SCHRÖDINGER OPERATORS

with complex potentials

J.-C. Cuenin, Loughborough University

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Setting

$$-\Delta + \underset{\mathbb{C}}{\underset{\cap}{V(x)}} \quad \text{in } L^2(\mathbb{R}^d)$$

V decaying : $V \in L^q(\mathbb{R}^d)$
for some $q < \infty$

Aim: Quantitative bounds on eigenvalues

Many results by <u>many authors</u> :

Aslanyan - Abramov - Davies, Frank - Laptev - Lieb - Seiringer, Demuth - Hansmann - Katriel, Laptev - Safronov, Frank - Simon, Fanelli - Krejcirik - Vega, Mizutani, Lee - Seo, Cossetti, Bögli - Stampach, ...

For real potentials,

$$\sum_j |E_j|^\gamma \lesssim \int_{\mathbb{R}^d} V_-(x)^{\frac{d}{2}+\gamma}\, dx$$

$\forall \gamma > 0$ if $d \geq 2$.

For complex potentials, Frank proved

$$|E_j|^\gamma \lesssim \int_{\mathbb{R}^d} |V(x)|^{\frac{d}{2}+\gamma}\, dx$$

$\forall\, 0 < \gamma \leq \frac{1}{2}$ if $d \geq 2$.

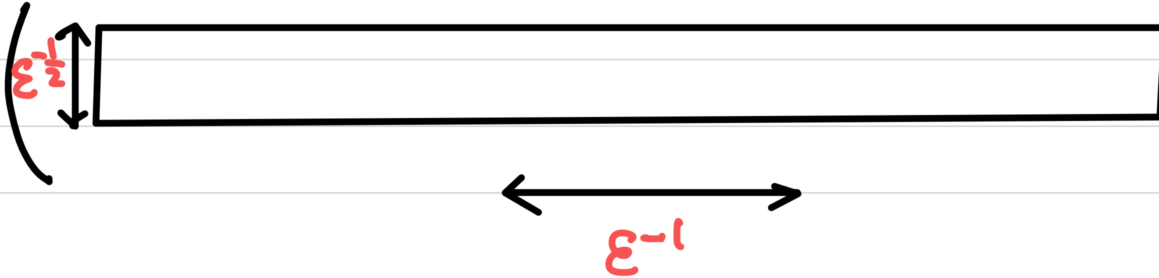
Conjecture (Laptev - Safronov 2009)

$$|E_j|^\gamma \lesssim \int_{\mathbb{R}^d} |V(x)|^{\frac{d}{2}+\gamma}\, dx$$

$\forall \ \frac{1}{2} \le \gamma \le \frac{d}{2}$ if $d \geq 2$.

Counterexample (Bögli - C. 2021)

$V_\varepsilon \sim \varepsilon \, \mathbb{1} \Big($



$\Big)$

Optimality

Counterexample also shows optimality of :

$$* \ |E|^{q-\frac{d}{2}} \lesssim \int |V|^q \quad \forall q \leq \frac{d+1}{2}$$

$$* \ d(E,\mathbb{R}_+)^{q-\frac{d+1}{2}} |E|^{\frac{1}{2}} \lesssim \int |V|^q \quad \forall q > \frac{d+1}{2}$$

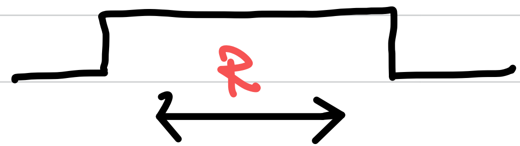
$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{(Frank)}$$

$$* \ |E|^{\frac{1}{2}} \lesssim \sup_y \int \exp(-\operatorname{Im}\sqrt{E}\,|x-y|)\, |V(x)|^{\frac{d+1}{2}} dx \qquad \text{(C.)}$$

Q: Under what structural asspts. on V do we have improvements ?

