Propagation bounds for the Bose-Hubbard model

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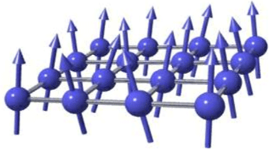
Review of standard Lieb-Robinson bounds

General quantum spin system: Spins fixed to sites of a finite lattice Λ. Local and bounded interactions h_{xy}.



Hamiltonian: On Hilbert space $\bigotimes_{j\in\Lambda}\mathbb{C}^d$, consider

$$H_\Lambda = \sum_{x\sim y} h_{xy}, \qquad x \sim y \text{ nearest-neighbors}$$

(Also OK: Sufficiently rapidly decaying interactions and/or unbounded on-site interactions.)

Dynamics: For an observable A, set $A(t) = e^{itH_\Lambda} A e^{-itH_\Lambda}$

Lieb-Robinson bound: For any observables A and B

$$\boxed{\|[A(t), B]\| \leq C\|A\|\|B\|e^{\xi(vt-d(A,B))}}$$

where $d(A,B) = \mathrm{dist}(\mathrm{supp}\, A, \mathrm{supp}\, B)$.

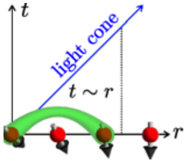
Interpretation of Lieb-Robinson bounds

Lieb-Robinson (LR) bound:

$$\|[A(t), B]\| \leq C\|A\|\|B\|e^{\xi(vt-d(A,B))}$$

Note that LHS $= 0$ at time $t = 0$ (if $d(A, B) > 0$)

Interpretation: Correlations between A and B stay localized within an effective light cone $d(A, B) \leq vt$ up to exponentially small errors.



Foss-Feig et al., PRL 114 (2014)

→ Quantum spin systems mimick the "region of causality" of relativistic systems. The underlying lattice is crucial for this.

Remarks: (i) Proof uses that interactions are bounded and local. In particular, the constants C, v, ξ depend on $\max_{x\sim y} \|h_{xy}\|$
(ii) v is called the "Lieb-Robinson velocity" (Of course, $v \ll c$.)

Short history of Lieb-Robinson bounds

- 1974: First proof by Lieb & Robinson
- ...(crickets)...
- 2004: Hastings uses and extends LR bounds as a tool in the proof of higher-dimensional Lieb-Schultz-Mattis theorem
- 2005: Nachtergaele & Sims widely extend LR bounds; use them as a tool to prove exponential clustering (independently: Hastings-Koma)
- 2006: Nachtergaele-Ogata-Sims use LR bounds as a tool to prove existence of infinite-volume dynamics
- 2006: Bravyi-Hastings-Verstraete identify several useful corollaries of LR bounds (e.g., bounds on dynamical generation of entanglement and topological order)
- 2007: Hastings proof of area law for gapped 1D spin chains using LR bounds as a tool
- 2007-today: many extensions and diverse applications of LR bounds (e.g., to lattice fermions by Nachtergaele-Sims-Young)

Unreasonable effectiveness of Lieb-Robinson bounds

Corollary (example): Local operators spread at most with speed v

$$\|A(t) - A_r(t)\| \leq Ce^{\xi(vt-r)}$$

where $A_r(t) = \mathrm{Tr}_{\Lambda\setminus(\mathrm{supp}\, A+r)} A(t)$ is supported in $\mathrm{supp}\, A + r$.

Main message: Lieb-Robinson bounds are an extremely versatile analytical tool for many body physics with decisive applications in, e.g.,

- quantum information theory (1D area law)
- condensed-matter physics (classification of quantum phases)
- high-energy physics (fast scrambling)

Question: *Why are Lieb-Robinson bounds so useful?* In a nutshell:

$$\text{local and bounded interactions} \overset{\text{LRBs}}{\Longrightarrow} \text{locality of dynamics}$$

How restrictive are the assumptions on the interaction?

Restriction: To prove LR bound, the two assumptions on the interaction between different sites were critical:

(a) local (short-ranged)

(b) bounded

...but there are many relevant physical systems for which these fail!

Remove (a) $\to$ long-range bounded interactions: Massive research effort in the last 10 years has essentially resolved this problem. (Experimentally relevant, e.g., for Rydberg atoms)

Remove (b) $\to$ unbounded interactions: Much less understood! (But experimentally observed, e.g., for ultracold bosons in optical traps.) Most results for the paradigmatic Bose-Hubbard model.

$$\boxed{H_{BH} = \sum_{x,y} J_{xy} b_x^\dagger b_y + \sum_x V(n_x)}$$

Prototypical case: $J_{xy} = \delta_{x\sim y}$ and $V(n_x) = \frac{U}{2} n_x(n_x - 1) - \mu n_x$

Brief literature review of bosonic Lieb-Robinson bounds

Key restriction: Lieb-Robinson bound only known for special initial states. (Absence of particles helps because $\|n_x\| = \infty$.)

