Transport Equation on Metric Graphs

Mathematical aspects of the Physics with non-self-Adjoint Operators, BIRS, July 2022

Marjeta Kramar Fijavž (University of Ljubljana)

Joint work with Klaus-Jochen Engel (University of L'Aquila)

Univerza *v Ljubljani*
Fakulteta za *gradbeništvo in geodezijo*

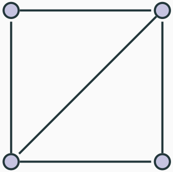
Institute of Mathematics, Physics and Mechanics

Mathematical models for interacting dynamics on networks

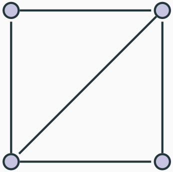
The Setting

The Setting

The Setting

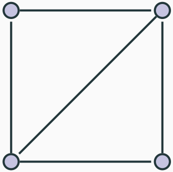


The Setting



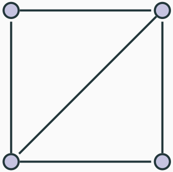
- finite connected
  compact metric graph

The Setting



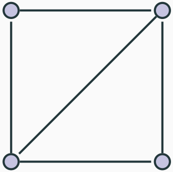
- finite connected compact metric graph
- edges $\sim [0, 1]$

The Setting



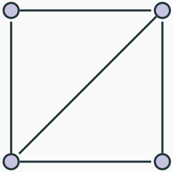
- finite connected compact metric graph
- edges $\sim [0, 1]$

The Setting



- finite connected compact metric graph
- edges $\sim [0, 1]$

The Setting

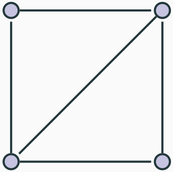


- finite connected compact metric graph
- edges $\sim [0, 1]$

- transport equation on the edges:

$$\partial_t u_j(t, s) = c_j(s) \partial_s u_j(t, s)$$

The Setting



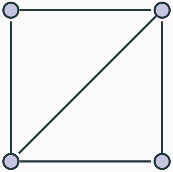
- finite connected compact metric graph
- edges $\sim [0, 1]$

- transport equation on the edges:

$$\partial_t u_j(t, s) = c_j(s)\partial_s u_j(t, s)$$

- standard boundary conditions in the vertices

The Setting



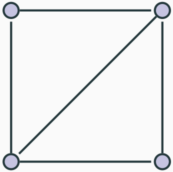
- finite connected compact metric graph
- edges $\sim [0,1]$

- transport equation on the edges:

$$\partial_t u_j(t,s) = c_j(s)\partial_s u_j(t,s)$$

- standard boundary conditions in the vertices
    - distribution rules

The Setting



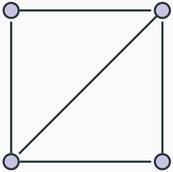
- finite connected compact metric graph
- edges $\sim [0, 1]$

- transport equation on the edges:

$$\partial_t u_j(t,s) = c_j(s)\partial_s u_j(t,s)$$

- standard boundary conditions in the vertices
  - distribution rules
  - $\sum$ in-flow $= \sum$ out-flow

The Setting



- finite connected compact metric graph
- edges $\sim [0,1]$

- transport equation on the edges:

$$\partial_t u_j(t,s) = c_j(s)\partial_s u_j(t,s)$$

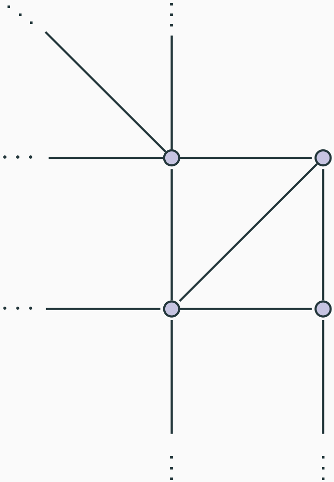
- standard boundary conditions in the vertices
    - distribution rules
    - $\sum$ in-flow $= \sum$ out-flow

