A New Direction: The Oriented Chromatic Number of Random Graphs of Bounded Degree

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SOON ➡ Toronto Metropolitan University

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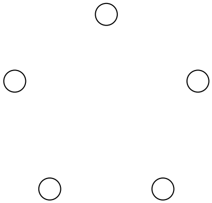

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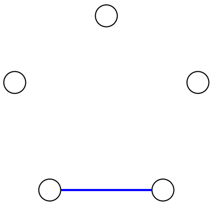

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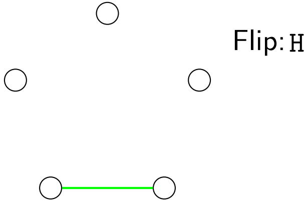

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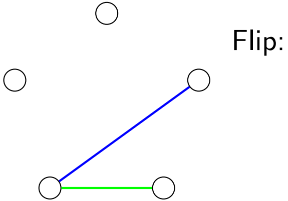

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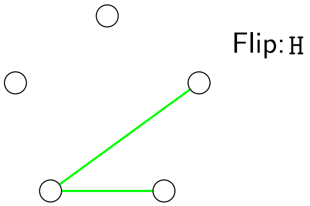

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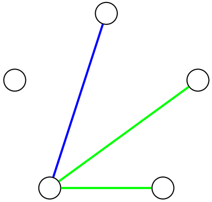

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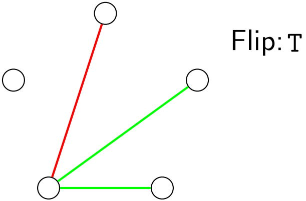

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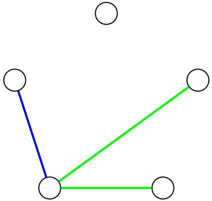

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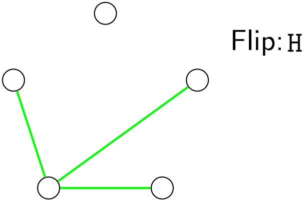

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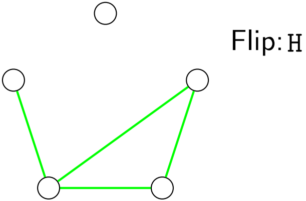

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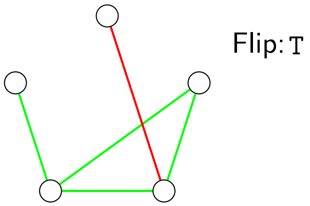

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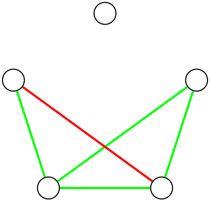

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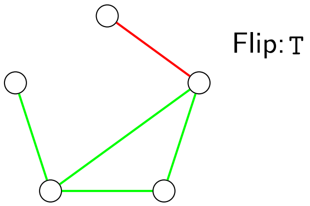

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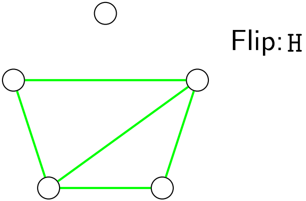

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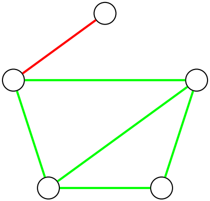

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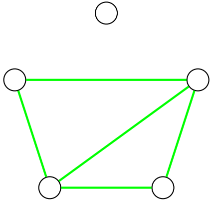

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Different random graph models produce different distributions.

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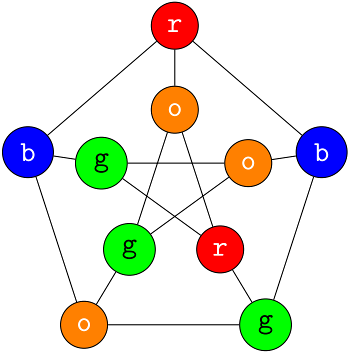

