Periodic Minimizers of a Ternary Non-Local Isoperimetric Problem

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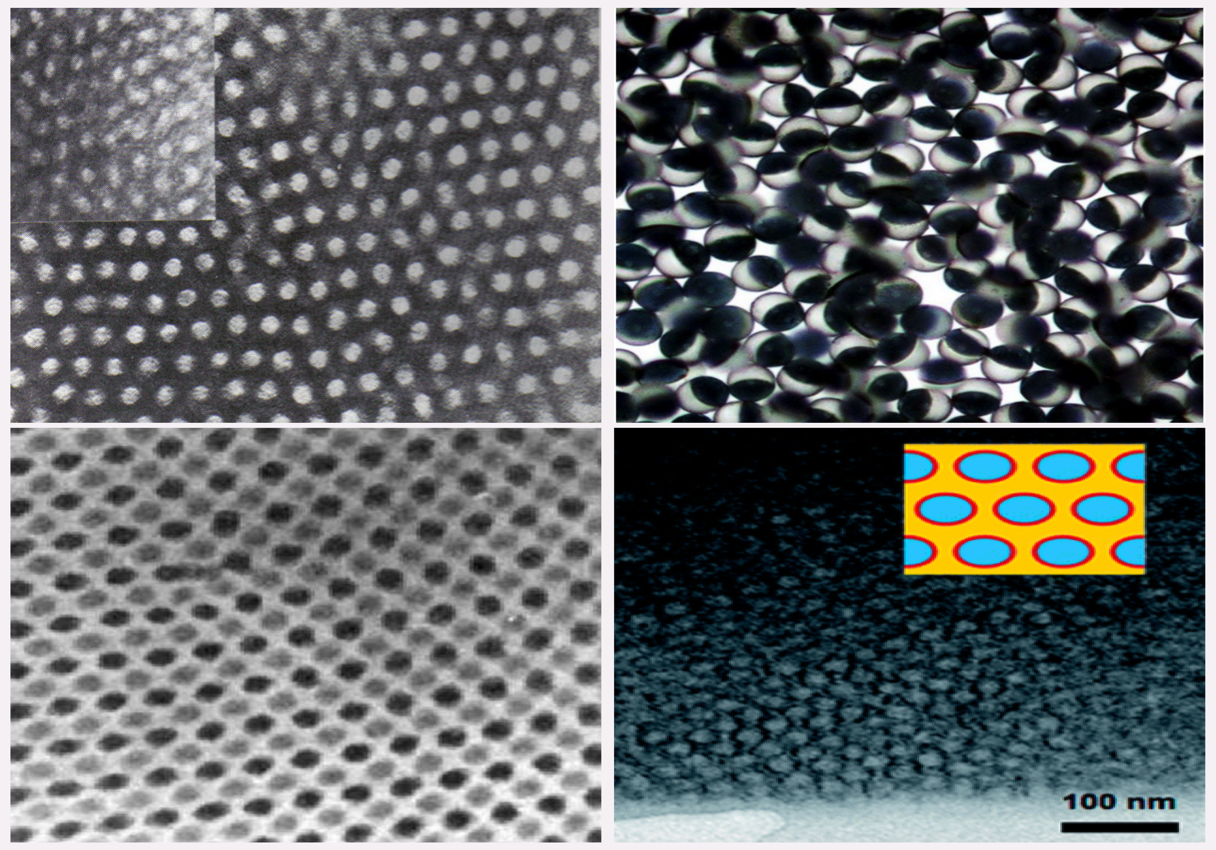
Diverse Patterns



Block Copolymers

When two or different monomers unite together to polymerize, their result is called a **copolymer**. Copolymers can be classified based on how the monomers are arranged along the chain.

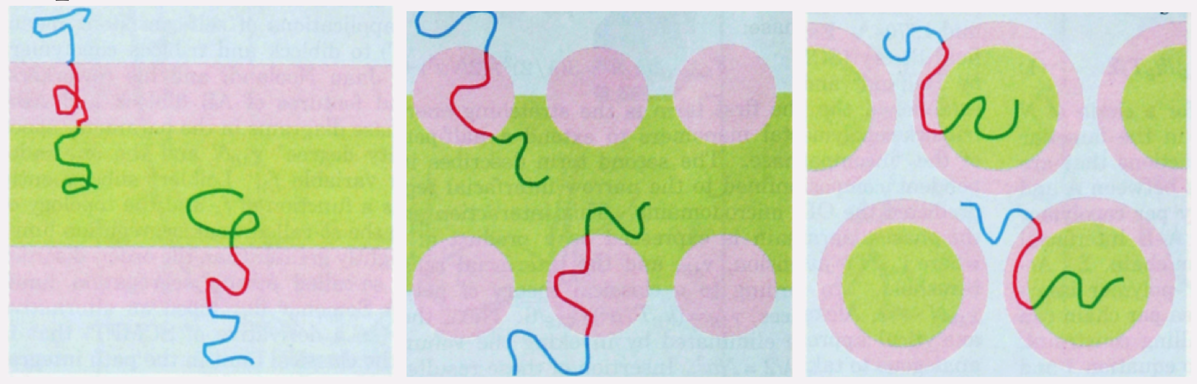
- **Alternating copolymers**
- **Random copolymers**
- **Block copolymers**