Studied by

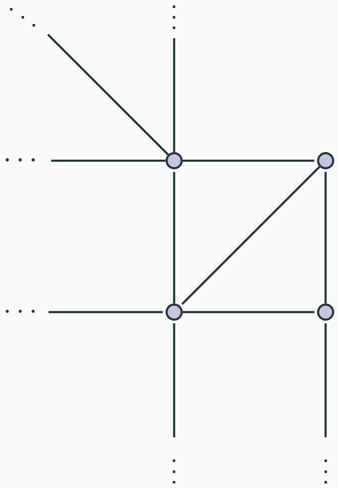
Barletti, Sikolya, KF, Klöss, Engel, Radl, Dorn, Bayazit, Banasiak, Puchalska, Namayanja, Błoch, Jacob, Zwart, Le Gorrec, Maschke, Villegas, Kaiser, Wegner, ...

The Setting

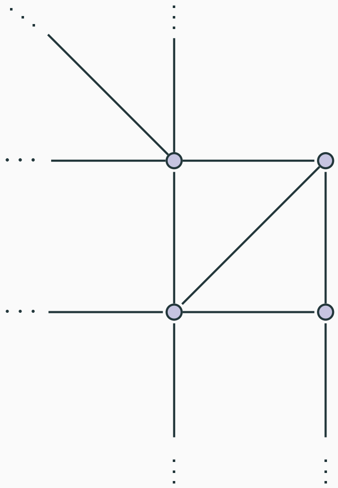
The Setting



The Setting

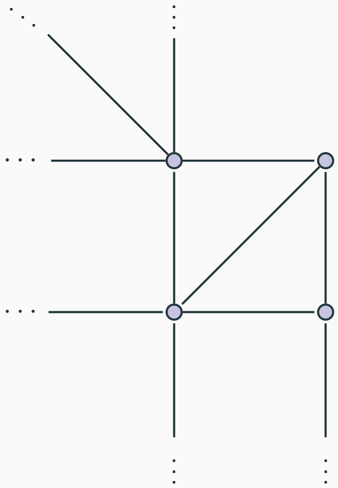


The Setting



Our aims

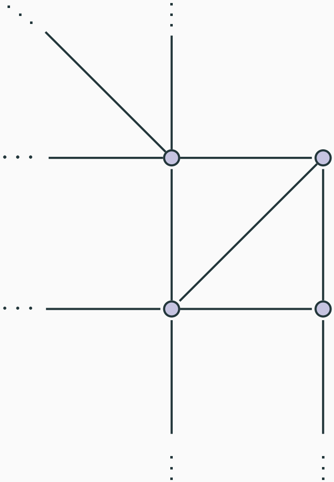
The Setting



Our aims

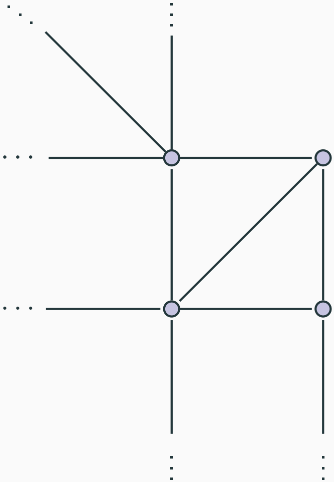
1. consider non-compact graphs

The Setting



Our aims

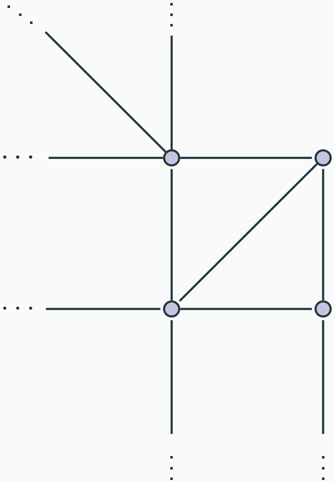
1. consider non-compact graphs
2. take general coefficients $c(\bullet)$



