RATIONAL HOMOTOPY, SMALL COCHAIN MODELS AND THE TORAL RANK CONJECTURE

BERNHARD HANKE

ABSTRACT. We develop the part of rational homotopy theory due to Sullivan which is required to compute the (stable) free rank of symmetry of products of spheres.

1. OVERVIEW

Let $G = (S^1)^r$ and let X be a finite free G-CW complex. Halperin's toral rank conjecture predicts $\dim H^*(X;\mathbb{Q}) \geq 2^r$. One appraoch to this question is as follows: Since G acts freely, we get $X_G = EG \times_G X \simeq X/G$, which is a finite CW complex. In particular, $\dim H^*(X_G;\mathbb{Q}) < \infty$. The cohomology $H^*(X_G;\mathbb{Q})$ can be studied by means of the Leray-Serre spectral sequence for the fibration $X \hookrightarrow X_G \to BG$. We have $E_2 \cong H^*(X;\mathbb{Q}) \otimes \mathbb{Q}[t_1, \ldots, t_r]$ and $\dim E_\infty^{*,*} < \infty$, which might imply the predicted lower bound for $H^*(X;\mathbb{Q})$. However, in general one does not have enough control of the differentials in the spectral sequence in order resolve the Halperin conjecture in this way. So we need a more precise understanding how $H^*(X_G;\mathbb{Q})$ and $H^*(X;\mathbb{Q})$ are related.

For this aim let us rethink the following basic problem in algebraic topology:

Given a topological space X, compute its cohomology ring $H^*(X;\mathbb{Q})$.

If X is a CW complex, then the additive, but not the multiplicative structure, of $H^*(X;\mathbb{Q})$ can be computed from the cellular cochain complex of X. In order to compute the multiplicative structure as well, we apply a different approach which closely reflects the homotopy type of X.

If π is a group π and $k \geq 1$, let $K(\pi, k)$ denote an Eilenberg-MacLane space of type (π, k), i.e., $K(\pi, k)$ is a path connected CW-complex with $\pi_i(K(\pi, k)) = 0$ for $i \neq k$ and $\pi_k(K(\pi, k)) \cong \pi$. The space $K(\pi, k)$ is unique up to homotopy equivalence. For consecutive k, these spaces are related by a path loop fibration with contractible total space

$$K(\pi, k) = \Omega K(\pi, k+1) \to PK(\pi, k+1) \to K(\pi, k+1). \tag{1.1}$$

Assume that X is a *simple* topological space, that is, X is path connected, $\pi_1(X)$ is abelian and $\pi_1(X)$ acts trivially on the higher homotopy groups $\pi_k(X)$ for $k \geq 2$. The homotopy type of X can then be described by its *Postnikov tower* $(X_k, p_k, \phi_k)_{k \geq 0}$, that is, $X_0 = *$, $p_k \colon X_k \to X_{k-1}$, $k \geq 1$, and $\phi_k \colon X \to X_k$, $k \geq 0$, are continuous maps such that

(i) each ϕ_k is a k-equivalence, i.e., the induced maps $\pi_i(X) \to \pi_i(X_k)$ are bijections for $0 \leq i \leq k$ and a surjection for $i = k+1$,

(ii) $p_k \circ \phi_k = p_{k-1}$ for $k \geq 1$,

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(iii) each p_k fits into a pull back of fibrations

$$\begin{array}{ccc} X_k & \longrightarrow & PK(\pi_k(X), k+1) \\ \Big\downarrow{\scriptstyle p_k} & & \Big\downarrow \\ X_{k-1} & \xrightarrow{f_k} & K(\pi_k(X), k+1) \end{array}$$

In other words, $p_k\colon X_k \to X_{k-1}$ is a fibration with fibre $K(\pi_k(x), k)$ and classified by f_k. The space X_k can be constructed, up to homotopy equivalence, by attaching cells of dimension $\geq k+2$ to X in order to kill $\pi_i(X)$, $i \geq k+1$.

Let us now assume that $\pi_*(X)$ is finitely generated in each degree. By a theorem of Serre, this is equivalent to $H_*(X;\mathbb{Z})$ being finitely generated in each degree, compare [9, Thm. 5.7].

The rational cohomology rings of $K(\pi, k)$ with finitely generated abelian π were computed by Cartan and Serre. If V is a rational vector space and $k \geq 0$, we denote by $V^{(k)}$ the graded vector space V concentrated in degree k.

Proposition 1.2. *Let π be a finitely generated abelian group. Then, for each $k \geq 1$, there exists an isomorphism of $\mathbb{Q}$-algebras*

$$H^*(K(\pi, k);\mathbb{Q}) \cong \Lambda^*(\mathrm{Hom}(\pi, \mathbb{Q})^{(k)}).$$

In degree k it restricts to the identity $H^*(K(\pi, k);\mathbb{Q}) = \mathrm{Hom}(\pi_k(K(\pi, k), \mathbb{Q})) = \mathrm{Hom}(\pi, \mathbb{Q})$.

Proof. Write $\pi \cong T \oplus \mathbb{Z}^r$ for some $r \geq 0$ where T is a finitely generated torsion abelian group. We have

$$\tilde{H}^*(K(T, k);\mathbb{Q}) = 0, \qquad H^*(K(\mathbb{Z}, k);\mathbb{Q}) = \Lambda^*(\mathbb{Q}^{(k)}).$$

Both assertions are clear for $k = 1$ and for higher k follow by analysing the Leray-Serre spectral sequence, including its multiplicative properties, for the path loop fibration (1.1) for $\pi = T$ and $\pi = \mathbb{Z}$.

From this the assertion of Proposition 1.2 follows from the Künneth theorem. □

We can now try to compute the cohomology rings $H^*(X;\mathbb{Q})$ inductively along a Postnikov decomposition of X. For this aim, it remains to resolve the following problem. Let π be a finitely generated abelian group, let $k \geq 1$ and let $p\colon E \to B$ be a fibration fitting into a pull back diagram

(1.3)
$$\begin{array}{ccc} E & \longrightarrow & PK(\pi, k+1) \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow \\ B & \xrightarrow{f} & K(\pi, k+1) \end{array}$$

Problem 1.4. Compute the cohomology ring $H^*(E;\mathbb{Q})$ in terms of $H^*(B;\mathbb{Q})$, $H^*(K(\pi, k);\mathbb{Q}) = \Lambda^*(\mathrm{Hom}(\pi, \mathbb{Q})^{(k)})$ and the map f.

We will present an efficient solution of this problem going back to Dennis Sullivan [13] and use this to verify the Halperin conjecture if X is a product of spheres.