—A—B—A—B—A—B—A—B—A—B—

—A—B—B—B—A—B—A—B—A—A—

—B—B—B—B—B—A—A—A—A—A—

Due to the repulsion between the unlike monomers, the different type sub-chains tend to segregate, but as they are chemically bonded in chain molecules, segregation of sub-chains cannot lead to a macroscopic phase separation. Only a local micro-phase separation occurs.



Commercial Uses

Wine Bottle Stoppers, Jelly Candles, Outdoor Covering for Optical Fibre Cables, Adhesives, Bitumen Modifiers, or in Artificial Organ Technology

Isoperimetric Problems

An Isoperimetric Problem:
Find a subset Ω of D, such that $|\Omega| = \omega|D|$ and the perimeter of Ω in D, $\mathcal{P}_D(\Omega)$, is the smallest.
(Here $D \subset \mathbb{R}^n$: a bounded domain. $\omega \in (0,1)$: a parameter. $\mathcal{P}_D(\Omega) := \int_D |\nabla \chi_\Omega|$, where

$\int_D |\nabla \chi_\Omega| := \sup \left\{ \int_D \chi_\Omega \ \operatorname{div} \mathbf{g}\, dx : \mathbf{g} = (g_1, \cdots, g_n) \in C_c^1(D, \mathbb{R}^n), \text{ and } |\mathbf{g}(x)| \le 1 \text{ for } x \in D \right\}$.)

A Binary Non-Local Isoperimetric Problem:
Find a subset Ω of D, such that $|\Omega| = \omega|D|$ to minimize

$$\mathcal{J}_B(\Omega) = \mathcal{P}_D(\Omega) + \frac{\gamma}{2}\int_D\int_D G(x,y)(\chi_\Omega(x) - \omega)(\chi_\Omega(y) - \omega)\, dxdy.$$

(Here $D \subset \mathbb{R}^n$: a bounded domain. $\omega \in (0,1), \gamma > 0$: two parameters. $G(x,y)$: the Green's function of $-\triangle$. $-\triangle G(\cdot, y) = \delta(\cdot - y) - \frac{1}{|D|}$ in D, $\partial_\nu G(\cdot, y) = 0$ on ∂D, $\int_D G(x,y)dx = 0$.)

A Ternary Non-Local Isoperimetric Problem:
Find $\Omega_1 \subset D, \Omega_2 \subset D$, such that $|\Omega_1| = \omega_1|D|$, $|\Omega_2| = \omega_2|D|$, $|\Omega_1 \cap \Omega_2| = 0$, to minimize

$$\mathcal{J}_T(\Omega_1, \Omega_2) = \frac{1}{2}\sum_{i=1}^{3}\mathcal{P}_D(\Omega_i) + \sum_{i,j=1}^{2}\frac{\gamma_{ij}}{2}\int_D\int_D G(x,y)(\chi_{\Omega_i}(x) - \omega_i)(\chi_{\Omega_j}(y) - \omega_j)\, dxdy.$$

(Here $D \subset \mathbb{R}^n$: a bounded domain. ω_1, ω_2 both in $(0,1)$. Moreover $\omega_3 = 1 - (\omega_1 + \omega_2) \in (0,1)$. $\gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{12} & \gamma_{22} \end{bmatrix}$: a 2 by 2 symmetric matrix.)