Our aims

1. consider non-compact graphs
2. take general coefficients $c(\bullet)$
3. consider operators on L^p-spaces, $p \in [1, \infty)$

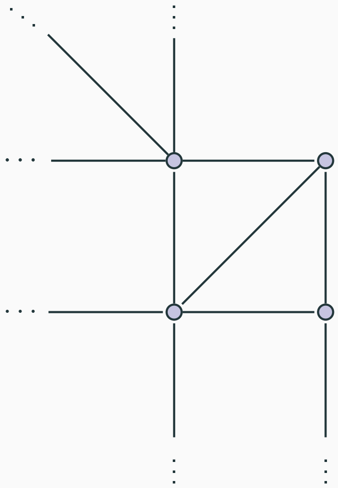
The Setting



Our aims

1. consider non-compact graphs
2. take general coefficients $c(\bullet)$
3. consider operators on L^p-spaces, $p \in [1, \infty)$
4. treat general boundary conditions

The Setting



Our aims

1. consider non-compact graphs
2. take general coefficients $c(\bullet)$
3. consider operators on L^p-spaces, $p \in [1, \infty)$
4. treat general boundary conditions
5. obtain necessary and sufficient conditions for well-posedness

Main idea: Boundary Perturbations

Main idea: Boundary Perturbations

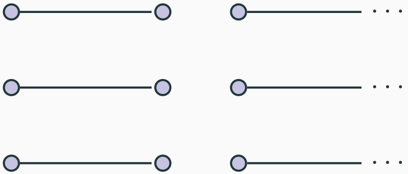
Consider graph as boundary perturbation of a set of intervals.

Main idea: Boundary Perturbations

Consider graph as boundary perturbation of a set of intervals.

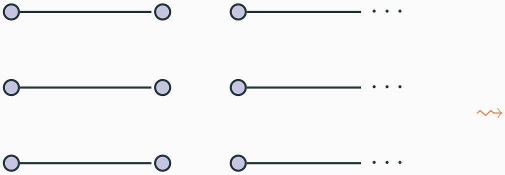
Main idea: Boundary Perturbations

Consider graph as boundary perturbation of a set of intervals.



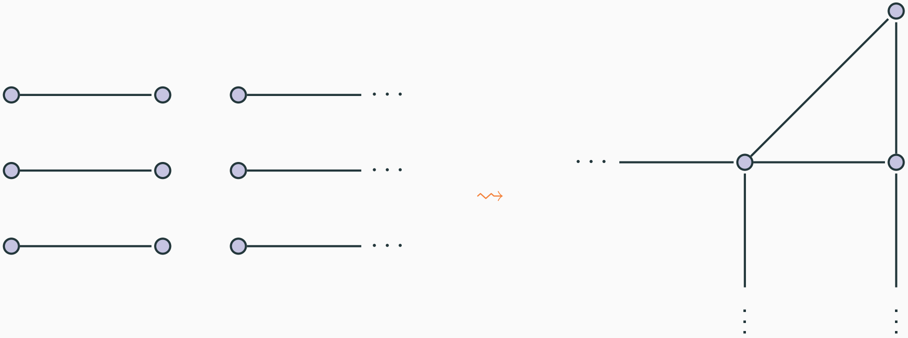
Main idea: Boundary Perturbations

Consider graph as boundary perturbation of a set of intervals.



Main idea: Boundary Perturbations

Consider graph as boundary perturbation of a set of intervals.