2. Sullivan-de Rham theorem

Recall that given a smooth manifold M, the real cohomology ring $H^*(M;\mathbb{R})$ can be computed by means of the cochain complex $\Omega^*(M)$ of smooth differential forms on M. The ring structure on

$H^*(M;\mathbb{R})$ is induced by the wedge product of differential forms which makes $\Omega^*(M)$ a real differential graded commutative algebra (DGCA). Dennis Sullivan in [13] generalized this construction to arbitrary topological spaces.

If V is a graded vector space we denote by $\Lambda^*(V)$ the free rational GCA generated by V. Consider the free rational DGCA

$$\Lambda^*(t_0,\dots,t_n,dt_0,\dots,dt_n) := \Lambda^*(\mathrm{Span}(t_0,\dots,t_n,dt_0,\dots,dt_n))$$

with generators $t_0,\dots,t_n$ in degree 0 and $dt_1,\dots,dt_n$ in degree 1 and coboundary given by $t_i \mapsto dt_i$, $dt_i \mapsto 0$. We obtain the DGCA

$$T_n^* := \Lambda^*(t_0,\dots,t_n,dt_0,\dots,dt_n)/(t_0+\dots+t_n-1, dt_0+\dots+dt_n)$$

which we regard as the algebra of rational polynomial forms on the n-simplex

$$\Delta^n := \{(t_0,\dots,t_n)\in\mathbb{R}^{n+1} \mid 0\le t_i\le 1, t_0+\dots+t_n=1\}.$$

The inclusion of the i-th face into Δ^n and the i-th collapse onto Δ^n, $0\le i\le n$, are given by

$$\begin{aligned}
&\Delta^{n-1}\to\Delta^n, \quad (t_0,\dots,t_{n-1})\mapsto(t_0,\dots,t_{i-1},0,t_{i+1},\dots,t_{n-1}),\\
&\Delta^{n+1}\to\Delta^n, \quad (t_0,\dots,t_{n+1})\mapsto(t_0,\dots,t_{i-1},t_i+t_{i+1},t_{i+2},\dots,t_{n+1})\,.
\end{aligned}$$

Via pullback of forms, these maps induce DGCA maps $\partial_i : T_n^* \to T_{n-1}^*$ and $s_i : T_n^* \to T_{n+1}^*$ that satisfy the simplicial identities. In other words, $T^* := (T_n^*)_{n\in\mathbb{N}}$ is a simplicial rational DGCA.

Definition 2.1. Let X be a topological space and let $\mathsf{Sing}(X)$,

$$(\mathsf{Sing}(X))_n = \mathrm{Mor}_{\mathsf{Top}}(\Delta^n, X),$$

be the simplicial set of singular simplices in X. The rational GCDA

$$\mathcal{A}^*(X) := \mathrm{Mor}_{\mathsf{SimplSet}}(\mathsf{Sing}(X), T^*)$$

is called the *Sullivan-de Rham cochain algebra* of X.

We think of $\mathcal{A}^k(X)$ as compatible polynomial k-forms with rational coefficients on the simplices of a triangulation of X.

Let $\xi\in\mathcal{A}^k(X)$ and let $\sigma\colon\Delta^n\to X$ be a singular simplex. Then $\xi(\sigma)\in T_n^k$ is a rational polynomial k-form ω on the geometric n-simplex $\Delta^n\subset\mathbb{R}^{n+1}$ and we set

$$\Psi_\xi(\sigma) := \int_{\Delta^n}\omega \in \mathbb{Q}.$$

This is zero for $k\neq n$. We can thus regard $\Psi_\xi\in C^k_{\mathsf{sing}}(X;\mathbb{Q})$ and hence obtain a $\mathbb{Q}$-linear map $\Psi^k\colon\mathcal{A}^k(X)\to C^k_{\mathsf{sing}}(X;\mathbb{Q})$, $\xi\mapsto\Psi_\xi$. Stokes' theorem implies that $\Psi^*\colon\mathcal{A}^*(X)\to C^*_{\mathsf{sing}}(X;\mathbb{Q})$ is a cochain map.

Theorem 2.2 (Sullivan-de Rham comparison theorem)**.** *The map Ψ^* induces a multiplicative isomorphism*

$$H^*(\mathcal{A}^*(X)) \cong H^*(C^*_{\mathsf{sing}}(X;\mathbb{Q})) = H^*_{\mathsf{sing}}(X;\mathbb{Q}).$$

For a proof see [2, Sections 2 and 3] and [6, Section 9].

3. HIRSCH LEMMA

We will now come back to Problem 1.4. Let

$$f^\sharp\colon H^{k+1}(K(\pi,k+1);\mathbb{Q}) = \mathrm{Hom}(\pi,\mathbb{Q}) \to \mathcal{A}^{k+1}(B)$$

be cochain representative of f^* in degree $k+1$. It is uniquely determined up to cochain homotopy, that is, a linear map $\mathrm{Hom}(\pi,\mathbb{Q}) \to \mathcal{A}^{k+1}(B)$ with values in the coboundaries of $\mathcal{A}^{k+1}(B)$.

Definition 3.1. Let (B^*, d_B) be a rational GCDA, let V be a vector space which is concentrated in degree $k \geq 1$ and let $\tau\colon V \to B^{k+1}$ be a $\mathbb{Q}$-linear map with $d_B \circ \tau = 0$.

The *free Hirsch extension*[1] $(B^* \otimes_\tau \Lambda^*(V), d)$ is the rational DGCA equal to $B^* \otimes \Lambda^*(V)$ as a GCA and equipped with the differential d which acts as a derivation and satisfies

$$d(b\otimes 1) = d_B(b)\otimes 1, \qquad d(1\otimes v) = \tau(v)\otimes 1.$$

If two maps $\tau,\tau'\colon V \to B^{k+1}$ with $d_B\circ\tau = 0 = d_B\circ\tau'$ induce the same maps $V \to H^{k+1}(B^*)$, then there exists a DGCA isomorphism $B^*\otimes_\tau \Lambda^*(V) \cong B^* \otimes_{\tau'} \Lambda^*(V)$ which restricts to the identity on $B^*\otimes 1$. In particular, if τ induces the zero map $V \to H^{k+1}(B^*)$, then $B^*\otimes_\tau \Lambda^*(V)$ is isomorphic to $(B^*, d_B)\otimes \Lambda^*(V)$ with the zero differential on $1\otimes \Lambda^*(V)$.

The following result gives a satisfactory answer to Problem 1.4.

Theorem 3.2 (Hirsch lemma)**.** *There is a DGCA map*

$$\Gamma_f\colon \mathcal{A}^*(B)\otimes_{f^\sharp} \Lambda^*(\mathrm{Hom}(\pi,\mathbb{Q})^{(k)}) \to \mathcal{A}^*(E)$$

which induces an isomorphism in cohomology.

