## 1. Bernstein-Gel'fand-Gel'fand (BGG) Correspondence

1.1. **BGG in algebra.** Fix a field $k$. Throughout, we let $S_n$ be the polynomial ring in $n$ variables

$$S_n = k[t_1, \dots, t_n] \text{ with } \deg(t_i) = 2$$

and we let $E_n$ be the exterior algebra in $n$ variables

$$E_n = \Lambda_k(e_1, \dots, e_n) \text{ with } \deg(e_i) = -1$$

(Unless otherwise stated, all indexing is cohomological. Sometimes we'll write just $S$ or $E$ for $S_n$ or $E_n$.)

Given any graded ring $R$, a dg-$R$-module is a graded $R$-module $M$ equipped with a degree 1, $R$-linear endomorphism $d_M$ such that $d_M^2 = 0$. (This is not quite the same thing as a complex of graded $R$-modules, but the latter determines a dg-$R$-module via "totalization".)

The BGG correspondence is an equivalence of triangulated categories

$$D^b(dg - S_n - modules) \xrightarrow{\cong} D^b(dg - E_n - modules).$$

In lay terms: dg-$S_n$-modules and dg-$E_n$-modules are the same thing, up to quasi-isomorphism (weak equivalence). The correspondence is given by

$$M \mapsto k \otimes_S^{\mathbb{L}} M = K \otimes_S M$$

where $K = (E^* \otimes_k S, \sum_i e_i \otimes t_i)$ (the Koszul resolution of $k$).

For instance $k \longleftrightarrow E^* \cong \Sigma^{-n} E$ and $S \longleftrightarrow k$ under BGG.

The BGG correspondence induces an equivalence on subcategories

$$D^b(\text{dg-}S_n\text{-modules } M \text{ with } \dim_k H^*(M) < \infty) \xrightarrow{\cong} D^b(\text{perfect dg-}E_n\text{-modules}) = \text{Thick}(E)$$

1.2. **"topological BGG".** Let $T_n$ denote the $n$-dimensional torus

$$T_n = \overbrace{S^1 \times \cdots \times S^1}^{n}$$

where $S^1$ is the unit circle in the complex plane. We regard $T_n$ as a topological abelian group. (Sometimes we will write it as just $T$.)

Let $X$ be a simple, compact $T_n$-CW-complex. Set

$$X/T = \{pt\} \times_T X, \text{ the orbit space}$$

and

$$X_T = ET \times_T X = (ET \times X)/T, \text{ the homotopy orbit space.}$$

(Heuristic for algebraists: $ET \times_T X = \{pt\} \times_T^{\mathbb{L}} X$.)

"Topological BGG": The map $X \to \{pt\}$ induces a map $X_T \to \{pt\}_T = BT$, and the $T$-space $X$ may be recovered from $X_T \to BT$, by forming the pull-back of

$$\begin{array}{ccc} & & ET \\ & & \downarrow \\ X_T & \longrightarrow & BT. \end{array}$$

In fact, there is a bijection (up to weak equivalence):

$$\{\text{spaces with } T\text{-actions}\} \longleftrightarrow \{\text{spaces equipped with a map to } BT\}.$$

1.3. **Connection between the two BGGs.** The group law for $T$ makes $C_*(T, \mathbb{Q})$ (the rational chain complex) into a dga (= differential graded algebra) over $\mathbb{Q}$ and the action of $T$ on $X$ makes $C_*(X, \mathbb{Q})$ into a dg-$C_*(T, \mathbb{Q})$-module.

The map $X_T \to BT$ makes $C^*(X_T, \mathbb{Q})$ is a dg-$C^*(BT, \mathbb{Q})$-module (in fact, algebra). We could instead use Sulliven minimal models here, as was discussed in detail by Hanke.

**Proposition 1.1.** *$C_*(T_n, \mathbb{Q})$ and $C^*(BT_n, \mathbb{Q})$ are both formal: there are quasi-isomorphisms of dgas $E_n \to C_*(T_n, \mathbb{Q})$ and $S_n \to C^*(BT_n, \mathbb{Q})$. Thus, $C_*(X, \mathbb{Q})$ and hence $C^*(X, \mathbb{Q})$ is a dg-$E_n$-module and $C^*(X_T, \mathbb{Q})$ is a dg-$S_n$-module.*

So, starting with $X$ equipped with a $T_n$-action, we get a dg-$E_n$-module $C^*(X, \mathbb{Q})$. We may also associate to $X$ the dg-$S_n$-module $C^*(X_T, \mathbb{Q})$. These coincide under (algebraic) BGG.