Optimality II

A simpler counterexample in $d=1$

$V = i \cdot \Big($  $\Big)$

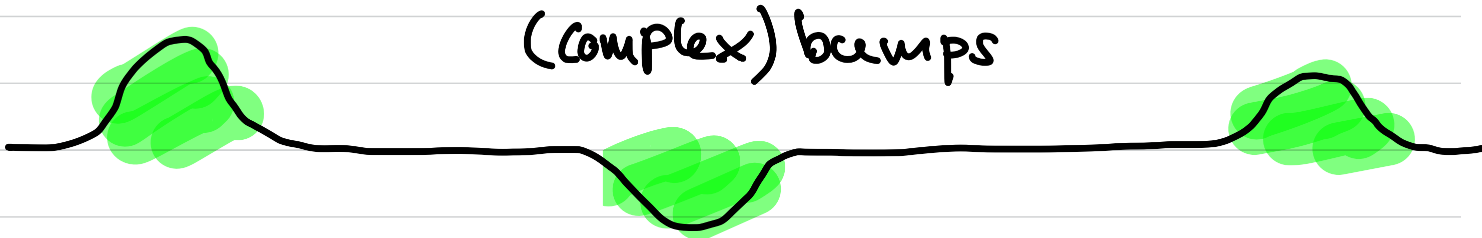
shows optimality of bounds for sums / number of eigenvalues (Bögli–Stampach, C.), e.g. (Frank–Laptev–Safronov)

$$\#\,\text{eigenvalues} \lesssim \inf_{\varepsilon>0} \frac{1}{\varepsilon^2}\left(\int e^{\varepsilon|x|}\,|V(x)|\,dx\right)^2$$

Improved bounds

* V radial $\Rightarrow$ L-S conj. true (Frank-Simon)
* V sparse
* V random
* ...

Sparse Potentials

$V =$ 

Bögli constructed examples $\forall q > d$ $\forall \varepsilon > 0$ s.t. $\|V\|_q + \|V\|_\infty < \varepsilon$ and $-\Delta + V$ has eigenvalues accumulating to every point in $\mathbb{R}_+$.

Almost orthogonality

Exp. decay of Green's fct. + large separation $\Rightarrow$

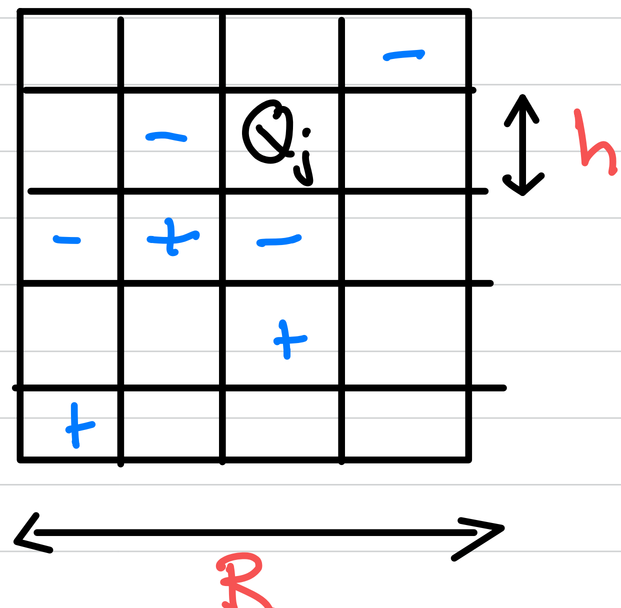
$$-\Delta + V \sim \bigoplus_j (-\Delta + V_j) + \text{small error}$$

$\Rightarrow$ Can replace $\int |V|^q$ by $\sup_j \int |V_j|^q$ in Frank's bounds.

Random Potentials

$$V_\omega(x) = \omega_j \, v_j \qquad \forall x \in Q_j$$

$$\boxed{P(\omega_j = \pm 1) = \tfrac{1}{2}} \qquad \boxed{\sum_j |v_j|^q < \infty}$$



<u>Thm</u> (C.–Merz '22)

For $q \le d+1$, every eigenvalue z of $-\Delta + V_\omega$ satisfies

$$\frac{|z|^{1-\frac{d}{2q}}}{\langle |z|^{1/2} h\rangle \left[\log\langle |z|^{1/2} R\rangle\right]^{7/2}} \lesssim M \Big(\sum_j |v_j|^q\Big)^{\frac{1}{q}}$$

except for ω in a set of measure $\le \exp(-cM^2)$.

