# Fine structure of BV functions on fractals (a preliminary report)

Alexander Teplyaev

University of Connecticut

BIRS Banff 22w5080 - Smooth Functions on Rough Spaces and Fractals with Connections to Curvature Functional Inequalities,

November 20 – 25, 2022

# 7th Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals: June 9–13, 2020

Home » 7th Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals

### Welcome!

Planning has begun for Fractals 7 (June 9-13, 2020). The purpose of this conference, held every three years, is to bring together mathematicians who are already working in the area of analysis and probability on fractals with students and researchers from related areas. Information will be posted here as it becomes available.

Financial support will be provided to a limited number of participants to cover the cost of housing in Cornell single dormitory rooms partially support other travel expenses. Students and junior researchers from underrepresented groups in STEM are particularly encouraged to apply for travel funding. Well-established researchers are encouraged to use their own travel funding; the NSF expec that most funds will be expended on otherwise unfunded mathematicians.

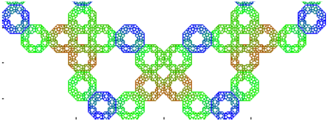
Registration details will be publicized once available.

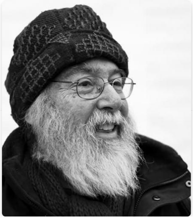
All general inquiries can be sent to: fractals_math@cornell.edu

Conference Organizers:

- Robert Strichartz (chair), Cornell University
- Patricia Alonso Ruiz, Texas A&M University
- Michael Hinz, Bielefeld University
- Luke Rogers, University of Connecticut
- Alexander Teplyaev, University of Connecticut

# 7th Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals: June 4–8, 2022





In Memory of Professor Robert Strichartz

We will be dedicating the entire conference to Professor Strichartz. A special session will be scheduled during the conference for all to attend and reflect on their thoughts and memories of Bob. Bob is appreciated and recognized for his organizing of the Fractals Conference Community in 2002. He will be profoundly missed by family, friends, colleagues, and most of all, the students he mentored and influenced throughout his career.

A message from the Cornell Department of Mathematics Chair, Tara Holm:

Dear friends,

I am sad to share that our colleague and friend Professor Robert Strichartz died yesterday, 19 December 2021, after a long illness. He was 78.

# Plan of the talk:

## Dirichlet forms and diffusions of fractals

Introduction and examples of fractals

## Existence, uniqueness, heat kernel estimates

$F$-invariant Dirichlet forms

## Selected results: spectral analysis

Open problems and further directions

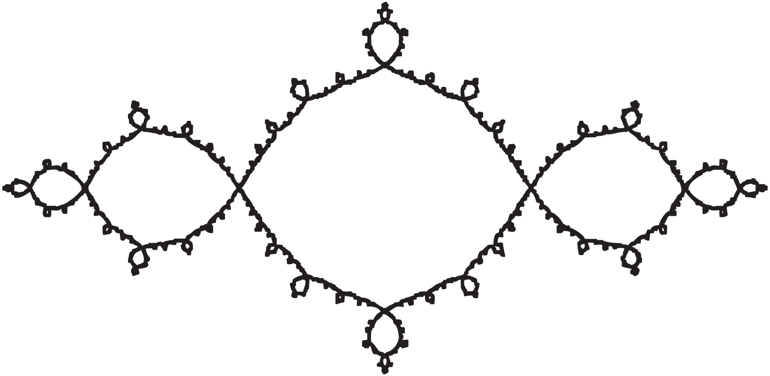
- Introduction
- Classical Curl
- Sierpinski carpets
- Non-closable curl
- Generalization

## BV and Besov spaces on fractals with Dirichlet forms

(Patricia Alonso-Ruiz, Fabrice Baudoin, Li Chen, Luke Rogers, Nages Shanmugalingam, T.)