Main challenge: Control positive density states e.g. Mott states

$$\bigotimes_{x\in\Lambda}(b_x^\dagger)^{\nu_x}|0\rangle_x, \qquad \text{with } \nu_x \in \{0,1,2,\ldots\} \text{ occupation no.'s}$$

Nachtergaele-Raz-Schlein-Sims ('07): LRB in oscillator systems

Eisert-Gross ('09): Construction of unbounded interaction where information spreads super-ballistically

Schuch-Harrison-Osborne-Eisert ('11): Initially all particles localized in finite region, control transport into empty space. Follow-up by Wang-Hazzard ('20).

Kuwahara-Saito ('21): Perturbations of stationary state with controlled average density spread at most (almost-)ballistically. → first meaningful result at positive density!

Yin-Lucas ('21): Bound on $\mathrm{Tr}(e^{-\mu N}[A(t), B])$.

Setup for the first result

For $\Lambda \subset \mathbb{Z}^d$, recall the Bose-Hubbard Hamiltonian

$$H_{BH} = \sum_{x,y} J_{xy} b_x^\dagger b_y + \sum_x V(n_x)$$

Question 1: Can we extend the previous result bounding transport *into* initially empty space to bounding transport *through* initially empty space?

Hopping assumption: For some integer $p \geq 2$,

$$\kappa_J^{(p)} = \sup_{x\in\Lambda} \sum_{y\in\Lambda} |x-y|^p |J_{xy}| \leq C \quad (C \text{ independent of } \Lambda)$$

Examples: (i) If $J_{xy} \lesssim |x-y|^{-\alpha}$, then $p = \alpha - d - 1$, so $\alpha \geq d+3$ works.
(ii) For nearest-neighbor hopping J_{xy}, can take any p.
(iii) We call $v_{\max} = \kappa_J^{(1)}$ the maximal propagation speed.

The first result

Theorem (Faupin-L-Sigal 2021)

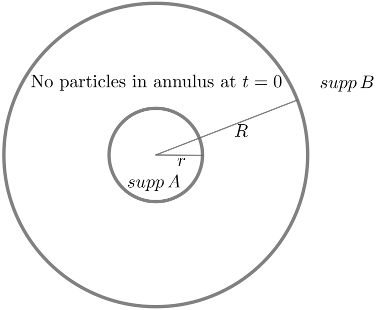
Let A, B commute with N and $\operatorname{supp} A \subset \mathcal{B}_r$, $\operatorname{supp} B \subset \Lambda \setminus \mathcal{B}_R^c$. *Suppose that $n_x \varphi = 0$ for $x \in \mathcal{B}_R \setminus \mathcal{B}_r$. Then*

$$\langle \varphi, [A(t), B]\varphi \rangle \leq C \left(\frac{v_{\max} t}{2(R-r)} \right)^{p-2} \|A\| \|B\| \langle \varphi, N\varphi \rangle$$

Interpretation: Transport through a region that initially has no particles happens at most at speed

$$v_{\max} = \sum_{xy} |J_{xy}||x-y|$$

$$\left(\overset{\text{for n.n.}}{=} 2d|J| \right)$$



Comments on first result

More general version allows for:

- Some particles inside annulus (but not fixed positive density)
- Unbounded observables A and B not necessarily commuting with N (e.g., polynomials in $b_x^\dagger, b_x$) $\to$ replace $\|A\|\|B\|$ by suitably N-weighted norms

Compared to result for spin systems this has three restrictions:

(i) matrix elements instead of norms (expected, if not necessary)

(ii) mild restrictions on observables

(iii) requires few particles inside annulus

$\to$ Result paves the way for adapting LR-based proofs to bosonic situations where these requirements are met.

Consequence (example): With A and φ as before and $\rho < \frac{R-r}{2}$,

$$\langle \varphi, (A(t) - A_\rho(t))\varphi \rangle \leq C \left(\frac{v_{\max} t}{2(R-r)} \right)^{p-2} \|A\| \langle \varphi, N\varphi \rangle,$$

where $A_\rho(t) = \mathrm{Tr}_{\Lambda \setminus (\mathrm{supp}\, A + rho)} A(t)$ is supported in $\mathrm{supp}\, A + \rho$.

The second result

Question 2: Can we treat general positive-density states if we only want to bound transport of macroscopic fraction of particles ("thermodynamic perspective")?

Normalized local particle number: For $X \subset \Lambda$,

$$\bar{N}_X = \frac{1}{N} \sum_{x \in X} n_x, \qquad X^c = \Lambda \setminus X.$$

Let $d_{XY} = \mathrm{dist}\,(X, Y)$ and $\psi_t = e^{-itH_\Lambda}\psi_0$.