If we sacrifice the endpoint, we can remove the compact support assumption.

<u>In particular</u>: $\forall q < d+1$,

$$\sup_{z} \frac{|z|^{q-\frac{d}{2}}}{\left(\sum_j |v_j|^q\right)^{1/q}} < \infty \quad \text{a.s.}$$

$\uparrow$

e.v. of $-\Delta + V_\omega$

Proof Sketch

① $(-\Delta + V_\omega - z) = (-\Delta - z)\left(1 + \underbrace{(-\Delta - z)^{-1}}_{R_0(z)} V_\omega\right)$

Born series:

$$(1+R_0 V)^{-1} = \sum_{n=0}^{\infty} (-1)^n [R_0 V]^n R_0$$

$$\downarrow$$

$$R_0 V R_0^{1/2} |R_0|^{1/2} V R_0^{1/2} |R_0|^{1/2} V \ldots.$$

$|R_0|^{\frac{1}{2}}$ is the Fourier multiplier

$$|\xi^2 - z|^{-\frac{1}{2}}$$

Localization in x-space ($\operatorname{supp} \gamma \subset B_R$)

$$\Rightarrow \quad \left(|\xi^2 - z| + 1/R\right)^{-\frac{1}{2}}$$

(smoothing in ξ-space)

② By spectral thm.

$$|R_0(z)| = \int |\lambda - z|^{-1} \, dE(\lambda)$$

$$\frac{dE(\lambda)}{d\lambda} = c\lambda^{\text{power}} \, \mathcal{E}(\lambda)^* \mathcal{E}(\lambda)$$

$$\boxed{\begin{aligned} \mathcal{E}(\lambda) : L^2(\lambda S^{d-1}, d\sigma_\lambda) &\to L^\infty(\mathbb{R}^d) \\ g &\longmapsto (g \, d\sigma_\lambda)^\vee \end{aligned}}$$

$$\int_{1-c}^{1+c} \left(|\lambda - 1| + \frac{1}{R}\right)^{-1} \lesssim \log \frac{1}{R}$$

Up to factors of $\log \frac{1}{R}$ it suffices to control the operators

$$\boxed{\mathcal{E}(\lambda)^* V_\omega \, \mathcal{E}(\lambda)} \; : L^2(\lambda S^{d-1}, d\sigma_\lambda) \circlearrowleft$$

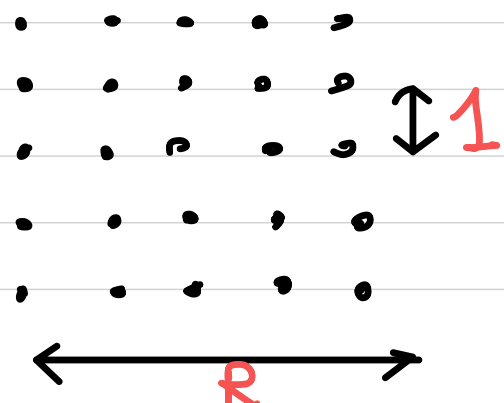
(3) Reductions :

- Suff. to consider $z = 1 + i\varepsilon$
- Localization in ξ-space (S^{d-1} compact)

$\Rightarrow$ smoothing of V on unit scale

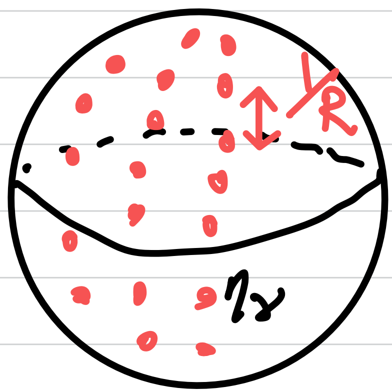
$\Rightarrow$ discretize

(# lattice pts. $\sim R^d$)



- Localization in x-space ($\operatorname{supp} \psi \subset B_R$)

$\Rightarrow$ discretize S^{d-1}:



$\{\eta_\nu\}$ $\frac{1}{R}$-net

$\mathcal{H} := \ell^2(\{1, \dots, R^{d-1}\})$

Discrete extension operator:

$$(Sg)(x) := \sum_{\nu} e^{2\pi i x \cdot \eta_\nu} \, g(\eta_\nu)$$

Reduced to control

$$S^* V_\omega S : \mathcal{H} \to \mathcal{H}$$

- By large deviation estimates suff. to control

$$\mathbb{E} \| S^* V_\omega S \| = \sup_{\|f\| = \|g\| = 1} | \langle V_\omega S f, S g \rangle |$$

- Replace sup by a sup over a finite set, pay an entropy cost.

Entropy bound

Consider a linear operator $S : \mathcal{H} \to \ell_m^\infty$, where

- $\mathcal{H}$ is a finite-dimensional Hilbert space and
- $\ell_m^\infty = \ell^\infty(\{1, \ldots, m\})$.

For $\epsilon > 0$ let $\mathcal{N}(\epsilon)$ be the minimal number of balls in ℓ_m^∞ of radius ϵ needed to cover the set $\{Sx : x \in \mathcal{H}, \|x\|_{\mathcal{H}} \leq 1\}$.
Using an entropy bound known as the "dual Sudakov inequality", [Pajor and Tomczak-Jaegermann 1986], Bourgain shows that

$$\log \mathcal{N}(\epsilon) \lesssim (\log m)\epsilon^{-2}\|S\|^2_{\mathcal{H}\to\ell_m^\infty}.$$

Crucial:

- Independent of $\dim \mathcal{H}$,
- Only log loss in m.

We apply the entropy bound to a discrete version of the Fourier extension operator ($\log m = \log R$):

$$S : \mathcal{H} \to \ell^\infty(\Lambda_1 \cap B_R), \quad \{a_\nu\} \mapsto \{\sum_{\nu \in \Lambda_R^*} a_\nu \mathrm{e}(\nu \cdot x)\}_x$$

where $\mathcal{H} := \ell^2(\Lambda_R^*)$ with norm $\|a\|_{\mathcal{H}} := R^{-\frac{d-1}{2}}(\sum_\nu |a_\nu|^2)^{1/2}$.

Discrete Stein–Tomas inequality

For each $R \geq 2$, each collection Λ_R^* consisting of $1/R$-separated points on S^{d-1}, each sequence $a_\nu \subset \mathbb{C}$, each ball B_R and each collection Λ_1 of 1-separated points in $\mathbb{R}^d$:

$$\|S\|_{\mathcal{H} \to \ell^{p'}(\Lambda_1 \cap B_R)} \lesssim 1 \quad \text{for} \quad p' \geq 2(d+1)/(d-1).$$

In particular, we have the trivial bound ($p' = \infty$)

$$\|S\|_{\mathcal{H} \to \ell^\infty(\Lambda_1 \cap B_R)} \lesssim 1.$$

Thus the entropy number satisfies the bound

$$\log \mathcal{N}(\epsilon) \lesssim (\log R)\epsilon^{-2}.$$

Consequence:

Chaining

Let $p' \geq 2(d+1)/(d-1)$. For every $k \in \mathbb{Z}_+$, there exist sets $\mathcal{F}_k \subset \ell^\infty(\Lambda_1 \cap B_R)$ with the following properties.

(a) $\log|\mathcal{F}_k| \lesssim \log(R)4^k$ (here $|\cdot|$ denotes counting measure).

(b) For $\xi \in \mathcal{F}_k$,

$$\|\xi\|_{\ell^\infty(\Lambda_1)} \lesssim 2^{-k}, \quad \|\xi\|_{\ell^{p'}(\Lambda_1)} \lesssim 1.$$

(c) For each $a \in \mathcal{H}$ with $\|a\|_{\mathcal{H}} \leq 1$ there is a representation

$$Sa = \sum_{k \in \mathbb{Z}_+} \xi^{(k)} \quad \text{for some} \quad \xi^{(k)} \in \mathcal{F}_k.$$

Thank you for your
attention !