Definitions of $BV$ and $wBE(\kappa)$

*This is a part of the broader program to develop probabilistic, spectral and vector analysis on singular spaces by carefully building approximations by graphs or manifolds.*



The basilica Julia set, the Julia set of $z^2 - 1$ and the limit set of the basilica group of exponential growth (Grigorchuk, Żuk, Bartholdi, Virág, Nekrashevych, Kaimanovich, Nagnibeda et al.).

# Asymptotic aspects of Schreier graphs and Hanoi Towers groups

Rostislav Grigorchuk[1], Zoran Šunić

*Department of Mathematics, Texas A&M University, MS-3368, College Station, TX, 77843-3368, USA*

Received 23 January, 2006; accepted after revision +++++

Presented by Étienne Ghys

**Abstract**

We present relations between growth, growth of diameters and the rate of vanishing of the spectral gap in Schreier graphs of automaton groups. In particular, we introduce a series of examples, called Hanoi Towers groups since they model the well known Hanoi Towers Problem, that illustrate some of the possible types of behavior. *To cite this article: R. Grigorchuk, Z. Šunić, C. R. Acad. Sci. Paris, Ser. I 344 (2006).*

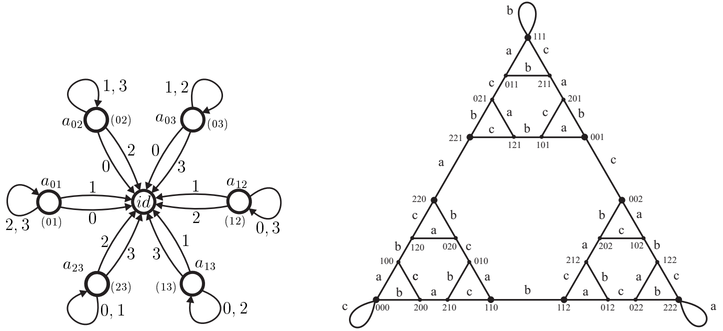
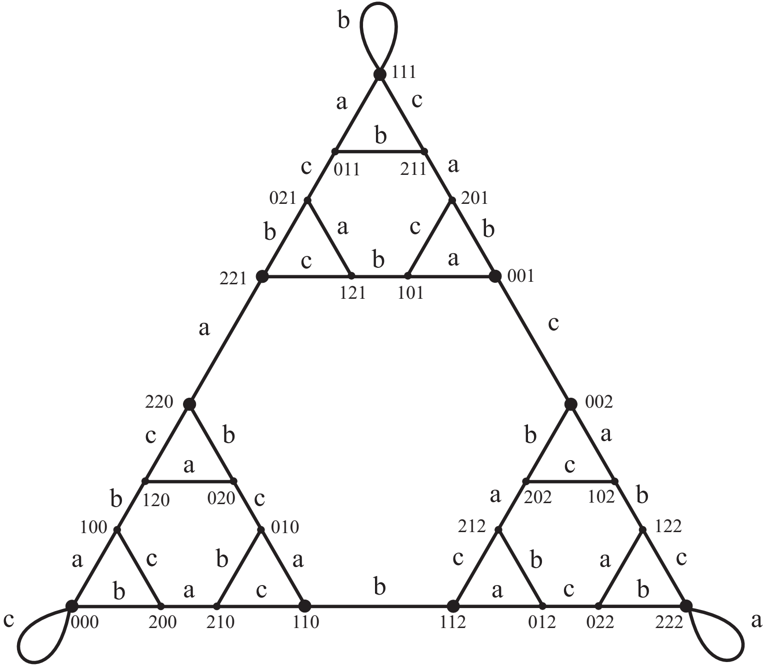