An important point: There is more structure on the topological side, that is ignored when passing to algebra. For instance, $C^*(X_T, \mathbb{Q})$ is a dg-$S_n$-*algebra* (not merely a dg-$S_n$-module).

## 2. To*al Rank Conjecture for $* \in \{r, t\}$

**To$t$al Rank Conjecture** [Avramov]. Suppose $M$ is a graded $S_n$-module (i.e., a dg-$S_n$-module with trivial differential) and $0 < \dim_k(M) < \infty$. Then $\sum_i b_i(M) \geq 2^n$ where $b_i(M)$ is the $i$-th Betti number of $M$. That is, if $F_* \xrightarrow{\sim} M$ is the minimal free resolution of $M$, then $\sum_i \operatorname{rank}_S(F_i) \geq 2^n$. Alternatively, $\dim_k H_*(M \otimes^{\mathbb{L}}_S k) \geq 2^n$.

*Remark* 2.1. This was originally stated in the local case.

**Generalized To$t$al Rank Conjecture** [F. Lore] Assume $F$ is a semi-free dg-$S_n$-module such that $0 < \dim_k H^*(F) < \infty$. Then $\dim_k H^*(F \otimes_S k) \geq 2^n$.

**Perfect dg-$E$-module Conjecture**: Let $P$ be a perfect dg-$E_n$-module. Then $\dim_k H^*(P) \geq 2^n$.

**To*r*al Rank Conjecture:** [Halperin] If $T_n$ acts freely on $X$ then $\dim_{\mathbb{Q}} H^*(X, \mathbb{Q}) \geq 2^n$; i.e., $\dim_{\mathbb{Q}} H^*(X, \mathbb{Q}) \geq 2^{\text{toral rank of } X}$.

These Conjectures are related as follows:

- The Generalized Total Rank Conjecture and the Perfect dg-$E$-module Conjectures are equivalent. This is a consequence of BGG.
- The Generalized Total Rank Conjecture implies the Total Rank Conjecture. This holds since a graded free resolution determines a dg-module.
- The Generalized Total Rank Conjecture implies the Toral Rank Conjecture. This holds since given a simple, free $T$-CW-complex $X$, $C^*(X_T)$ is a semi-free dg-$S_n$-module with finite dimensional homology.

## 3. A Theorem

**Theorem 3.1** (W, 2017)**.** *If* $\operatorname{char}(k) \neq 2$ *and* $F$ *is a semi-free dg-*$S_n$*-module such that* $0 < h(F) < \infty$*, then*

$$\operatorname{rank}_{S_n}(F) \geq 2^n \cdot \frac{|\chi(F)|}{h(F)}$$

*where*

$$\chi(F) := \sum_i (-1)^i \dim_k H^i(F) \text{ and } h(F) := \sum_i \dim_k H^i(F)$$

**Corollary 3.2.** *The To**t**al Rank Conjecture holds for graded modules, provided* $\operatorname{char}(k) \neq 2$*.*

**Theorem 3.3** (Topological Version of this Theorem)**.** *Assume* $T_n$ *acts freely on* $X$*, with* $X$ *a compact, simple* $T_n$*-CW-complex. Then*

$$h(X) \geq 2^n \cdot \frac{\chi(X_T)}{h(X_T)}$$

*where* $h(X) = \sum_i \dim_{\mathbb{Q}} H^i(X, \mathbb{Q})$ *and* $\chi(X) = \sum_i (-1)^i \dim_{\mathbb{Q}} H^i(X, \mathbb{Q})$*. (Note that* $X_T \sim X/T$ *under these assumptions.)*

**Corollary 3.4.** *The To**r**al Rank Conjecture holds for* $X$ *whenever* $H^{odd}(X_T, \mathbb{Q}) = 0$*.*

3.1. **Example: Rationally Elliptic Spaces.** Say $X$ is "rationally elliptic": this means both $H^*(X,\mathbb{Q})$ and $\pi_*(X,\mathbb{Q})$ are finite dimensional. Algebraically, this means the Sulliven model $\mathcal{M}(X)$ is a finite generated as a $\mathbb{Q}$-algebra and it has finite dimensional homology. Let

$$\chi_\pi(Y) = \sum_i (-1)^i \dim_{\mathbb{Q}} \pi_i(Y,\mathbb{Q}).$$

If $T_n$ acts freely on $X$ then Halperin has shown that $\chi_\pi(X) \leq -n$. We have

$$\chi_\pi(X_T) = \chi_\pi(X) - \chi_\pi(T_n) = \chi_\pi(X) + n \leq 0.$$

Halperin also shows that $\chi_\pi(X_T) = 0$ if and only if $H^{\mathrm{odd}}(X,\mathbb{Q}) = 0$.