Main idea: Boundary Perturbations

Abstract formulation

$$(ACP)\begin{cases}\dot{x}(t) = Ax(t),\\ x(0) = x_0,\end{cases} \qquad \text{on} \quad X = \mathrm{L}^p(\mathbb{R}_+, \mathbb{C}^\ell) \times \mathrm{L}^p\big([0,1], \mathbb{C}^m\big)$$

where

$$\begin{aligned} A &= \left(c_{ij}(\bullet)\partial_s\right)_{ij} \\ D(A) &= \left\{f \in \mathrm{W}^{1,p}(\mathbb{R}_+, \mathbb{C}^\ell) \times \mathrm{W}^{1,p}([0,1], \mathbb{C}^m) \mid \Phi f = 0\right\} \end{aligned}$$

and $\Phi\colon X \to \partial X \subset \mathbb{C}^{m+\ell}$ is boundary operator

Boundary perturbations of domains of generators

Greiner (1987), Weiss (1994), Staffans (2005),
Adler-Bombieri-Engel (2014), Hadd-Manzo-Rhandi (2015)

Boundary perturbations of domains of generators

- Greiner (1987), Weiss (1994), Staffans (2005), Adler-Bombieri-Engel (2014), Hadd-Manzo-Rhandi (2015)

Boundary perturbations of domains of generators

Greiner (1987), Weiss (1994), Staffans (2005), Adler-Bombieri-Engel (2014), Hadd-Manzo-Rhandi (2015)

- X and ∂X two Banach spaces

Boundary perturbations of domains of generators

Greiner (1987), Weiss (1994), Staffans (2005), Adler-Bombieri-Engel (2014), Hadd-Manzo-Rhandi (2015)

- X and ∂X two Banach spaces
- $A_m \colon D(A_m) \subseteq X \to X$ a closed, densely defined maximal operator

Boundary perturbations of domains of generators

Greiner (1987), Weiss (1994), Staffans (2005), Adler-Bombieri-Engel (2014), Hadd-Manzo-Rhandi (2015)

- X and ∂X two Banach spaces
- $A_m \colon D(A_m) \subseteq X \to X$ a closed, densely defined maximal operator
- boundary operator $\Phi = L - C \colon X \to \partial X$

Boundary perturbations of domains of generators

Greiner (1987), Weiss (1994), Staffans (2005), Adler-Bombieri-Engel (2014), Hadd-Manzo-Rhandi (2015)

- X and ∂X two Banach spaces
- $A_m \colon D(A_m) \subseteq X \to X$ a closed, densely defined maximal operator
- boundary operator $\Phi = L - C \colon X \to \partial X$
- $A_0, A \subset A_m$ with $D(A_0) = \ker(L)$, $D(A) = \ker(\Phi)$

Boundary perturbations of domains of generators

Greiner (1987), Weiss (1994), Staffans (2005), Adler-Bombieri-Engel (2014), Hadd-Manzo-Rhandi (2015)

- X and ∂X two Banach spaces
- $A_m \colon D(A_m) \subseteq X \to X$ a closed, densely defined maximal operator
- boundary operator $\Phi = L - C \colon X \to \partial X$
- $A_0, A \subset A_m$ with $D(A_0) = \ker(L)$, $D(A) = \ker(\Phi)$
- A_0 generates a C_0-semigroup $(T(t))_{t \geq 0}$ on X

Boundary perturbations of domains of generators

Greiner (1987), Weiss (1994), Staffans (2005), Adler-Bombieri-Engel (2014), Hadd-Manzo-Rhandi (2015)

- X and ∂X two Banach spaces
- $A_m \colon D(A_m) \subseteq X \to X$ a closed, densely defined maximal operator
- boundary operator $\Phi = L - C \colon X \to \partial X$
- $A_0, A \subset A_m$ with $D(A_0) = \ker(L)$, $D(A) = \ker(\Phi)$
- A_0 generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X

Problem

Find conditions on Φ so that A generates a C_0-semigroup on X.