Figure 1. The automaton generating $H^{(4)}$ and the Schreier graph of $H^{(3)}$ at level 3 / L'automate engendrant $H^{(4)}$ et le graphe de Schreier de $H^{(3)}$ au niveau 3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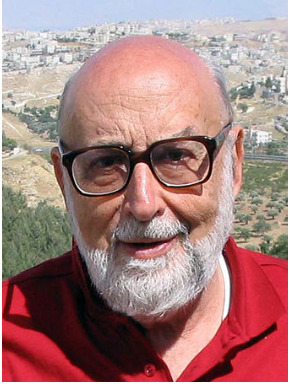
# François Englert

From Wikipedia, the free encyclopedia

**François Baron Englert** (French: [ɑ̃glɛʁ]; born 6 November 1932) is a Belgian theoretical physicist and 2013 Nobel prize laureate (shared with Peter Higgs). He is Professor emeritus at the Université libre de Bruxelles (ULB) where he is member of the Service de Physique Théorique. He is also a Sackler Professor by Special Appointment in the School of Physics and Astronomy at Tel Aviv University and a member of the Institute for Quantum Studies at Chapman University in California. He was awarded the 2010 J. J. Sakurai Prize for Theoretical Particle Physics (with Gerry Guralnik, C. R. Hagen, Tom Kibble, Peter Higgs, and Robert Brout), the Wolf Prize in Physics in 2004 (with Brout and Higgs) and the High Energy and Particle Prize of the European Physical Society (with Brout and Higgs) in 1997 for the mechanism which unifies short and long range interactions by generating massive gauge vector bosons. He has made contributions in statistical physics, quantum field theory, cosmology, string theory and supergravity.[4] He is the recipient of the 2013 Prince of Asturias Award in technical and scientific research,

**François Englert**



François Englert in Israel, 2007

Nuclear Physics B280 [FS 18] (1987) 147–180
North-Holland, Amsterdam

# METRIC SPACE-TIME AS FIXED POINT OF THE RENORMALIZATION GROUP EQUATIONS ON FRACTAL STRUCTURES

F. ENGLERT, J.-M. FRÈRE[1] and M. ROOMAN[2]

*Physique Théorique, C.P. 225, Université Libre de Bruxelles, 1050 Brussels, Belgium*

Ph. SPINDEL

*Faculté des Sciences, Université de l'Etat à Mons, 7000 Mons, Belgium*

Received 19 February 1986

We take a model of foamy space-time structure described by self-similar fractals. We study the propagation of a scalar field on such a background and we show that for almost any initial conditions the renormalization group equations lead to an effective highly symmetric metric at large scale.

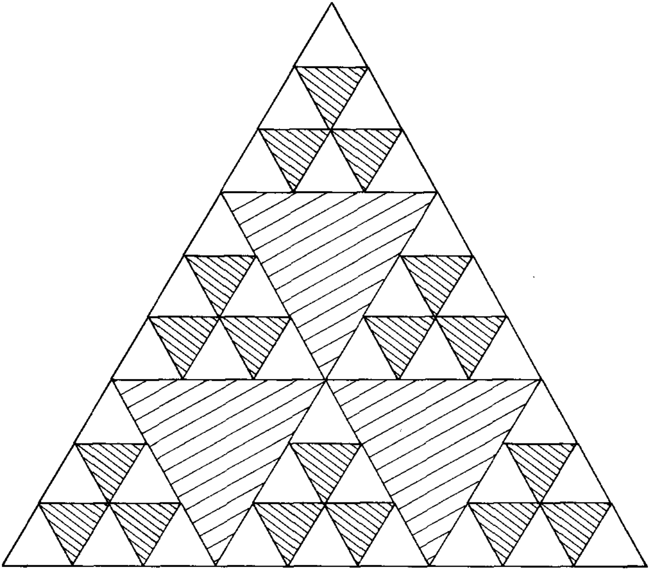


Fig. 1. The first two iterations of a 2-dimensional 3-fractal.