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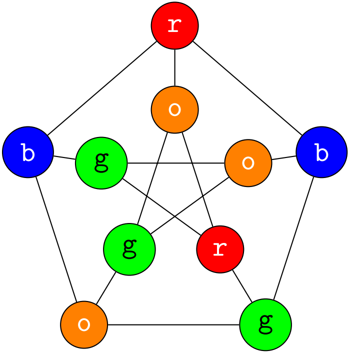

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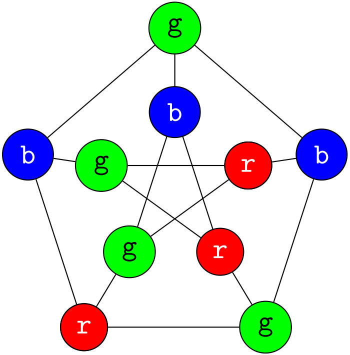

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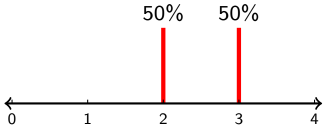

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For $p = \frac{1}{2}$, what is the probability M_n that $\mathcal{G}_{n,p}$ is **not** *a complete graph?*

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If P_n is the probability that property X holds for, say, $\mathcal{G}_{n,p}$ and

$$\lim_{n\to\infty} P_n \to 1$$

then property X holds **asymptotically almost surely** or a.a.s.

Undirected Graphs

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- Coja-Oghlan et al. (2013): Use ideas from statistical physics to show both models concentrated on one value (for d large enough).

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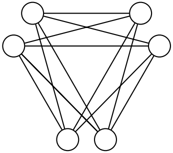
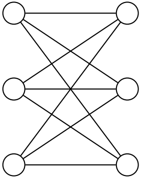
Key Idea (Overlap Matrices)

In order to calculate second moments, need to know which types of graphs permit many colourings.





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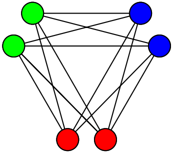

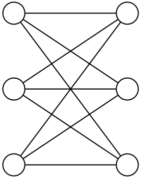

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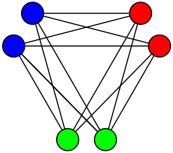

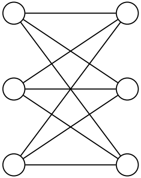

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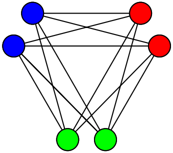

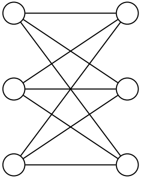

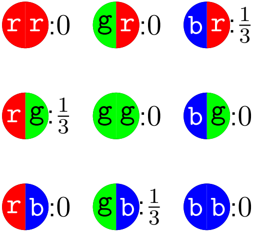

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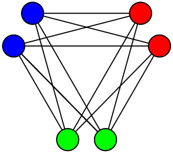

r r: 0	g r: 0	b r: $\frac{1}{3}$
r g: $\frac{1}{3}$	g g: 0	b g: 0
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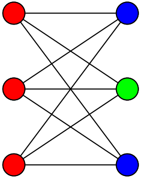

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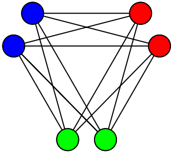

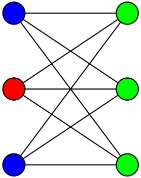

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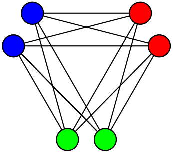

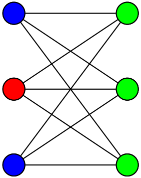

$$\begin{bmatrix} 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 \end{bmatrix}$$

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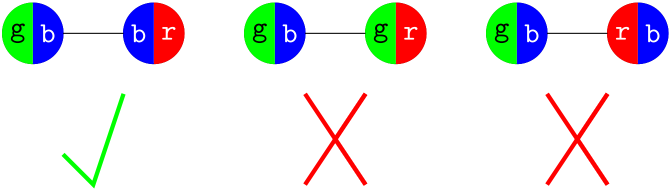
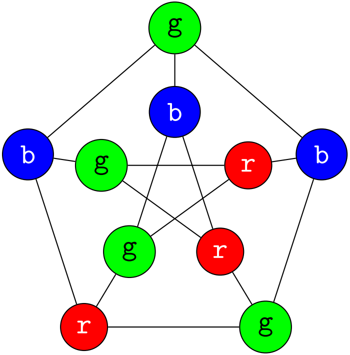