Special case: Boundary matrices

Special case: Boundary matrices

Assumptions

Special case: Boundary matrices

Assumptions

- coefficients $c_{ij}(\bullet)$ are bounded functions

Special case: Boundary matrices

Assumptions

- coefficients $c_{ij}(\bullet)$ are bounded functions
- $$c(\bullet) = \begin{pmatrix} q^e(\bullet) & 0 \\ 0 & q^i(\bullet) \end{pmatrix} \cdot \begin{pmatrix} \lambda^e(\bullet) & 0 \\ 0 & \lambda^i(\bullet) \end{pmatrix} \cdot \begin{pmatrix} q^e(\bullet) & 0 \\ 0 & q^i(\bullet) \end{pmatrix}^{-1}$$

Special case: Boundary matrices

Assumptions

- coefficients $c_{ij}(\bullet)$ are bounded functions
- $$c(\bullet) = \begin{pmatrix} q^e(\bullet) & 0 \\ 0 & q^i(\bullet) \end{pmatrix} \cdot \begin{pmatrix} \lambda^e(\bullet) & 0 \\ 0 & \lambda^i(\bullet) \end{pmatrix} \cdot \begin{pmatrix} q^e(\bullet) & 0 \\ 0 & q^i(\bullet) \end{pmatrix}^{-1}$$
- λ^e, λ^i are diagonal whose diagonal entries are functions with strictly constant sign

Special case: Boundary matrices

Assumptions

- coefficients $c_{ij}(\bullet)$ are bounded functions
- $$c(\bullet) = \begin{pmatrix} q^e(\bullet) & 0 \\ 0 & q^i(\bullet) \end{pmatrix} \cdot \begin{pmatrix} \lambda^e(\bullet) & 0 \\ 0 & \lambda^i(\bullet) \end{pmatrix} \cdot \begin{pmatrix} q^e(\bullet) & 0 \\ 0 & q^i(\bullet) \end{pmatrix}^{-1}$$
- λ^e, λ^i are diagonal whose diagonal entries are functions with strictly constant sign
- q is bounded, Lipschitz continuous with bounded inverse

Special case: Boundary matrices

Assumptions

- coefficients $c_{ij}(\bullet)$ are bounded functions
-
$$c(\bullet) = \begin{pmatrix} q^e(\bullet) & 0 \\ 0 & q^i(\bullet) \end{pmatrix} \cdot \begin{pmatrix} \lambda^e(\bullet) & 0 \\ 0 & \lambda^i(\bullet) \end{pmatrix} \cdot \begin{pmatrix} q^e(\bullet) & 0 \\ 0 & q^i(\bullet) \end{pmatrix}^{-1}$$
- λ^e, λ^i are diagonal whose diagonal entries are functions with strictly constant sign
- q is bounded, Lipschitz continuous with bounded inverse

Boundary space

Denote by $P_+^e, P_-^e \in \mathrm{M}_\ell(\mathbb{C})$ and $P_+^i, P_-^i \in \mathrm{M}_m(\mathbb{C})$ the spectral projections corresponding to positive/negative values of λ^e, λ^i, respectively. Then $\partial X = \mathrm{rg}(P_-^e) \times \mathbb{C}^m = \mathbb{C}^n \subseteq \mathbb{C}^{\ell+m}$.

Special case: Boundary matrices

Theorem

Let $\Phi := \left(V_0^e \delta_0, V_0^i \delta_0 - V_1^i \delta_1\right) - B$ for some $V_0^e \in M_{n\times\ell}(\mathbb{C})$, $V_0^i, V_1^i \in M_{n\times m}(\mathbb{C})$, and $B \in \mathcal{L}(X, \partial X)$. Then A generates a C_0-semigroup on X if and only if

$$\left(V_0^e q^e(0), V_1^i q^i(1) P_+^i - V_0^i q^i(0) P_-^i\right) \in \mathcal{L}\left(\partial X\right)$$

is invertible.

Special case: Boundary matrices

For compact graph and diagonal velocities we obtain

Corollary

Let $X = \mathrm{L}^p([0,1],\mathbb{C}^m)$ and $\Phi := V_0\delta_0 - V_1\delta_1 - B$ for some $V_0, V_1 \in \mathrm{M}_m(\mathbb{C})$ and $B \in \mathcal{L}(X,\mathbb{C}^m)$. Then A generates a C_0-semigroup on X if and only if

$$\det\big(V_1P_+ - V_0P_-\big) \neq 0.$$

Moreover, it generates C_0-group if and only if in addition

$$\det\big(V_1P_- - V_0P_+\big) \neq 0.$$

Standard boundary conditions

Standard boundary conditions

Compact graph

Standard boundary conditions

Compact graph

- Assume diagonal velocities and $\lambda(\bullet) < 0$.