Proof. The trickiest part is the construction of Γ_f. Diagram 1.3 induces a pull-back square of simplicial Kan fibrations

$$\begin{array}{ccc} \mathsf{Sing}(E) & \longrightarrow & \mathsf{Sing}(PK(\pi,k+1)) \\ \big\downarrow p & & \big\downarrow \\ \mathsf{Sing}(B) & \xrightarrow{f} & \mathsf{Sing}(K(\pi,k+1)) \end{array} \tag{3.3}$$

Given a cochain complex V^* we define the simplicial abelian group

$$\|V^*\| := \mathrm{Mor}_{\mathsf{CochainCompl}}(V^*, T^*),$$

called the *simplicial realisation* of V^*. For each topological space X, we obtain a canonical bijection

$$\mathrm{Mor}_{\mathsf{SimplSet}}(\mathsf{Sing}(X), \|V^*\|) \approx \mathrm{Mor}_{\mathsf{CochainCompl}}(V^*, \mathcal{A}^*(X)) \tag{3.4}$$

which is natural in X and V^* and preserves homotopies. This will help us to construct the map Γ_f.

The right hand vertical map in (3.3) is a simplicial analogue of the path look fibration $PK(\pi,k+1) \to K(\pi,k+1)$. After replacing π by $\pi\otimes\mathbb{Q}$ we will now construct an especially convenient model for this fibration.

Let $V^* := \mathrm{Hom}(\pi,\mathbb{Q})^{(k)}$, let ΣV^* with $(\Sigma V)^i = V^{i-1}$ be the suspension of V^* and let $\mathrm{cone}\, V^* = (V^*\oplus \Sigma V^*, d(v,w) := (0,v))$ be the cone of V^*, which has vanishing cohomology. We obtain a short exact sequence of cochain complexes

$$0 \longrightarrow \Sigma V^* \overset{v\mapsto(0,v)}{\longrightarrow} \mathrm{cone}\, V^* \overset{(v,w)\mapsto v}{\longrightarrow} V^* \longrightarrow 0.$$

[1] Named after Guy Hirsch (1915–1993) who also appears in the Leray-Hirsch theorem

Passing to simplicial realisations, we obtain a Kan fibration of simplicial groups

$$\|V^*\| \hookrightarrow \|\operatorname{cone} V^*\| \to \|\Sigma V^*\|$$

which is a model for the simplicial path-loop fibration

$$\hat{K}(G,k) \to P\hat{K}(G,k+1) \to \hat{K}(G,k+1)$$

where $G = \operatorname{Hom}(\operatorname{Hom}(\pi,\mathbb{Q}),\mathbb{Q}) = \pi \otimes \mathbb{Q}$ and where $\hat{K}$ denotes simplicial Eilenberg-MacLane complexes. Here we use the fact that the map of simplicial abelian groups $\|\operatorname{cone} V^*\| \to \|\Sigma V^*\|$ is surjective, hence a principal Kan fibration by [11, Lemma 18.2] whose kernel can be identified with the simplicial abelian group $\|V^*\|$ by an explicit calculation. Furthermore, $\|V^*\| = \hat{K}(G,k)$ since $(\pi \otimes \mathbb{Q}) \otimes T^*$ is a cohomology theory with coefficients $\pi \otimes \mathbb{Q}$ in the sense of Cartan [3] and by the inductive argument in the proof of [3, Théorème 1]. A similar argument shows that $\|\Sigma V^*\| = \hat{K}(G,k+1)$.

The canonical inclusion $\pi \to \pi \otimes \mathbb{Q}$ combined with diagram (3.3) induces a commutative diagram

$$\begin{array}{ccc} \mathsf{Sing}(E) & \longrightarrow & \|\operatorname{cone} V^*\| \\ \downarrow{\scriptstyle p} & & \downarrow \\ \mathsf{Sing}(B) & \xrightarrow{f} & \|\Sigma V^*\| \end{array}$$

and applying (3.4) to this diagram, we obtain the commutative diagram

(3.5)
$$\begin{array}{ccc} \mathcal{A}^*(E) & \longleftarrow & \operatorname{cone} V^* \\ \uparrow{\scriptstyle p^\sharp} & & \uparrow \\ \mathcal{A}^*(B) & \xleftarrow{f^\sharp} & \Sigma V^* \end{array}$$

where addig to $f^\sharp$ a linear map with values in the coboundaries $\mathcal{A}^{k+1}(B)$ amounts to replacing f by a homotopic map. In particular, we obtain an induced grading preserving linear map

$$\phi\colon\ \operatorname{Hom}(\pi,\mathbb{Q})^{(k)} \overset{v\mapsto(v,0)}{\longrightarrow} (\operatorname{cone} V)^k \longrightarrow \mathcal{A}^k(E)$$

which satisfies (since the upper horizontal map in (3.5) is a cochain map)

$$d_{\mathcal{A}^*(E)} \circ \phi = p^\sharp \circ f^\sharp.$$

Now the maps $p^\sharp$ and ϕ induce the required DGCA map

$$\Gamma_f\colon \mathcal{A}^*(B) \otimes_{f^\sharp} \Lambda^*(\operatorname{Hom}(\pi,\mathbb{Q})^{(k)}) \to \mathcal{A}^*(E).$$

It remains to show that it induces an isomorphism in cohomology. We may assume without loss of generality that B is a CW complex.

In a first step, we show that Γ_f induces an isomorphism in cohomology if f is constant. In this case, we have commutative diagram

$$\begin{array}{ccc} K(\pi,k) & \xrightarrow{\quad = \quad} & K(\pi,k) \\ \downarrow & & \downarrow \\ E = B \times K(\pi,k) & \longrightarrow & PK(\pi,k+1) \\ \downarrow{\scriptstyle p} & & \downarrow \\ B & \xrightarrow{\quad \text{const.} \quad} & K(\pi,k+1) \end{array}$$

and $\phi\colon \operatorname{Hom}(\pi,\mathbb{Q}) \to \mathcal{A}^k(E)$ factors as $\operatorname{Hom}(\pi,\mathbb{Q}) \to \mathcal{A}^k(K(\pi,k)) \to \mathcal{A}^k(E)$, where the first map induces the isomorphism $\operatorname{Hom}(\pi,\mathbb{Q}) \cong H^k(K(\pi,k),\mathbb{Q})$ and the second map is induced by the projection $E \to K(\pi,k)$. Hence the claim follows from the Künneth formula and Proposition 1.2.