We conclude:

**Corollary 3.5.** *If $T_n$ acts freely on a rationally elliptic $X$ and $\chi_\pi(X) = -n$ (the largest value possible) then the Toral Rank Conjecture holds for $X$:* $\dim_{\mathbb{Q}} H^*(X,\mathbb{Q}) \geq 2^n$

*Remark* 3.6. Halperin shows that if $\chi_\pi(X_T) < 0$, then $\chi(X_T) = 0$. So, if $\chi_\pi(X) > -n$, then the Theorem above *gives no information.*

**Naive Question**: If $X$ is elliptic and $\chi_\pi(X) = -n$ then does $X$ admit a free $T_n$-action?

The answer is likely "no". Examples where $\chi_\pi(X) = -n$ and $X$ does not admit a free $T_n$-action represent a place to look for counter-examples to the Toral Rank Conjecture.

## 4. $E$-module version of Theorem, and its proof

Under BGG, Theorem 3.1 is equivalent to:

**Theorem 4.1.** *Assume* $\mathrm{char}(k) \neq 2$. *Let $P$ be a perfect dg-$E_n$-module. Then* $h(P) \geq 2^n \cdot \frac{|\chi(\overline{P})|}{h(\overline{P})}$ *where* $\overline{P} = P \otimes_E k = P/(e_1,\dots,e_n)P$.

*Proof.* The central idea is to approximate $P \otimes_E P$ in two ways.

(1) (Easy part) $h(P \otimes_E P) \leq h(P)h(\overline{P})$.
(2) (Sneaky, but still pretty easy part) $2^n \cdot |\chi(\overline{P})| \leq h(P \otimes_E P)$

*Remark* 4.2. The topological version of these two facts are:

(1) $h(X \times_T X) \leq h(X)h(X_T)$
(2) $2^n \cdot |\mathrm{ch}(X_T)| \leq h(X \times_T X)$

I leave the proof of (1) to your imaginations. For (2), we use that $C_2 = \langle \tau \rangle$ acts on $P \otimes_E P$ by $\tau(\alpha \otimes \beta) = (-1)^{|\alpha||\beta|}\beta \otimes \alpha$ and thus (provided $\mathrm{char}(k) \neq 2$)

$$P \otimes_E P = S^2_E(P) \oplus \Lambda^2_E(P)$$

where $S^2_E(P) = (P \otimes_E P)^{(1)}$ and $\Lambda^2_E(P) = (P \otimes_E P)^{(-1)}$. Set

$$\Psi^2(P) = S^2_E(P) - \Lambda^2_E(P) \text{ in the Grothendieck group.}$$

and

$$\chi\Psi^2(P) = \chi(S^2_E(P)) - \chi(\Lambda^2_E(P)) \in \mathbb{Z}.$$

**Key Fact**: $\chi\Psi^2(P)) = 2^n\chi(\overline{P})$.

Sketch of Proof of Key Fact: $\chi\Psi^2$ enjoys the following properties:

- $\chi(\Psi^2(-))$ is additive on short exact sequences of perfect dg-$E_n$-modules,
- $\chi(\Psi^2(\Sigma P)) = -\Psi^2(\Sigma P))$.
- $\chi(\Psi^2(E)) = 1$.

I'll omit justification of the first two. For the last $E \otimes_E E \cong E$, but under this isomorphism $\tau$ acts as $\tau(\alpha) = (-1)^{|\alpha|}\alpha$. So $S^2_E(E) = E^{even}$ and $\Lambda^2_E(E) = E^{odd}$. Whence $\chi(\Psi^2(E)) = 2^{n-1} - (-2^{n-1}) = 2^n = 2^n\chi(\overline{E})$. The Key Fact follows from these three properties, since $P$ perfect means $P$ is built up from $E$ and its suspensions by a sequence of mapping cones constructions.

We can now complete the proof of Theorem 4.1:

$$\begin{aligned} h(P \otimes_E P) &= h(S^2_E(P)) + h(\Lambda^2_E(P)) \\ &\geq h^{even}(S^2_E(P)) + h^{odd}(\Lambda^2_E(P)) \\ &\geq \chi(S^2_E(P)) - \chi(\Lambda^2_E(P)) \\ &= \chi(\Psi^2_E(P)) \\ &= 2^n\chi(\overline{P}) \end{aligned}$$

(When $\chi(\overline{P}) < 0$, interchange roles of even and odd.) □

*Remark* 4.3. In fact $\chi(S^2_E(P)) = 2^{n-1}\chi(\overline{P})$ and $\chi(\Lambda^2_E(P)) = -2^{n-1}\chi(\overline{P})$. The topological version of the first of these

$$\chi(Sp^2(X)_T) = 2^{n-1}\chi(X_T)$$

where $Sp^2(X) = (X \times X)/C_2$, the second symmetric power of $X$.