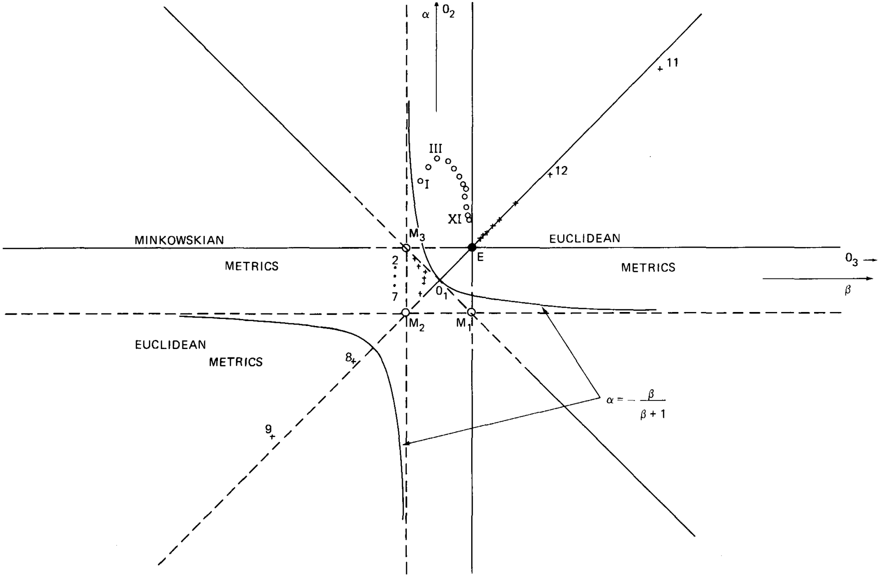



Fig. 5. The plane of 2-parameter homogeneous metrics on the Sierpinski gasket. The hyperbole $\alpha = -\beta/(\beta+1)$ separates the domain of euclidean metrics from minkowskian metrics and corresponds – except at the origin – to 1-dimensional metrics. $M_1, M_2, M_3$ denote unstable minkowskian fixed geometries while $E$ corresponds to the stable euclidean fixed point. The unstable fixed points $0_1$, $0_2$ and $0_3$ associated to 0-dimensional geometries are located at the origin and at infinity on the $(\alpha, \beta)$ coordinates axis. The six straight lines are subsets invariant with respect to the recursion relation but repulsive in the region where they are dashed. The first points of two sequences of iterations are drawn. Note that for one of them the 10th point ($\alpha = -56.4$, $\beta = -52.5$) is outside the frame of the figure.

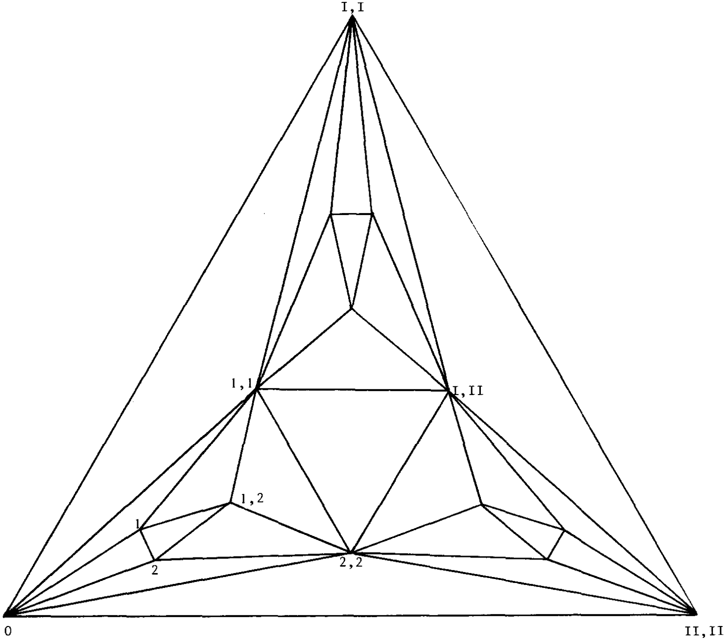



Fig. 10. A metrical representation of the two first iterations of a 2-dimensional 2-fractal corresponding to the euclidean fixed point. Vertices are labelled according to fig. 4.