Theorem (Faupin-L-Sigal 2021)

Let $v > v_{\max}$ and $0 \leq \eta < \xi \leq 1$. Let $P_{\bar{N}_{X^c} \leq \eta}\psi_0 = \psi_0$. Then

$$\boxed{\langle \psi_t, P_{\bar{N}_Y \geq \xi} \psi_t \rangle \leq C \left(\frac{vt}{d_{XY}} \right)^{p-1}}$$

Interpretation of second result

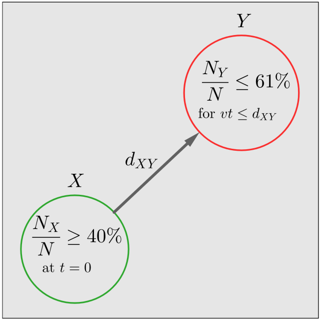
For $P_{\bar{N}_{X^c} \leq \eta} \psi_0 = \psi_0$, we have

$$\langle \psi_t, P_{\bar{N}_Y \geq \xi} \psi_t \rangle \leq C \left(\frac{vt}{d_{XY}} \right)^{p-1}$$

The transport of 1% of the particles from X to Y takes time proportional to $d(X, Y)$.

"A macroscopic cloud of particles moves at most at speed $v_{\max}$."

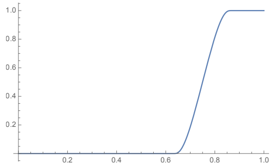
Result does not require any constraint on the local particle density.



Main proof idea for Result 2: ASTLOs

Technique: "adiabatic spacetime localization observables" (ASTLOs); inspired by technique first developed for one-body Schrödinger operators $-\Delta + V$ on $\mathrm{L}^2(\mathbb{R}^d)$.

Idea: Dynamically track local particle number outside of the light cone but in an adiabatically smeared-out way, where only particles at distance $\sim d(X, Y)$ from the light cone are fully counted.



Cutoff profile χ

Definition of ASTLO: Let $\chi : \mathbb{R}_+ \to [0, 1]$ be a really nice cutoff function. Set

$$\mathbb{A}_t = \frac{1}{N} \sum_{x \in \Lambda} \chi\left(\frac{|x| - \mathrm{diam}\, X - vt}{\epsilon d_{XY}}\right) n_x$$

for ϵ sufficiently small but fixed.

Second-order ASTLO

Heuristic: Adiabatic smearing leads to controllable time derivative and thus precise tracking of number of particles outside the light cone.

For result 1, this can be implemented. For result 2, we need to smear out the spectral projectors $P_{\bar{N}_Y \geq \xi}$ as well.

Second-order ASTLO: Let $f : \mathbb{R}_+ \to [0, 1]$ be a nice cutoff function such that $f = 0$ until η and $f = 1$ after ξ. Then set

$$\Phi(t) = f(\mathbb{A}_t)$$

With $\langle \cdot \rangle_t \equiv \langle \cdot \rangle_{\psi_t}$, we have

$$\langle \Phi(0) \rangle_0 = 0, \qquad \langle P_{\bar{N}_Y \geq \xi} \rangle_t \leq \langle \Phi(t) \rangle_{\psi_t} = \int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau} \langle \Phi(\tau) \rangle_\tau \mathrm{d}\tau$$

so it suffices to control growth rate of $\langle \Phi(t) \rangle_t$ in time.

Key estimates on time derivative

We calculate the time derivative and recall $\Phi(t) \equiv f(\mathbb{A}_t)$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\Phi(t)\rangle_t = \langle D\Phi(t)\rangle_t, \qquad D\Phi(t) \equiv \Phi'(t) + i[H, \Phi(t)].$$

Key technical estimate: Given cutoff functions f, χ there exist $\tilde{f}, \tilde{\chi}$ such that we have the differential inequality

$$Df(\mathbb{A}_t) \leq -\frac{v - v_{\max}}{s} f'(\mathbb{A}_t)\mathbb{A}_t' + \frac{C}{s^2}\tilde{f}'(\tilde{\mathbb{A}}_t)\tilde{\mathbb{A}}_t' + \frac{C}{s^p}. \qquad (1)$$

Proved by iterated commutator expansion of $[H, \Phi(t)]$ using resolvents (starting from Helffer-Sjöstrand formula) and some analytical tricks to get operator inequalities.

Observation: The leading and subleading terms in (1) are of the same structure → iteration possible!

$$\int_0^t \langle f'(\mathbb{A}_r)\mathbb{A}_r'\rangle_r \leq \frac{C}{s}\underbrace{\int_0^t \langle \tilde{f}'(\tilde{\mathbb{A}}_r)\tilde{\mathbb{A}}_r'\rangle_r}_{\leq \frac{C}{s}\text{etc.}} + \frac{C}{s^{p-1}}$$

Summary and open problems

Summary: New Lieb-Robinson bounds for Bose-Hubbard model
Result 1: LRB through initially particle-free region.
Result 2: Bound on transport of macroscopic particle clouds for general initial states.

New analytical proof tool: Adiabatic space-time localization observables (ASTLO) $\Phi(t)$

Two key properties: (i) $\Phi(t)$ dynamically tracks particles far (namely at distance $\gtrsim \epsilon d_{XY}$) outside of light cone
(ii) $\langle \Phi(t) \rangle_t$ can be shown to be slowly varying by commutator expansion; its growth can then be controlled by iteration trick.

Open problems:

- Use Result 1 to develop (suitably restricted) bosonic analog of LPPL principle, quasi-adiabatic evolution, etc.
- Macroscopic transport of other physical quantities, e.g., entanglement?

Thank you for your attention!