In a next step, we show that Γ_f induces an isomorphism in cohomology if $f \simeq \mathrm{const}$. In order to prove this, let $H\colon B \times [0,1] \to K(\pi,k+1)$ be a homotopy from f to $\mathrm{const.}$ and notice that the restrictions of $\mathcal{A}^*(B \times [0,1]) \otimes_{H^\sharp} \Lambda^*(\operatorname{Hom}(\pi,\mathbb{Q})^{(k)})$ to $\mathcal{A}^*(B) \otimes_{f^\sharp} \Lambda^*(\operatorname{Hom}(\pi,\mathbb{Q})^{(k)})$ and $\mathcal{A}^*(B) \otimes_0 \Lambda^*(\operatorname{Hom}(\pi,\mathbb{Q})^{(k)})$ induce isomorphisms in cohomology. Thus we are reduced to the case $f = \mathrm{const.}$.

We now show that Γ_f induces an isomorphism in cohomology by induction on $\dim B$. If $\dim B = 0$, then $f \simeq \mathrm{const.}$ and hence this case is clear. In the induction, step we write

$$B^k = B^{k-1} \bigcup_{\alpha_i} \coprod_{i \in I} D^k$$

with attaching maps $\alpha_i\colon \partial D^k \to B^{k-1}$. Let $A_i\colon D^k \to B^k$ be the induced characteristic maps.

Then $\Gamma_{\tilde{f}}$ induce isomorphisms for

- ▷ $\tilde{f} = f|_{B^{k-1}}\colon B^{k-1} \to K(\pi,k+1)$, by the induction hypothesis,
- ▷ $\tilde{f} = f \circ A_i\colon D^k \to K(\pi,k+1)$ which is homotopic to a constant map,
- ▷ $\tilde{f} = f \circ \alpha_i\colon S^{k-1} \to K(\pi,k+1)$ since $\dim S^{k-1} = k-1$.

Hence, Γ_f is an isomorphism by a Mayer-Vietoris argument and the five lemma, keeping in mind that our construction of Γ_f is natural with respect to precomposing $f\colon B \to K(\pi_k(X),k+1)$ with maps $B' \to B$. This finishes the proof of Theorem 3.2. $\square$

Our exposition is inspired by [6]. However, the Hirsch lemma in [6, Section 16] is proven in a different and, in our opinion, less conceptual way.

4. Minimal models via Postnikov decompositions

Assume that X is a path connected simple topological space such that $\pi_*(X)$ is finitely generated in each degree. Using the Postnikov decomposition$(X_k, p_k, \phi_k)_{k\geq 0}$ of X (see Section 1) and the Hirsch lemma, we will replace the Sullivan-de Rham algebra $\mathcal{A}^*(X)$ by a smaller DGCA which closely reflects the homotopy type of X.

For $k \geq 1$ we will construct a finitely generated free rational DGCA $\mathcal{M}_k^*$ together with a DGCA map

$$\psi_k\colon \mathcal{M}_k^* \to \mathcal{A}^*(X_k)$$

with $\mathcal{M}_0^* := \mathbb{Q}$ and the following properties for $k \geq 1$:

▷ ψ_k induces an isomorphism in rational cohomology,
▷ $\mathcal{M}_k^* = \mathcal{M}_{k-1}^* \otimes_{\tau_k} \Lambda^*(\mathrm{Hom}(\pi_k(X), \mathbb{Q})^{(k)})$, where the twisting map τ_k is induced by f_k,
▷ the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}_{k-1}^* \otimes_{\tau_k} \Lambda^*(\mathrm{Hom}(\pi_k(X), \mathbb{Q})^k) & \xrightarrow{\psi_k} & \mathcal{A}^*(X_k) \\ \uparrow {\scriptstyle \zeta \mapsto \zeta \otimes 1} & & \uparrow {\scriptstyle p_k^*} \\ \mathcal{M}_{k-1}^* & \xrightarrow{\psi_{k-1}} & \mathcal{A}^*(X_{k-1}) \end{array}$$

Assume that $\psi_{k-1} \colon \mathcal{M}_{k-1}^* \to \mathcal{A}^*(X_{k-1})$ has been constructed. The Hirsch lemma gives a DGCA map

$$\Gamma_{f_k} = \mathcal{A}^*(X_{k-1}) \otimes_{f_k^\sharp} \Lambda^*(\mathrm{Hom}(\pi_k(X), \mathbb{Q})^{(k)}) \to \mathcal{A}^*(X_k)$$

which induces an isomorphism in cohomology.

Since $\psi_{k-1} \colon \mathcal{M}_{k-1}^* \to \mathcal{A}^*(X_{k-1})$ induces an isomorphism in cohomology, there is (possibly after replacing $f^\sharp$ by a cochain homotopic map) a $\mathbb{Q}$-linear map

$$\tau_k \colon \mathrm{Hom}(\pi_k(X), \mathbb{Q}) \to \mathcal{M}_{k-1}^{k+1}$$

whose image lies in the cocycles of $\mathcal{M}_{k-1}^{k+1}$ and such that $\psi_{k-1} \circ \tau_k = f_k^\sharp$. We now set

$$\mathcal{M}_k^* := \mathcal{M}_{k-1}^* \otimes_{\tau_k} \Lambda^*(\mathrm{Hom}(\pi_k(X), \mathbb{Q})^{(k)})$$

and $\psi_k = \Gamma_{f_k} \circ (\psi_{k-1} \otimes \mathrm{id}) \colon \mathcal{M}_{k-1}^* \otimes_{\tau_k} \Lambda^*(\mathrm{Hom}(\pi_k(X), \mathbb{Q})^{(k)}) \to \mathcal{A}^*(X_k)$. A spectral sequence argument shows that $\psi_{k-1} \otimes \mathrm{id}$ induces an isomorphism in cohomology and hence the same is true for ψ_k.

We finally set

$$\mathcal{M}^*(X) := \mathrm{colim}_k\, \mathcal{M}_k^*, \qquad \psi := \mathrm{colim}_k\, \psi_k \colon \mathcal{M}^*(X) \to \mathcal{A}^*(X).$$

By construction, the map ψ induces an isomorphism in cohomology.

Definition 4.1. We call $\psi \colon \mathcal{M}^*(X) \to \mathcal{A}^*(X)$ the *Sullivan minimal model* of $\mathcal{A}^*(X)$.

Remark 4.2. ▷ The Sullivan model of X determines the rational homotopy groups $\pi_*(X) \otimes \mathbb{Q}$.
▷ The Sullivan minimal model of X can be characterised in an axiomatic way and is determined up to isomorphism by $\mathcal{A}^*(X)$ alone. In particular, $\mathcal{A}^*(X)$ determines $\pi_*(X) \otimes \mathbb{Q}$.