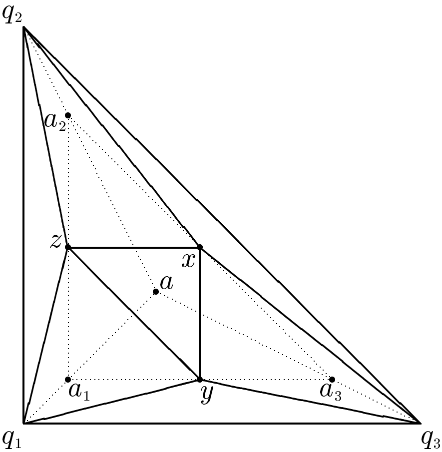



**Figure 6.4.** Geometric interpretation of Proposition 6.1.

# The Spectral Dimension of the Universe is Scale Dependent

J. Ambjørn,[1,3,*] J. Jurkiewicz,[2,†] and R. Loll[3,‡]

[1]*The Niels Bohr Institute, Copenhagen University, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark*
[2]*Mark Kac Complex Systems Research Centre, Marian Smoluchowski Institute of Physics, Jagellonian University, Reymonta 4, PL 30-059 Krakow, Poland*
[3]*Institute for Theoretical Physics, Utrecht University, Leuvenlaan 4, NL-3584 CE Utrecht, The Netherlands*

(Received 13 May 2005; published 20 October 2005)

We measure the spectral dimension of universes emerging from nonperturbative quantum gravity, defined through state sums of causal triangulated geometries. While four dimensional on large scales, the quantum universe appears two dimensional at short distances. We conclude that quantum gravity may be "self-renormalizing" at the Planck scale, by virtue of a mechanism of dynamical dimensional reduction.

DOI: 10.1103/PhysRevLett.95.171301 PACS numbers: 04.60.Gw, 04.60.Nc, 98.80.Qc

*Quantum gravity as an ultraviolet regulator?*—A shared hope of researchers in otherwise disparate approaches to quantum gravity is that the microstructure of space and time may provide a physical regulator for the ultraviolet infinities encountered in perturbative quantum field theory. *tral dimension*, a diffeomorphism-invariant quantity obtained from studying diffusion on the quantum ensemble of geometries. On large scales and within measuring accuracy, it is equal to four, in agreement with earlier measurements of the large-scale dimensionality based on the other hand, the "short-distance spectral dimension," obtained by extrapolating Eq. (12) to $\sigma \rightarrow 0$ is given by

$$D_S(\sigma = 0) = 1.80 \pm 0.25, \tag{15}$$

and thus is compatible with the integer value two.

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Published: *December 2, 2011*

# Fractal space-times under the microscope: a renormalization group view on Monte Carlo data

**Martin Reuter and Frank Saueressig**

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Abstract: The emergence of fractal features in the microscopic structure of space-time is a common theme in many approaches to quantum gravity. In this work we carry out a detailed renormalization group study of the spectral dimension $d_s$ and walk dimension $d_w$ associated with the effective space-times of asymptotically safe Quantum Einstein Gravity (QEG). We discover three scaling regimes where these generalized dimensions are approximately constant for an extended range of length scales: a classical regime where $d_s = d, d_w = 2$, a semi-classical regime where $d_s = 2d/(2+d), d_w = 2+d$, and the UV-fixed point regime where $d_s = d/2, d_w = 4$. On the length scales covered by three-dimensional Monte Carlo simulations, the resulting spectral dimension is shown to be in very good agreement with the data. This comparison also provides a natural explanation for the apparent puzzle between the short distance behavior of the spectral dimension reported from Causal Dynamical Triangulations (CDT), Euclidean Dynamical Triangulations (EDT), and Asymptotic Safety.