Standard boundary conditions

Compact graph

- Assume diagonal velocities and $\lambda(\bullet) < 0$.
- Standard boundary conditions:

$$\Phi f = V_0 f(0) - V_1 f(1) = 0$$

Standard boundary conditions

Compact graph

- Assume diagonal velocities and $\lambda(\bullet) < 0$.
- Standard boundary conditions:

$$\Phi f = V_0 f(0) - V_1 f(1) = 0$$

- A is generator $\iff V_0$ invertible.

Standard boundary conditions

Compact graph

- Assume diagonal velocities and $\lambda(\bullet) < 0$.
- Standard boundary conditions:

$$\Phi f = V_0 f(0) - V_1 f(1) = 0$$

- A is generator $\iff$ V_0 invertible.
- $V_0 f(0) = V_1 f(1) \iff f(0) = V_0^{-1} V_1 f(1)$

Standard boundary conditions

Non-compact graph

Standard boundary conditions

Non-compact graph

- Assume diagonal velocities and $\lambda^i(\bullet) < 0$.

Standard boundary conditions

Non-compact graph

- Assume diagonal velocities and $\lambda^i(\bullet) < 0$.
- Standard boundary conditions:

$$\left(V_0^e, V_0^i\right)\begin{pmatrix} f^e(0) \\ f^i(0) \end{pmatrix} = V_1^i f^i(1)$$

Standard boundary conditions

Non-compact graph

- Assume diagonal velocities and $\lambda^i(\bullet) < 0$.
- Standard boundary conditions:

$$\left(V_0^e, V_0^i\right)\begin{pmatrix} f^e(0) \\ f^i(0) \end{pmatrix} = V_1^i f^i(1)$$

- A is generator $\iff$ $\left(V_0^e, V_0^i\right)$ has full rank.

Standard boundary conditions

Non-compact graph

- Assume diagonal velocities and $\lambda^i(\bullet) < 0$.
- Standard boundary conditions:

$$\left(V_0^e, V_0^i\right)\begin{pmatrix} f^e(0) \\ f^i(0) \end{pmatrix} = V_1^i f^i(1)$$

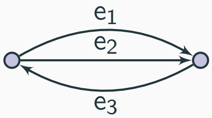
- A is generator $\iff$ $\left(V_0^e, V_0^i\right)$ has full rank.
- $\left(V_0^e, V_0^i\right)$ is Moore-Penrose invertible and boundary conditions can be equivalently written as

$$\begin{pmatrix} f^e(0) \\ f^i(0) \end{pmatrix} = \left(V_0^e, V_0^i\right)^+ V_1^i f^i(1)$$

Examples

Examples

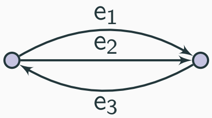
Compact graph, diagonal velocities



$$A \subseteq \lambda(\bullet)\frac{d}{ds},\ \lambda_1(\bullet), \lambda_2(\bullet), \lambda_3(\bullet) < 0$$

Examples

Compact graph, diagonal velocities



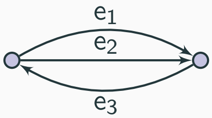
$$A \subseteq \lambda(\bullet)\frac{d}{ds},\ \lambda_1(\bullet), \lambda_2(\bullet), \lambda_3(\bullet) < 0$$

- Standard conditions:

$$\begin{array}{rcl} u_1(0) & = & \alpha u_3(1) \\ u_2(0) & = & (1-\alpha)u_3(1) \\ u_3(0) & = & u_1(1) + u_2(1) \end{array} \iff V_0 = I, \quad V_1 = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 1-\alpha \\ 1 & 1 & 0 \end{pmatrix}$$

