

In the original paper by Angulo, Concho, Hakkaev, the authors proved that dnoidal solutions are orbitally stable for $\beta \leq 0$ and for $\beta > 0$ and $8\beta\sigma - 3c(1-\beta c)^2 \leq 0$. This is achieved by evaluating the number of negative eigenvalues of the operator of linearization around the periodic waves and number of positive eigenvalues of the Hessian of $d(\omega, c) = E(u, v) - \frac{c}{4}P(u, v) - \frac{\omega}{2}M(u, v)$. We extend this result herein to the whole domain of the parameters.

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Our next result concerns the instability of the snoidal waves.

### Theorem

*(Instability of the snoidal solutions)*
*Let* $\omega \in \mathbf{R}^1$ *and* $c \neq 0, \beta > \frac{1}{c}, \sigma < 0$. *Then, the Benney system has a family of snoidal solutions*

$$\left(e^{i\omega t} e^{i\frac{c}{2}(x-ct)} \varphi(x-ct), -\frac{1}{c}\varphi^2(x-ct) + \sigma + \frac{c^2}{4} - \omega\right)$$

*These waves are periodic exactly when*

$$cK(\kappa)\frac{\sqrt{1+\kappa^2}}{\sqrt{-\sigma}} \in \pi\mathbb{Z}. \tag{13}$$

*The snoidal periodic waves are spectrally unstable (with at least one real and positive eigenvalue) for all values of the parameters* $\omega \in \mathbf{R}^1, \sigma < 0, \beta > \frac{1}{c}, \kappa \in (0,1)$, *subject to* (13).

Index theory implies that we need a determination of a basis of $gker(\mathcal{JH})$. We introduce another Schrödinger operator $L = -\partial_x^2 + \sigma + 3\left(\beta - \frac{1}{c}\right)\varphi^2$.

## Proposition

*We have the following:*

- *In both the dnoidal and snoidal cases, the Hill operator $L$, equipped with periodic boundary conditions on $[-T, T]$, has Morse index $n(L) = 1$ and $Ker[L] = span[\varphi']$.*
- *In the dnoidal case, the operator $L_2$ has Morse index $n(L_2) = 0$, $Ker[L_2] = span[\varphi]$.*
- *In the snoidal case, the operator $L_2$ has Morse index $n(L_2) = 2$, $Ker[L_2] = span[\varphi]$.*

## Proposition

*The kernel of $\mathcal{H}$ is two dimensional, namely*

$$Ker[\mathcal{H}] = span[\begin{pmatrix} \varphi' \\ -\frac{2}{c}\varphi\varphi' \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \varphi \end{pmatrix}]. \qquad (14)$$

*In addition, under the assumption $\langle L^{-1}\varphi, \varphi\rangle \neq 0$, we can identify all the generalized eigenvectors.*

$$gKer(\mathcal{J}\mathcal{H}) \ominus Ker(\mathcal{H}) = span\left[\begin{pmatrix} \frac{1}{2c(c\beta-1)}\varphi \\ -\frac{\beta}{c(c\beta-1)}\varphi^2 \\ \\ L_2^{-1}\varphi'. \end{pmatrix}, \begin{pmatrix} -L^{-1}\varphi \\ \frac{2}{c}\varphi L^{-1}\varphi \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right].$$

## Proposition

*We have the following formula for the Morse index $n(\mathcal{H})$,*

- *If $\varphi$ is the dnoidal wave, then $n(\mathcal{H}) = 1$.*
- *For the snoidal case, we have $n(\mathcal{H}) = 3$.*

For the dnoidal waves, since $n(\mathcal{H}) = 1$, the stability analysis reduces to establishing that $n(D) = 1$. Indeed, in such a case, the right-hand side of (2) is zero, which would rule out all potential instabilities on the left-hand side.
We proceed to evaluating the elements of the matrix $D$. In fact, we only need to compute $D_{22} = \langle L^{-1}\varphi, \varphi\rangle$, which we show is negative.

For the snoidal waves, to get $D$ we compute $\langle L_2^{-1}\varphi', \varphi'\rangle$, $\langle L^{-1}\varphi, \varphi\rangle$ and $\int \varphi^2, \int \varphi^4$. Consider $\det(D)$, in the regime $\beta = \frac{1}{c} + \epsilon, 0 < \epsilon << 1$.

## Proposition

*Fix $c \neq 0, \sigma < 0$. Then, there exists $\epsilon_0 = \epsilon_0(c, \sigma) > 0$, so that for all $0 < \epsilon < \epsilon_0$ and $\beta = \frac{1}{c} + \epsilon$, we have that $det(D) > 0$.*

To show that the snoidal waves are spectrally unstable, we argue as follows - for very small $\epsilon$, we have that $\det(D) > 0$, whence the symmetric matrix $D$ has either two negative eigenvalues and a positive one ($n(D) = 2$), or 3 positive eigenvalues, $n(D) = 0$.
We have either $k_{Ham} = 3 - 2 = 1$ or $k_{Ham} = n(\mathcal{L}) - n(D) = 3 - 0 = 3$.
This implies that there is at least one real instability.
In fact, for systems with $k_{Ham} = 1$, this is obvious.
If $k_{Ham} = 3$, the possibilities are: 3 real instabilities, one real instability and 2 complex/oscillatory instabilities and one real instability and a pair of purely imaginary eigenvalues of negative Krein signature.