Keywords: Models of Quantum Gravity, Renormalization Group, Lattice Models of Gravity, Nonperturbative Effects

JHEP12(2011)012

# Fractal space-times under the microscope: A Renormalization Group view on Monte Carlo data

Martin Reuter and Frank Saueressig

a classical regime where $d_s = d, d_w = 2$, a semi-classical regime where $d_s = 2d/(2+d), d_w = 2+d$, and the UV-fixed point regime where $d_s = d/2, d_w = 4$. On the length scales covered

# Norbert Wiener

From Wikipedia, the free encyclopedia

**Norbert Wiener** (November 26, 1894, Columbia, Missouri – March 18, 1964, Stockholm, Sweden) was an American mathematician.

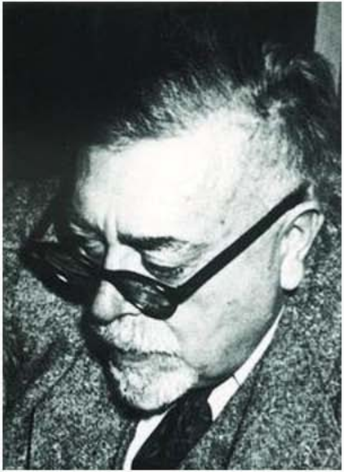
A famous child prodigy, Wiener (*pronounced WEE-nur*) later became an early studier of stochastic and noise processes, contributing work relevant to electronic engineering, electronic communication, and control systems.

Wiener is wrongly regarded as the originator of cybernetics(see Ştefan Odobleja), a formalization of the notion of feedback, with many implications for engineering, systems control, computer science, biology, philosophy, and the organization of society.

**Contents** [hide]



**Norbert Wiener**



| | |
|---|---|
| **Born** | November 26, 1894<br>Columbia, Missouri, U.S. |
| **Died** | March 18, 1964 (aged 69) |

# Andrey Kolmogorov

From Wikipedia, the free encyclopedia

**Andrey Nikolaevich Kolmogorov** (Russian: Андре́й Никола́евич Колмого́ров) (25 April 1903 – 20 October 1987) was a Soviet Russian mathematician, preeminent in the 20th century, who advanced various scientific fields, among them probability theory, topology, intuitionistic logic, turbulence, classical mechanics and computational complexity.

**Andrey Kolmogorov**

| | |
|---|---|
| **Born** | 25 April 1903<br>Tambov, Imperial Russia |
| **Died** | 20 October 1987 (aged 84) |

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# Wacław Sierpiński

From Wikipedia, the free encyclopedia

**Wacław Franciszek Sierpiński** (Polish pronunciation: [ˈvatswaf franˈtɕiʂɛk ɕɛrˈpʲiɲskʲi]) (March 14, 1882, Warsaw — October 21, 1969, Warsaw) was a Polish mathematician. He was known for outstanding contributions to set theory (research on the axiom of choice and the continuum hypothesis), number theory, theory of functions and topology. He published over 700 papers and 50 books.

Three well-known fractals are named after him (the Sierpinski triangle, the Sierpinski carpet and the Sierpinski curve), as are Sierpinski numbers and the associated Sierpiński problem.

**Contents** [hide]

1 Education

**Wacław Sierpiński**


| | |
|---|---|
| **Born** | March 14, 1882<br>Warsaw, Poland |
| **Died** | October 21, 1969<br>(aged 87) |

ANALYSE MATHÉMATIQUE. — *Sur une courbe dont tout point est un point de ramification.* Note [1] de M. **W. Sierpinski**, présentée par M. Émile Picard.

Le but de cette Note est de donner un exemple d'une courbe cantorienne et jordanienne en même temps, dont tout point est un point de ramification. (Nous appelons *point de ramification* d'une courbe $\mathcal{C}$ un point $p$ de cette courbe, s'il existe trois continus, sous-ensembles de $\mathcal{C}$, ayant deux à deux le point $p$ et seulement ce point commun.)