Examples

Compact graph, diagonal velocities



$$A \subseteq \lambda(\bullet)\tfrac{d}{ds},\ \lambda_1(\bullet), \lambda_2(\bullet), \lambda_3(\bullet) < 0$$

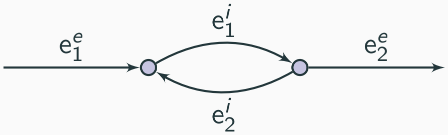
- Standard conditions:

$$\begin{array}{rcl} u_1(0) & = & \alpha u_3(1) \\ u_2(0) & = & (1-\alpha)u_3(1) \\ u_3(0) & = & u_1(1) + u_2(1) \end{array} \iff V_0 = I, \quad V_1 = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 1-\alpha \\ 1 & 1 & 0 \end{pmatrix}$$

- A generates a C_0-semigroup but not a group

Examples

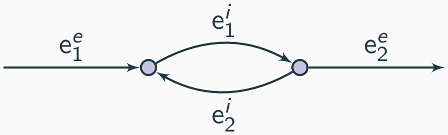
Non-compact graph, diagonal velocities



$$A \subseteq \lambda(\bullet)\frac{d}{ds},\ \lambda_1^i(\bullet), \lambda_2^i(\bullet) < 0,\ \lambda_1^e(\bullet) > 0,\ \lambda_2^e(\bullet) < 0$$

$$V_0^i = \begin{pmatrix} a & 0 \\ 0 & d \\ 0 & 0 \end{pmatrix}, \quad V_1^i = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \\ 0 & 0 \end{pmatrix}, \quad V_0^e = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \\ 0 & \mu \end{pmatrix}$$

Non-compact graph, diagonal velocities



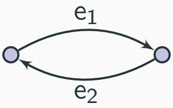
$$A \subseteq \lambda(\bullet)\tfrac{d}{ds},\ \lambda_1^i(\bullet), \lambda_2^i(\bullet) < 0,\ \lambda_1^e(\bullet) > 0,\ \lambda_2^e(\bullet) < 0$$

$$V_0^i = \begin{pmatrix} a & 0 \\ 0 & d \\ 0 & 0 \end{pmatrix}, \quad V_1^i = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \\ 0 & 0 \end{pmatrix}, \quad V_0^e = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \\ 0 & \mu \end{pmatrix}$$

- A is generator $\iff ad\mu \neq 0$.

Examples

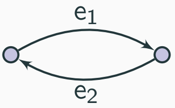
Compact graph, non-diagonal velocities



$$A \subseteq c(\bullet)\frac{d}{ds} \text{ with } q(s) = \begin{pmatrix} 2-s & s-1 \\ 1-s & s \end{pmatrix}$$

Examples

Compact graph, non-diagonal velocities

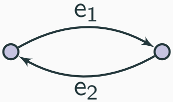


$$A \subseteq c(\bullet)\tfrac{d}{ds} \text{ with } q(s) = \left(\begin{smallmatrix} 2-s & s-1 \\ 1-s & s \end{smallmatrix}\right)$$

- Recall: A is generator $\iff$ $V_1 q(1) P_+ - V_0 q(0) P_-$ invertible.

Examples

Compact graph, non-diagonal velocities

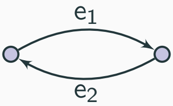


$$A \subseteq c(\bullet)\frac{d}{ds} \text{ with } q(s) = \begin{pmatrix} 2-s & s-1 \\ 1-s & s \end{pmatrix}$$

- Recall: A is generator $\iff V_1 q(1) P_+ - V_0 q(0) P_-$ invertible.
- If $\lambda_1(\bullet), \lambda_2(\bullet)$ both positive/negative, invertibility of V_1/V_1 yields C_0-semigroup.