Graph Colouring



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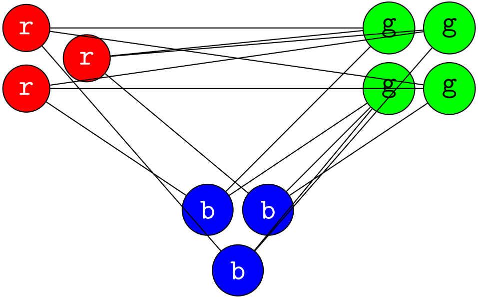
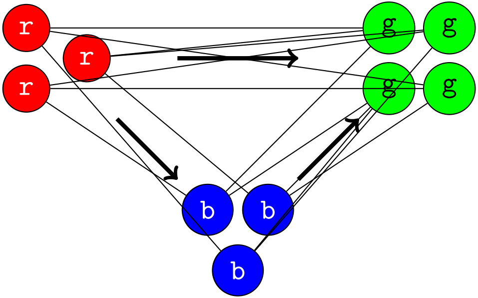


Directed Graphs

Oriented Colourings

Oriented Graph Colouring



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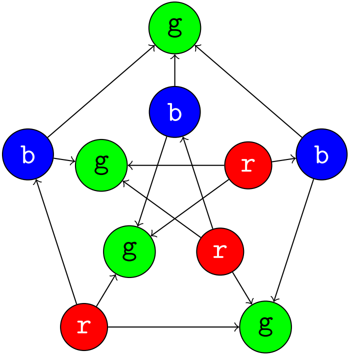

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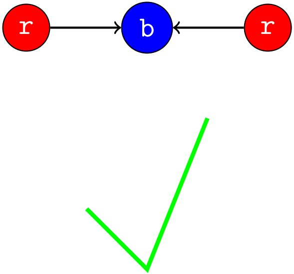

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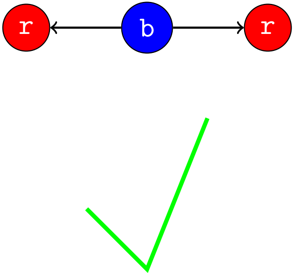

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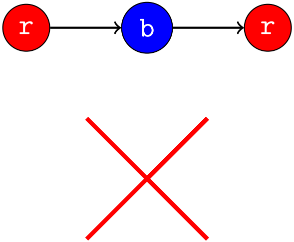

Directed Graphs
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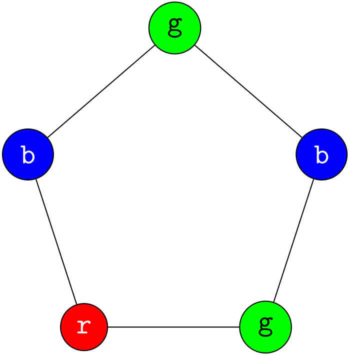

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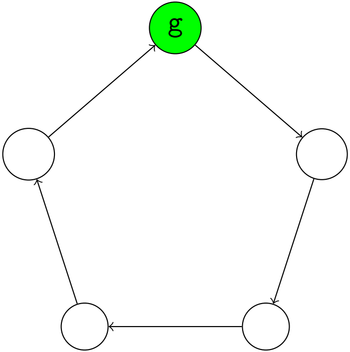

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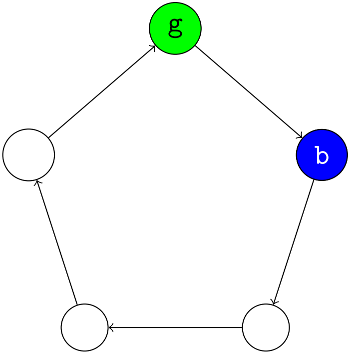

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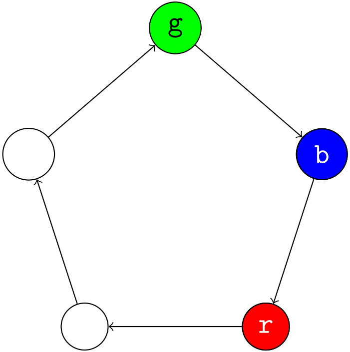