We consider the following Zakharov system of nonlinear PDEs

$$\begin{cases} v_{tt} - v_{xx} = \frac{1}{2}(|u|^2)_{xx} \\ iu_t + u_{xx} - uv = 0, \end{cases} \tag{15}$$

Here again, $v$ is a real-valued function and $u$ is complex-valued. The problem (15) was introduced by Zakharov to describe Langmuir turbulence in a plasma.

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Here again, $v$ is a real-valued function and $u$ is complex-valued. The problem (15) was introduced by Zakharov to describe Langmuir turbulence in a plasma.
We consider the spectral stability of periodic travelling wave solutions of the form

$$\begin{cases} v(t,x) = \psi(x-ct) \\ u(t,x) = e^{-i\omega t} e^{i\frac{c}{2}(x-ct)}\phi(x-ct), \end{cases} \tag{16}$$

where $\psi,\ \phi : \mathbf{R}^1 \to \mathbf{R}^1$ are smooth,periodic functions with fixed period $2T$, and $\omega, c \in \mathbf{R}^1$. To ensure that the traveling wave $u$ above is $2T$ periodic, we require that there is an integer $l$, so that $cT = 2\pi$.

## Proposition

*(Existence of dnoidal solutions)*
*Let* $1 - c^2 > 0, \sigma > 0$. *Assume that the quadratic equation* $r^2 - 4\sigma(1 - c^2)r - a_1 = 0$ *has two positive roots, denoted by* $\phi_0^2 > \phi_1^2$. *Then, the periodic traveling wave solution is given by*

$$\phi(x) = \phi_0 dn(\alpha x, \kappa), \tag{17}$$

*where*

$$\kappa^2 = \frac{\phi_0^2 - \phi_1^2}{\phi_0^2} = \frac{2\phi_0^2 - 4\sigma(1 - c^2)}{\phi_0^2}, \quad \alpha^2 = \frac{1}{4(1 - c^2)}\phi_0^2 = \frac{\sigma}{2 - \kappa^2}. \tag{18}$$

*In addition, the fundamental period of* $\phi$ *is* $2T = \frac{2K(k)}{\alpha}$.

For $\vec{U} = (p_2, p_1, q, h)$, the above system can be written in the form

$$\vec{U}_t = \mathcal{J}\mathcal{H}\vec{U}, \tag{19}$$

where

$$\mathcal{J} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\partial_x \\ 0 & 0 & -\partial_x & 0 \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} \mathcal{L}_- & 0 & 0 & 0 \\ 0 & \mathcal{L}_- & \phi & 0 \\ 0 & \phi & 1 & -c \\ 0 & 0 & -c & 1 \end{pmatrix} \tag{20}$$

$$\mathcal{L}_- = -\partial_x^2 + \sigma + \psi = -\partial_x^2 + \sigma - \frac{\phi^2}{2(1-c^2)}. \tag{21}$$

Clearly $\mathcal{J}^* = -\mathcal{J}$, whereas $\mathcal{H}^* = \mathcal{H}$, where we associate to the operators $\mathcal{J}, \mathcal{H}$ the following domains on the periodic functions

$$D(\mathcal{J}) = (L^2[-T,T])^2 \oplus (H^1[-T,T])^2$$
$$D(\mathcal{H}) = (H^2[-T,T])^2 \oplus L^2[-T,T] \oplus L^2_0[-T,T].$$

Note that $L^2_0[-T,T] = \{f \in L^2[-T,T] : \int_{-T}^T f(x)dx = 0\}$.

## Theorem

*Periodic traveling waves of dnoidal type of (15) are spectrally stable for all natural values of the parameters.*

Note: Orbital stability of periodic waves of dnoidal type was proved by Angulo and Brango. They proved that for all $\varepsilon > 0$, there is $\delta > 0$ s.t. for initial data $(v_0, V_0, u_0) \in L^2[-T,T] \times L^2_0[-T,T] \times H^1[-T,T]$ satisfying

$$\|v_0 - \psi\|_{L^2[-T,T]} < \delta, \quad \|V_0 - \varphi\|_{L^2[-T,T]} < \delta, \quad \|u_0 - \phi\|_{H^1[-T,T]} < \delta \tag{22}$$

then

$$\begin{cases} \inf_{y\in\mathbf{R}} \|v(\cdot + y, t) - \psi\|_{L^2[-T,T]} < \varepsilon, \quad \inf_{y\in\mathbf{R}} \|V(\cdot + y, t) - \psi\|_{L^2[-T,T]} < \varepsilon, \\ \inf_{(\theta,y)\in[0,2\pi)\times\mathbf{R}} \|e^{i\theta}u(\cdot + y, t) - \phi\|_{H^1[-T,T]} < \varepsilon \end{cases} \tag{23}$$

if

$$\int_0^T v_0(x)dx \leq \int_0^T \psi(x)dx. \tag{24}$$

Thank you!