Example 4.3. Applying the procedure from the previous section to the n-sphere S^n and (only) using the known cohomology computation for $H^*(S^n; \mathbb{Q})$, we obtain

(a) $\mathcal{M}^*(S^n) \cong \mathbb{Q}[\tau] \otimes \Lambda^*(\eta)$ where $\deg(\tau) = n$, $\deg(\eta) = 2n-1$, $d_{\mathcal{M}}(\eta) = \tau^2$, for even n,
(b) $\mathcal{M}^*(S^n) \cong \Lambda^*(\sigma)$ where $\deg(\sigma) = n$, for odd n.

For $X = BS^1$ we have $\mathcal{M}^*(X) = \mathbb{Q}[t]$ with $\deg(t) = 2$, hence $\mathcal{M}^*(B((S^1)^r)) = \mathbb{Q}[t_1, \ldots, t_r]$.

5. Small cochain models for torus actions

Let $G = (S^1)^r$ and let X be a finite connected G-CW complex which is a simple topological space. Let $X \hookrightarrow X_G \to BG$ be the Borel construction. Note that X_G is simple and $\pi_*(X_G) \otimes \mathbb{Q}$ is finitely generated in each degree.

By attaching G-cells to X for killing homotopy groups of X, we obtain the Postnikov decomposition of X_G *relative to* BG, leading to a commutative diagram

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
X_2 & \longrightarrow & (X_G)_2 & \longrightarrow & BG \\
\downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle P_2} & & \| \\
X_1 & \longrightarrow & (X_G)_1 & \xrightarrow{P_1} & BG \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle P_1} & & \| \\
X_0 = * & \longrightarrow & (X_G)_0 = BG & = & BG
\end{array}$$

where for all $k \geq 1$, the complexes X_k and $(X_G)_k$ are k-th stages of the Postnikov decompositions of X and X_G, each row is a fibration with fibre X_k and the vertical maps p_k and P_k are fibrations whose fibres are Eilenberg-MacLane complexes of type $(\pi_k(X), k)$.

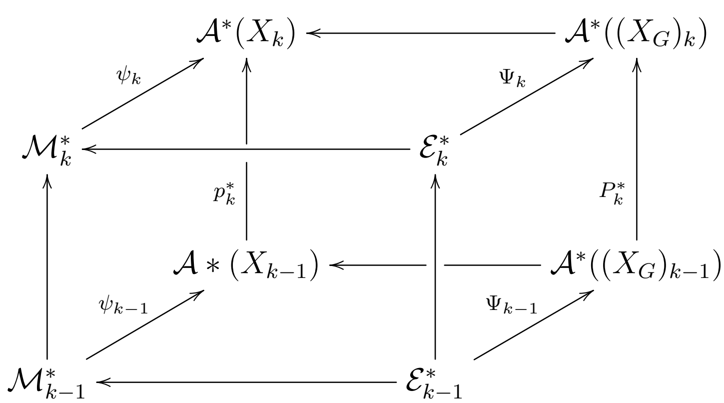
Carrying out the previous construction in this relative situation and using Example 4.3 we obtain a commutative diagram of rational DGCAs

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{M}_2^* & \longleftarrow & \mathcal{E}_2^* & \longleftarrow & \mathbb{Q}[t_1, \ldots, t_r] \\
{\scriptstyle p_2^*}\uparrow & & {\scriptstyle P_2^*}\uparrow & & \| \\
\mathcal{M}_1^*. & \longleftarrow & \mathcal{E}_1^* & \xleftarrow[P_1^*]{} & \mathbb{Q}[t_1, \ldots, t_r] \\
{\scriptstyle p_1^*}\uparrow & & {\scriptstyle P_1^*}\uparrow & & \| \\
\mathcal{M}_0^* = \mathbb{Q} & \longleftarrow & \mathcal{E}_0^* & = & \mathbb{Q}[t_1, \ldots, t_r]
\end{array}$$

Furthermore, we have

$$\mathcal{M}_k^* = \mathcal{M}_{k-1}^* \otimes_{\tau_k} \Lambda^*(\mathrm{Hom}(\pi_k(X), \mathbb{Q})^{(k)}), \qquad \mathcal{E}_k^* = E_{k-1}^* \otimes_{\tau_k} \Lambda^*(\mathrm{Hom}(\pi_k(X), \mathbb{Q})^{(k)})$$

where the twisting map τ_k are induced by the map $X_k \to (X_G)_k \to K(\pi_k(X), k{+}1)$ classifying the fibrations p_k and P_k. We also have DCGA maps $\psi_k : \mathcal{M}_k \to \mathcal{A}^*(X_k)$ and $\Psi_k : \mathcal{E}_k \to \mathcal{A}^*((X_G)_k)$ which induce isomorphisms in cohomology and fit into commutative diagrams



Setting $\mathcal{M}^* := \operatorname{colim}_k \mathcal{M}_k^*$ and $\mathcal{E}^* := \operatorname{colim}_k \mathcal{E}_k^*$, we arrive at the following theorem:

Theorem 5.1. *There are rational DGCAs $(\mathcal{E}^*, d_E)$ and $(\mathcal{M}^*, d_M)$ with the following properties:*

1) $\mathcal{E}^* = \mathbb{Q}[t_1, \ldots, t_r] \otimes \mathcal{M}^*$ as graded algebras where $\deg(t_i) = 2$,
2) d_E is zero on $\mathbb{Q}[t_1, \ldots, t_r]$ and the map $\mathcal{E}^* \to \mathcal{M}^*$, $t_i \mapsto 0$, is a cochain map,
3) $\mathcal{M}^*$ is free as a graded algebra. As generators in degree $k \geq 1$ we can take the elements of a basis of the $\mathbb{Q}$-module $\operatorname{Hom}(\pi_k(X), \mathbb{Q})$.,
4) there is a commutative diagram

$$\begin{array}{ccc} \mathcal{M}^* & \longrightarrow & \mathcal{A}^*(X) \\ \uparrow{\scriptstyle t_i \mapsto 0} & & \uparrow{\scriptstyle p^*} \\ \mathcal{E}^* & \longrightarrow & \mathcal{A}^*(X_G) \end{array}$$

and the horizontal maps induce isomorphisms in cohomology.

Example 5.2. Let $X = S^{2n-1}$ with the standard free S^1-action, $n \geq 1$. Then $\mathcal{M}^* = \Lambda^*(\sigma)$, $\deg(\sigma) = 2n - 1$ and $\mathcal{E}^* = \mathbb{Q}[t] \otimes \Lambda(\sigma)$, $d_E(\sigma) = t^n$.

6. The toral rank of products of spheres

We apply Theorem 5.1 to verify the toral rank conjecture for products of spheres.