Examples

Compact graph, non-diagonal velocities

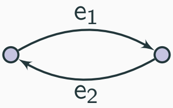


$$A \subseteq c(\bullet)\frac{d}{ds} \text{ with } q(s) = \begin{pmatrix} 2-s & s-1 \\ 1-s & s \end{pmatrix}$$

- Recall: A is generator $\iff V_1 q(1) P_+ - V_0 q(0) P_-$ invertible.
- If $\lambda_1(\bullet), \lambda_2(\bullet)$ both positive/negative, invertibility of V_1/V_1 yields C_0-semigroup.
- If $\lambda_1(\bullet) > 0 > \lambda_2(\bullet)$ and $V_0 = V_1 = Id$, this matrix is singular!

Examples

Compact graph, non-diagonal velocities



$$A \subseteq c(\bullet)\frac{d}{ds} \text{ with } q(s) = \begin{pmatrix} 2-s & s-1 \\ 1-s & s \end{pmatrix}$$

- Recall: A is generator $\iff V_1 q(1) P_+ - V_0 q(0) P_-$ invertible.
- If $\lambda_1(\bullet), \lambda_2(\bullet)$ both positive/negative, invertibility of V_1/V_1 yields C_0-semigroup.
- If $\lambda_1(\bullet) > 0 > \lambda_2(\bullet)$ and $V_0 = V_1 = Id$, this matrix is singular!
- However, if $q(\bullet) = Id$, we obtain the generation of a C_0-group.

Examples

General non-local boundary conditons

$X = \mathrm{L}^p[0,1]$, $\partial X = \mathbb{C}$, $A \subseteq \frac{d}{ds}$, $\Phi = \delta_1 - B$ where

$$Bf := \int_0^1 h(s)f(s)\,ds \quad \text{for some } h \in \mathrm{L}^q[0,1], \frac{1}{p} + \frac{1}{q} = 1.$$

Examples

General non-local boundary conditons

$X = \mathrm{L}^p[0,1]$, $\partial X = \mathbb{C}$, $A \subseteq \frac{d}{ds}$, $\Phi = \delta_1 - B$ where

$$Bf := \int_0^1 h(s) f(s)\, ds \quad \text{for some } h \in \mathrm{L}^q[0,1], \frac{1}{p} + \frac{1}{q} = 1.$$

- A generates a C_0-semigroup but not a group on $\mathrm{L}^p[0,1]$.

References

K.-J. Engel, MKF, *Flows on metric graphs with general boundary conditions*, J. Math. Anal. Appl 513 (2022), 126214.

References

- K.-J. Engel, MKF, *Flows on metric graphs with general boundary conditions*, J. Math. Anal. Appl 513 (2022), 126214.
- MKF, A. Puchalska, *Semigroups for dynamical processes on metric graphs*, Phil. Trans. R. Soc. A. 378: 20190619 (2020).

References

K.-J. Engel, MKF, *Flows on metric graphs with general boundary conditions*, J. Math. Anal. Appl 513 (2022), 126214.

MKF, A. Puchalska, *Semigroups for dynamical processes on metric graphs*, Phil. Trans. R. Soc. A. 378: 20190619 (2020).

COST Action CA18232, *Mathematical models for interacting dynamics on networks*, www.mat-dyn-net.eu

References

- K.-J. Engel, MKF, *Flows on metric graphs with general boundary conditions*, J. Math. Anal. Appl 513 (2022), 126214.
- MKF, A. Puchalska, *Semigroups for dynamical processes on metric graphs*, Phil. Trans. R. Soc. A. 378: 20190619 (2020).
- COST Action CA18232, *Mathematical models for interacting dynamics on networks*, www.mat-dyn-net.eu

References

K.-J. Engel, MKF, *Flows on metric graphs with general boundary conditions*, J. Math. Anal. Appl 513 (2022), 126214.

MKF, A. Puchalska, *Semigroups for dynamical processes on metric graphs*, Phil. Trans. R. Soc. A. 378: 20190619 (2020).

COST Action CA18232, *Mathematical models for interacting dynamics on networks*, www.mat-dyn-net.eu

Thank you!