Rescaling

$$\mathcal{E}(u) = \frac{1}{2}\sum_{i=0}^{2}\int_{\mathbb{T}^2}|\nabla u_i| + \sum_{i,j=1}^{2}\frac{\gamma_{ij}}{2}\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} G(x-y)\, u_i(x)\, u_j(y) dxdy.$$

Regime with two **vanishing** minority constituents: $\int_{\mathbb{T}^2} u_i = \eta^2 M_i$, $i = 1,2$, $\eta \ll 1$.
Rescale u_i as $v_{i,\eta} = \eta^{-2}u_i$. Thus $\int_{\mathbb{T}^2} v_{i,\eta} = M_i$. Choose $\gamma_{ij} = \frac{1}{|\log\eta|\eta^3}\Gamma_{ij}$.
Rescaled Energy:

$$E_\eta(v_\eta) = \frac{1}{\eta}\mathcal{E}(u) = \frac{\eta}{2}\sum_{i=0}^{2}\int_{\mathbb{T}^2}|\nabla v_{i,\eta}| + \sum_{i,j=1}^{2}\frac{\Gamma_{ij}}{2|\log\eta|}\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} G(x-y) v_{i,\eta}(x) v_{j,\eta}(y) dxdy.$$

Let $z^k_{i,\eta}(x) = \eta^2 v^k_{i,\eta}(\eta x + \xi^k)$, calculation yields

$$E_\eta(v_\eta) = \sum_{k=1}^{\infty}\left(\frac{1}{2}\sum_{i=0}^{2}\int_{\mathbb{R}^2}|\nabla z^k_{i,\eta}| + \sum_{i,j=1}^{2}\frac{\Gamma_{ij}}{4\pi}m_i^k\, m_j^k\right) + O(|\log\eta|^{-1}).$$

Consider $\overline{e_0}(M) = \inf\left\{\sum_{k=1}^{\infty} e_0(m^k) : m^k = (m_1^k, m_2^k),\ m_i^k \ge 0, \sum_{k=1}^{\infty} m_i^k = M_i, i = 1,2\right\}$, where $e_0(m) = p(m_1, m_2) + \sum_{i,j=1}^{2}\frac{\Gamma_{ij}m_i m_j}{4\pi}$, $m = (m_1, m_2)$.

Difficulties

No explicit formula for the perimeter of double bubbles.

Results

(**Coexistence**) Given Γ_{11}, Γ_{22}, K_1 and $K_2 > 0$, and $\Gamma_{12} = 0$, there exist M_1 and M_2 such that any minimizing configuration has at least K_1 double bubbles and K_2 single bubbles.

(**All single bubbles**) Given $\Gamma_{11} > 0$, $\Gamma_{22} > 0$, $M_1 > 4M_1^*$, $M_2 > 4M_2^*$, there exists a threshold Γ_{12}^* such that for all $\Gamma_{12} > \Gamma_{12}^*$, any minimizing configuration has no double bubbles.

(**One double bubble**) Given Γ_{ii}, $M_i < \min\{m_i^*, \pi\Gamma_{ii}^{-2/3}\}$, $i = 1,2$, and sufficiently small $\Gamma_{12} > 0$ such that $\frac{\Gamma_{12}}{2\pi}M_1 M_2 + p(M_1, M_2) < 2\sqrt{\pi}(\sqrt{M_1} + \sqrt{M_2})$, then there is a unique minimizer made of one double bubble.

Behaviour as $\eta \to 0$:

- First level: $E_\eta \xrightarrow{\Gamma} E_0$.
- Second level: $F_\eta \xrightarrow{\Gamma} F_0$.

Minimizers at η level:

Let $v_\eta^* = \eta^{-2}\chi_{\Omega_\eta}$ be minimizers of E_η for all $\eta > 0$. Then, there exists a subsequence $\eta \to 0$ and $K \in \mathbb{N}$ such that:

1. There exist connected clusters $A^1, \dots, A^K$ in $\mathbb{R}^2$ and points $x_\eta^k \in \mathbb{T}^2$, $k = 1, \dots, K$, for which $\eta^{-2}\left|\Omega_\eta \triangle \bigcup_{k=1}^K (\eta A^k + x_\eta^k)\right| \xrightarrow{\eta\to 0} 0$;

2. Each A^k, $k = 1, \dots, K$ is a minimizer of $e_0(m^k)$, $m^k = |A^k|$; Moreover, $\overline{e_0}(M) = \lim_{\eta\to 0} E_\eta(v_\eta) = \sum_{k=1}^K e_0(m^k)$.

3. $x_\eta^k \xrightarrow{\eta\to 0} x^k$, $\forall k = 1\dots, K$. $\{x^1, \dots, x^K\}$ attains the minimum of $\mathcal{F}_K(y^1, \dots, y^K; \{m^1, \dots, m^K\})$ over all $\{y^1, \dots, y^K\}$ in $\mathbb{T}^2$.

Current and Future Work

Quaternary systems. Higher dimensions.