Theorem 6.1. *Let $r \geq 1$, let $n_1, \ldots, n_k \geq 1$, let $G = (S^1)^r$ and let X be a finite free G-CW complex homotopy equivalent to $S^{n_1} \times \cdots \times S^{n_k}$. Then $r \leq \sharp\{n_j \text{ odd}\}$.*

We denote by $X_G = EG \times_G X$ the Borel construction of X. Since G acts freely, we have $X_G \simeq X/G$. In particular $H^*(X_G; \mathbb{Q})$ is a finite dimensional vector space. Let k_o denote the number of odd n_i and k_e denote the number of even n_i.

Using Theorem 5.1 and Example 4.3 we obtain the following.

Proposition 6.2. *There are finitely generated free DGCAs $(\mathcal{E}^*, d_E)$ and $(\mathcal{M}^*, d_M)$ over $\mathbb{Q}$ such that*

- ▷ *$\mathcal{M}^* = \Lambda^*(\tau_1, \ldots \tau_{k_e}, \eta_1, \ldots, \eta_{k_e}, \sigma_1, \ldots, \sigma_{k_o})$, where the degrees of τ_j correspond to the even n_j, the degrees of σ_j correspond to the odd n_j, $\deg(\eta_j) = 2\deg(\tau_j) - 1$, and $d_M(\eta_i) = \tau_i^2$,*
- ▷ *$\mathcal{E}^* = \mathcal{M}^* \otimes \mathbb{Q}[t_1, \ldots, t_r]$ as graded commutative algebras where $\deg(t_i) = 2$,*
- ▷ *d_E is $\mathbb{Q}[t_1, \ldots, t_r]$-linear and the projection $\mathcal{E}^* \to \mathcal{M}^*$ given by evaluating $t_1, \ldots, t_r$ at 0 is a cochain map,*
- ▷ *$H^*(\mathcal{E}^*) \cong H^*(X_G; \mathbb{Q})$, in particular, the total dimension of $H^*(\mathcal{E}^*)$ is finite.*

We claim that $\mathcal{E}^*$ must have at east as many odd degree generators as even degree generators. Hence $k_e + r \leq k_e + k_o$ which implies Theorem 6.1.

Inspired by the construction of *pure towers* in [8], we deform d_E to another differential δ_E on $\mathcal{E}^*$ as follows: δ_E is a derivation that vanishes on $\mathbb{Q}[t_1, \ldots, t_r, \tau_1, \ldots, \tau_{k_e}]$ and satisfies

$$\delta_E(\sigma_j) = \pi(d_E(\sigma_j)), \qquad \delta_E(\eta_j) = \pi(d_E(\eta_j))\,.$$

where $\pi : \mathcal{E}^* \to \mathcal{E}^*$ is the projection onto $\mathbb{Q}[t_1, \ldots, t_r, \tau_1, \ldots, \tau_{k_e}]$ given by evaluating the odd degree generators η_j, σ_j at 0. It is easy to verify that $\delta_E^2 = 0$.

For $\ell \geq 0$ let $\Sigma^\ell \subset \mathcal{E}^*$ be the $\mathbb{Q}[t_1, \ldots, t_r]$-linear subspace generated by the monomials in $\mathcal{M}^*$ containing exactly ℓ of the odd degree generators σ_j, η_j. In particular, $\Sigma^\ell = 0$ for $\ell > k$ by the graded commutativity of the product. We set $\Sigma^+ := \bigoplus_{\ell \geq 1} \Sigma^\ell$. This is a nilpotent ideal in E^*.

Lemma 6.3. *For all $\ell \geq 1$, the differential δ_E maps Σ^ℓ to $\Sigma^{\ell-1}$. Furthermore, the image of $\delta_E - d_E$ is contained in Σ^+.*

Proof. The first assertion holds by the definition and derivation property of δ_E.

The second assertion holds for the generators σ_j and η_j, because $\mathrm{im}(\mathrm{id} - \pi) \subset \Sigma^+$, it holds for the generators t_i, because δ_E and d_E send these elements to zero and it holds for the generators τ_j, because each $d_E(\tau_j)$ is of odd degree and therefore contained in Σ^+. This implies the second assertion in general, since Σ^+ is an ideal in F^* and $\delta_E - d_E$ is a derivation. □

The elements t_i, $1 \leq i \leq r$, and τ_j, $1 \leq j \leq k_e$, represent cocycles in $(\mathcal{E}^*, \delta_E)$. Let $[t_i]$ and $[\tau_j]$ be the corresponding cohomology classes.

Proposition 6.4. *The classes $[t_i]$ and $[\tau_j]$ are nilpotent in $H^*(\mathcal{E}^*, \delta_E)$.*

Proof. We claim that each monomial in $t_1, \ldots, t_r$ of cohomological degree at least $\dim X \geq \dim X_G + 1$ represents the zero class in $H^*(\mathcal{E}^*)$. In particular, the classes $[t_i] \in H^*(\mathcal{E}, \delta_E)$ are nilpotent. Let m be such a monomial and write $m = d_E(c)$ for a cochain $c \in \mathcal{E}^*$.

By Lemma 6.3, we have $\delta_E(c) = m + \omega$ where $\omega \in \Sigma^+$. Let c_1 be the component of c in Σ^1. Lemma 6.3 and the fact that $m \in \Sigma^0$ imply the equation $\delta_E(c_1) = m$. This shows that m is a coboundary in $(\mathcal{E}^*, \delta_E)$.

The cochain algebra $(\mathcal{E}^*, \delta_E)$ has a decreasing filtration given by

$$\mathcal{F}^*_\gamma = \mathbb{Q}[t_1, \ldots, t_r]^{\geq \gamma} \otimes \mathcal{M}^*$$

where $\gamma \in \mathbb{N}$ denotes the cohomological degree. Our previous argument and the fact that each τ_j is a cocycle in $(\mathcal{E}^*, \delta_E)$ imply that each element in $\Sigma^0 \subset \mathcal{E}^*$ in filtration level at least $\dim X$ is a coboundary in $(\mathcal{E}^*, \delta_E)$.

Now pick a $j \in \{1, \ldots, k_e\}$. By Proposition 5.1, we have

$$d_E(\eta_j) = \tau_j^2 \mod \mathcal{F}^*_2 .$$

By the definition of δ_E, we have

$$\delta_E(\eta_j) = \pi(\tau_j^2) = \tau_j^2 \mod \mathcal{F}^*_2$$

since the map π preserves the ideal $(t_1, \ldots, t_r) = \mathcal{F}^*_2$. This implies that τ_j^2 is δ_E-cohomologous to a cocycle $c \in \mathcal{F}^*_2$. Hence $(\tau_j^2)^{\dim X}$ is δ_E-cohomologous to $c^{\dim X} \in \mathcal{F}^*_{2\dim X}$. We can split $c^{\dim X}$ into a sum $c_0 + c^+$ where $c_0 \in \Sigma^0 \cap \mathcal{F}^*_{2\dim X}$ and $c^+ \in \Sigma^+ \cap \mathcal{F}^*_{2\dim X}$. As noted earlier, c_0 is δ_E-cohomologous to zero. Because Σ^+ is nilpotent, the element c^+ is nilpotent.