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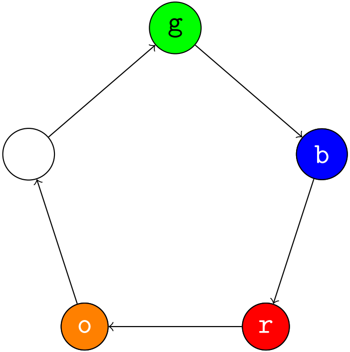

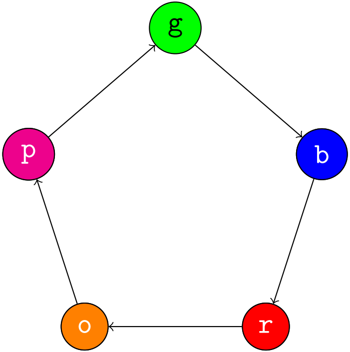
g
b
r
o
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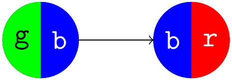
Overlap matrices become more complicated:



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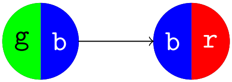
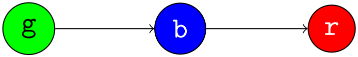


requires



Question

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Directed Graphs
Doubly Regular Tournaments

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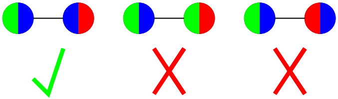
The product of a doubly-regular tournament with itself is strongly regular and has (unsigned) adjacency matrix

$$\frac{1}{2}\big(M \otimes M + (J - I) \otimes (J - I)\big).$$

More intricate constraints on overlap matrices:



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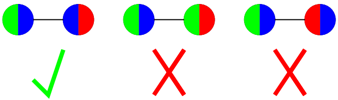

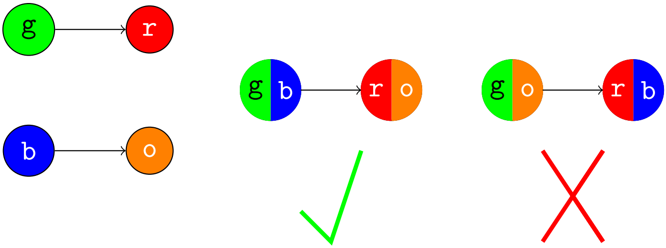

Directed Graphs

Results

Theorem (Gunderson-N., 2022+)

The oriented chromatic numbers $\chi_o(\vec{\mathcal{G}}_{n,p=d/n})$ *and* $\chi_o(\vec{\mathcal{G}}_{n,d})$ *are concentrated in the window*

Directed Graphs
Results

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$$(2^{d/2}, 6e^{d/2} + 6d + 17].$$

Directed Graphs
Results

Maybe the exponential gap isn't our fault?

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Theorem

For $\vec{G} \sim \vec{\mathcal{G}}_{n,2}$, with high probability, $\chi_o(\vec{G}) \in \{4, 5\}$, but each occurs with positive probability.

Directed Graphs
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Theorem

For $\vec{G} \sim \vec{\mathcal{G}}_{n,2}$, with high probability, $\chi_o(\vec{G}) \in \{4, 5\}$, but each occurs with positive probability.

Proof idea: with positive probability, $\vec{G}$ has no oriented 5-cycle.

Directed Graphs

Where to next?

Next steps:

- Adapt statistical physics models to directed case.

Directed Graphs
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Directed Graphs

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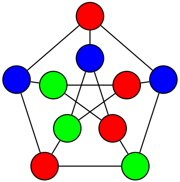
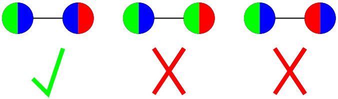
Directed Graphs
Where to next?

Next steps:

- Adapt statistical physics models to directed case. Challenge: finding the right type of colouring.
- Lower bound on concentration window length?
- Focus on small cases, like $d = 3$. Great workshop problem!

Thanks!





Thank you!