**Question 4.4.** *Is there a space $X$ with a free $T_n$-action such that $h(X \times_T X) < 2^n h(X_T)$? Any counter-example to the Toral Rank Conjecture would have to have this property (but just having it doesn't make it a counter-example). How about $h(Sp^2(X)_T) < 2^{n-1}h(X_T)$?*

*Algebraical version of this question: Is there is dg-$S_n$-*algebra *$A$ such that $h(A \otimes_{S_n} A) < 2^n h(A)$? Or $h(Symm^2_{S_n}(A)) < 2^{n-1}h(A)$?*

## 5. A counter-example to the Generalized Total Rank Conjecture

**Theorem 5.1.** *(Iyengar-W, 2018) The Generalized Total Rank Conjecture is false if $n \geq 8$ and $\mathrm{char}(k) \neq 2$.*

*Proof.* For simplicity, take $n = 8$. We disprove the Perfect dg-$E_8$-module Conjecture. Let $\omega = e_1e_2 + e_3e_4 + e_5e_6 + e_7e_8 \in E^{-2}$ and set

$$P = \mathrm{cone}(E(2) \xrightarrow{\omega} E).$$

The map $E(2) \xrightarrow{\omega} E$ has "highest possible rank" — in each degree, it is either injective or surjective. It follows that

$$h(P) = 8+\binom{8}{1}+(\binom{8}{2}-1)+(\binom{8}{3}-\binom{8}{1})+(\binom{8}{4}-\binom{8}{2})+(\binom{8}{4}-\binom{8}{6})+(\binom{8}{5}-\binom{8}{7})+(\binom{8}{6}-1)+8 = 252 < 256 = 2^8.$$

□

*Remark* 5.2. Under BGG, the corresponding dg-$S_n$-module $F$ satisfies $H^0(F) = k = H^3(F)$ and $H^j(F) \neq 0$ for all other $j$. It follows that $F$ cannot be homotopy equivalent to a commutative dg-algebra and thus it cannot be of the form $C^*(X_T)$ for a space $X$ with a free $T_n$-action. That is, we have *not* given a counter example to the Toral Rank Conjecture.

## 6. $p$-torus actions

Fix a prime $p$ and let $V = (\mathbb{Z}/p)^{\times n}$, a elementary abelian $p$-group of rank $n$. Assume $X$ a compact $V$-CW-complex such that $V$ acts freely on $X$. Then there is a finite free chain complex $C_*^V(X, \mathbb{F}_p)$ of $\mathbb{F}_p[V]$-modules whose homology is $H_*(X, \mathbb{F}_p)$.

**Conjecture 6.1.** *(Carlsson) If the action $V$ on $X$ is free then $h(X, \mathbb{F}_p) \geq 2^n$, where $h(X, \mathbb{F}_p) = \sum_i \dim_{\mathbb{F}_p} H_i(X, \mathbb{F}_p)$.*

"Algebraic analogue" of this conjecture:

**Conjecture 6.2.** *Let $F$ be any finite free complex of $\mathbb{F}_p[V]$-modules. Then $h(F, \mathbb{F}_p) \geq 2^n$.*

**Theorem 6.3.** *(Iyengar-W) The algebraic conjecture is false for $p \geq 3$ and $n \geq 8$.*

*Proof.* For simplicity, assume $n = 8$. Set

$$R = \mathbb{F}_p[V] \cong \mathbb{F}_p[y_1, \ldots, y_8]/(y_1^p, \ldots, y_8^p).$$

The only properties used are that $R$ is a complete intersection of codimension 8. Let $K = \mathrm{Kos}_R(y_1, \ldots, y_8)$. Then $K$ is a dg-$R$-algebra and $H_*(K) \cong \Lambda_k^*(e_1, \ldots, e_8)$ with $\deg(e_i) = 1$ (using homological indexing now). Let $z = K_2$ be a cycle representing $\omega = e_1e_2 + e_3e_4 + e_5e_6 + e_7e_8$, and set $F = \mathrm{cone}(K(-2) \xrightarrow{z} K)$. Then by considering long exact sequences in homology we get

$$h(F) = h(\mathrm{cone}(\Lambda_k^*(e_1, \ldots, e_8) \xrightarrow{\omega} \Lambda_k^*(e_1, \ldots, e_8))) = 252 < 256.$$

□

*Remark* 6.4. A very similar construction gives the counter-example to the (original) Betti-degree conjecture mentioned by Peeva.

**Theorem 6.5** (Rüping-Stephan)**.** *The example above does not come from a space with a $V$-action.*