We conclude that $\tau_j^{2\dim X}$ is δ_E-cohomologous to a nilpotent cocycle in $(\mathcal{E}^*, \delta_E)$. □

Together with Proposition 6.4, we see that the elements t_i, $1 \leq i \leq r$, and τ_j, $1 \leq j \leq k_e$, define nilpotent classes in $H^*(\mathcal{E}, \delta_E)$. This implies that $H^*(\mathcal{E}, \delta_E)$ is a finite dimensional $\mathbb{Q}$-vector space.

Consider the ideal

$$I = \big(\delta_E(\eta_1), \ldots, \delta_E(\eta_{k_e}), \delta_E(\sigma_1), \ldots, \delta_E(\sigma_{k_o})\big) \subset \mathbb{Q}[t_1, \ldots, t_r, \tau_1, \ldots, \tau_{k_e}]$$

contained in $\mathrm{im}(\delta_E)$ and obtain an inclusion

$$\mathbb{F}_p[t_1, \ldots, t_r, \tau_1, \ldots, \tau_{k_e}]/I \subset H^*(\mathcal{E}^*, \delta_E) .$$

Here we use the fact that the coboundaries in $(\mathcal{E}^*, \delta_E)$ are contained in the ideal $I \cdot \mathcal{E}^*$, whose intersection with $\mathbb{Q}[t_1, \ldots, t_r, \tau_1, \ldots, \tau_{k_e}]$ is equal to I. We conclude that $\mathbb{Q}[t_1, \ldots, t_r, \tau_1, \ldots, \tau_{k_e}]/I$ is a finite dimensional $\mathbb{Q}$-vector space.

Because I is generated by homogenous elements of positive degree, it does not contain a unit of $\mathbb{Q}[t_1, \dots, t_r, \tau_1, \dots, \tau_{k_e}]$ and hence there is a minimal prime ideal $\mathfrak{p} \subset \mathbb{Q}[t_1, \dots, t_r, \tau_1, \dots, \tau_{k_e}]$ containing I. The quotient $\mathbb{Q}[t_1, \dots, t_r, \tau_1, \dots, \tau_{k_e}]/\mathfrak{p}$ is both a finite dimensional $\mathbb{Q}$-vector space and an integral domain. Hence $\mathfrak{p} = (t_1, \dots, t_r, \tau_1, \dots, \tau_{k_e})$ and consequently $\mathrm{height}(\mathfrak{p}) = r + k_e$. By Krull's height theorem, see [5, Theorem 10.2], the number of generators of I must be at least $r + k_e$. From the definition of I we derive the inequality $k_e + k_o \geq r + k_e$. This implies $k_o \geq r$ and finishes the proof of Theorem 6.1.

Remark 6.5. Let $G = (S^1)^r$, let X is a free finite G-CW complex which is a simple topological space and assume that $\pi_*(X) \otimes \mathbb{Q}$ is zero in all but finitely many degrees. We obtain the *homotopy Euler characteristic*

$$\chi_\pi(X) := \sum_{k \geq 1} (-1)^k \dim(\pi_k(X) \otimes \mathbb{Q}).$$

It follows from [8, Theorem T] that $r \leq -\chi_\pi(X)$. This implies our Theorem 6.1 as a special case.

For further information about the relation of rational homotopy theory and torus actions we refer to [1, Chapters 2 and 4].

7. Cenkl-Porter theorem

We wish to prove a version of Theorem 6.1 for $G = (\mathbb{Z}/p)^r$. Since $\tilde{H}^*(BG; \mathbb{Q}) = 0$, we need to refine the previous constructions to subrings $R \subset \mathbb{Q}$ without inverting the prime p.

The Sullivan-deRham theorem does not generalize to integral coefficients in an obvious way since the integration map introduces denominators as in

$$\int_{[0,1]} t^{k-1} dt = \frac{1}{k}.$$

However, a closer look shows that the denominators are controlled by the *weights* of polynomial forms to be integrated. More precisely, defining the weight of a monomial $t_0^{\alpha_0} dt_0^{\varepsilon_0} \cdots t_n^{\alpha_n} dt_n^{\varepsilon_n}$, $\alpha_i \geq 0$, $0 \leq \varepsilon_i \leq 1$, as $\max_i\{\alpha_i + \varepsilon_i\}$, we get

$$\int_{[0,1]^k} \omega \in \mathbb{Q}_q$$

if ω is an k-form of weight at most q and $\mathbb{Q}_q \subset \mathbb{Q}$ is the smallest subring where all integers smaller than or equal to q are inverted.

Starting from this observation, Cenkl-Porter in [4] replace the simplicial DGCA T^* by a filtered simplicial DGCA $T^{*,*}$, where $(T^{*,q})_n$, $q \geq 0$, is the simplicial DGCA over $\mathbb{Q}_q$ consisting of polynomial forms with coefficients $\mathbb{Q}_q$ and weight at most q on a cubical decomposition of Δ^n. This leads to the *filtered Sulllivan-de Rham* cochain algebra $\mathcal{A}^{*,*}(X)$ with

$$\mathcal{A}^{*,q}(X) := \mathrm{Mor}_{\mathsf{SimplSet}}(\mathsf{Sing}(X), T^{*,q})$$

together with integration maps

$$\Psi^{*,q} \colon \mathcal{A}^{*,q}(X) \to C^*_{\mathsf{sing}}(X; \mathbb{Q}_q).$$

For the following result, see [4, Theorems 4.1 and 4.2].

Theorem 7.1. *For $q \geq 1$, the map $\Psi^{*,q}$ induces a linear isomorphism*

$$H^*(\mathcal{A}^{*,q}(X)) \cong H^*(C^*_{\mathsf{sing}}(X;\mathbb{Q}_q)) = H^*_{\mathsf{sing}}(X;\mathbb{Q}_q).$$

These maps are compatible with the multiplication maps $\mathcal{A}^{*,q_1}(X) \otimes \mathcal{A}^{*,q_1}(X) \to \mathcal{A}^{*,q_1+q_2}(X)$ and $C^*_{\mathsf{sing}}(X;\mathbb{Q}_{q_1}) \otimes C^*_{\mathsf{sing}}(X;\mathbb{Q}_{q_2}) \to C^*_{\mathsf{sing}}(X;\mathbb{Q}_{q_1+q_2})$ induced by muliplication of forms and the cup product of singular cochains.

Note in particular, that the Cenkl-Porter theorem gives a description of the $\mathbb{Z}$-module $H^*_{\mathsf{sing}}(X;\mathbb{Z})$ in terms of polynomial forms.

8. Tame Hirsch Lemma

Let p be a prime. By a computation due to Cartan and Serre, $H^*(K(\mathbb{Z},k);\mathbb{F}_p)$ is a DGCA over $\mathbb{F}_p$ in one generator of degree k and further generators of degrees at least $k + 2(p-1)$. This corresponds to the fact that the first reduced Steenrod power operation for the prime p raises degrees by $2(p-1)$. Hence, up to degree $k+2q-1$, we have $H^*(K(\mathbb{Z},k);\mathbb{Q}_q) \cong \Lambda^*(\mathrm{Hom}(\mathbb{Z},\mathbb{Q}_q)^{(k)})$, analogous to Proposition 1.2, whereas such an isomorphism does no longer hold in higher degrees.

This implies that with coefficients $\mathbb{Q}_q$ instead of $\mathbb{Q}$, the map Γ_f from the Hirsch lemma 3.2 can induce an isomorphism only up to degree $k(q)$ where $\lim_{q\to\infty} k(q) = \infty$. The precise formulation and the proof of such a "tame" Hirsch lemma can be found in [10].

9. The stable free rank of symmetry of products of spheres

Theorem 9.1. *Let $r \geq 1$, let $n_1, \ldots, n_k \geq 1$, let $G = (\mathbb{Z}/p)^r$ and let X be a finite free G-CW complex homotopy equivalent to $S^{n_1} \times \cdots \times S^{n_k}$. Then, assuming that p is sufficiently large with respect to $\dim X$, we obtain $r \leq \sharp\{n_j \,\mathrm{odd}\}$.*

Remark 9.2. It is shown in [10] that the conclusion of Theorem 9.1 holds for $p > 3\dim X$.

We denote by $X_G = EG \times_G X$ the Borel construction of X. Since G acts freely, we have $X_G \simeq X/G$. In particular, as in the case of free torus actions, we obtain $\dim_{\mathbb{F}_p} H^*(X_G;\mathbb{F}_p) < \infty$.

Using the Cenkl-Porter theorem and the tame Hirsch lemma one obtains the following version of Proposition 6.2, compare [10, Theorem 5.5].

Proposition 9.3. *If p is sufficiently large with respect to $\dim X$, there are finitely generated free DGCAs $(\mathcal{E}^*, d_E)$ and $(\mathcal{M}^*, d_M)$ over $\mathbb{F}_p$ such that*

- ▷ *$\mathcal{M}^* = \Lambda^*(\tau_1, \ldots \tau_{k_e}, \eta_1, \ldots, e_{k_e}, \sigma_1, \ldots, \sigma_{k_o})$ as in Proposition 6.2 with $d_M(\eta_j) = \tau_j^2$,*
- ▷ *$\mathcal{E}^* = \mathcal{M}^* \otimes \mathbb{F}_p[t_1, \ldots, t_r] \otimes \Lambda^*(s_1, \ldots, s_r)$ as graded commutative algebras, where $\deg(t_i) = 2$ and $\deg(s_i) = 1$,*
- ▷ *d_E is $\mathbb{F}_p[t_1, \ldots, t_r] \otimes \Lambda^*(s_1, \ldots, s_r)$-linear and the projection $\mathcal{E}^* \to \mathcal{M}^*$ given by evaluating $t_1, \ldots, t_r, s_1, \ldots, s_r$ at 0 is a cochain map,*
- ▷ *each monomial in $t_1, \ldots, t_r$ of cohomological degree at least $\dim X + 1$ represents the zero class in $H^*(\mathcal{E}^*)$. However, the cohomology algebra $H^*(\mathcal{E}^*)$ is not isomorphic to $H^*(X_G;\mathbb{F}_p)$ in large degrees,*

Note that $H^*(B(\mathbb{Z}/p)^r;\mathbb{F}_p) \cong \mathbb{F}_p[t_1, \ldots, t_r] \otimes \Lambda^*(s_1, \ldots, s_r)$. Now replace $\mathcal{E}^*$ by $\mathcal{E}^*/(s_1, \ldots, s_r)$ with the induced differential and denote this DGCA $(\mathcal{E}^*, d_E)$ again. Arguing as in the proof of Proposition 6.4, one shows that all t_i and τ_j represent nilpotent cohomology classes in $H^*(\mathcal{E}^*, \delta_E)$ so that this cohomology is finite dimensional over $\mathbb{F}_p$. Using a commutative algebra argument as in Section 6, this implies $k_e + r \leq k_e + k_o$, as required. More details can be found in [10].

References

[1] C. Allday, V. Puppe, *Cohomological methods in transformation groups*, Cambridge University Press (1993).
[2] A. Bousfield, V. Gugenheim, *On PL de Rham theory and rational homotopy type*, Mem. Amer. Math. Soc. **179** (1976).
[3] H. Cartan, *Théories cohomologiques*, Invent. Math. **35** (1976), 261-271.
[4] B. Cenkl, R. Porter, *De Rham theorem with cubical forms*, Pacific J. Math. **112** (1984), 35-48.
[5] D. Eisenbud, *Commutative Algebra*, Graduate Texts in Mathematics **150** (1996), Springer-Verlag.
[6] P. Griffiths, J. Morgan, *Rational Homotopy Theory and Differential Forms*, Second Edition, Birkhäuser, 2013.
[7] Y. Félix, S. Halperin, J.-C. Thomas, *Rational homotopy theory*, Springer Graduate Texts in Mathematics **205**, Springer-Verlag 2001.
[8] S. Halperin, *Finiteness in the minimal models of Sullivan*, Trans. Amer. Math. Soc. **230** (1977), 173-199.
[9] A. Hatcher, *Notes on spectral sequences*, availabe at https://pi.math.cornell.edu/ hatcher/AT/ATch5.pdf
[10] B. Hanke, *The stable free rank of symmetry of products of spheres*, Inv. Math. **178** (2009), 265–298.
[11] P. May, *Simplicial objects in algebraic topology*, Chicago Lecture Notes in Mathematics (1992).
[12] T. Sörensen, *Zahme Homotopietheorie einfacher Kan-Mengen und p-Torus-Operationen*, Diplomarbeit (1992), FU Berlin, available at ftp.math.fu-berlin.de/pub/math/publ/pre/2008/index.html.
[13] D. Sullivan, *Infinitesimal computations in topology*, Publ. Math. Inst. Hautes Études Sci. **47** (1977), 269-331.

Institute of Mathematics, University of Augsburg, Germany
Email address: hanke@math.uni-augsburg.de