Soient T un triangle régulier donné; A, B, C respectivement ses sommets : gauche, supérieur et droit. En joignant les milieux des côtés du triangle T, nous obtenons quatre nouveaux triangles réguliers (*fig.* 1), dont trois, $T_0$, $T_1$, $T_2$, contenant respectivement les sommets A, B, C, sont situés parallèlement à T et le quatrième triangle U contient le centre du triangle T; nous exclurons tout l'intérieur du triangle U.

Les sommets des triangles $T_0$, $T_1$, $T_2$ nous les désignerons respectivement :

(1) Séance du 1er février 1915.

triangles $U_0$, $U_1$, $U_2$, situés parallèlement à U, dont les intérieurs seront

Fig. 1.

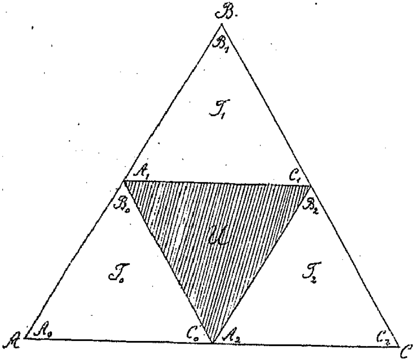
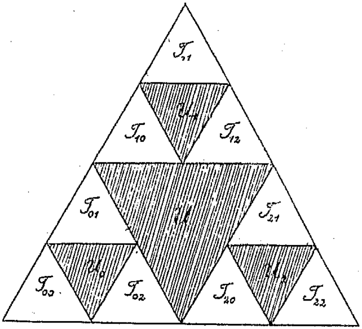



Fig. 2.




exclus (*fig.* 2). Avec chacun des triangles $T_{\lambda_1\lambda_2}$ procédons de même et ainsi

Fig. 3.

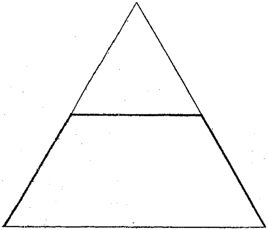
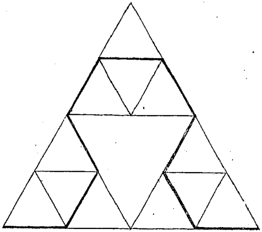


Fig. 4.



d'eux se rencontrent quatre segments différents, situés entièrement sur l'ensemble $\mathfrak{S}$.

Donc, tous les points de la courbe $\mathfrak{S}$, sauf peut-être les points A, B, C, sont ses points de ramification.

Pour obtenir une courbe dont tous les points sans exception sont ses

Fig. 5.

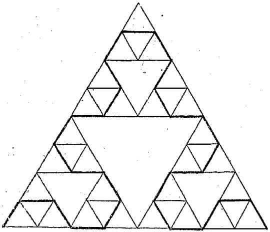
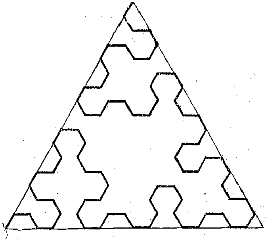


Fig. 6.



points de ramification, il suffit de diviser un hexagone régulier en six triangles réguliers et dans chacun d'eux inscrire une courbe $\mathfrak{S}$

Fig. 5.

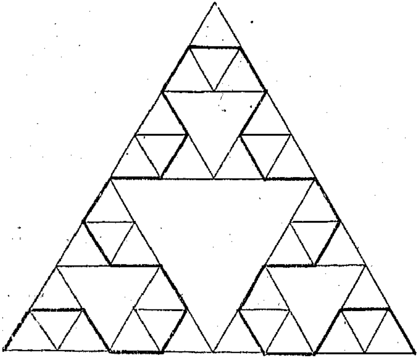
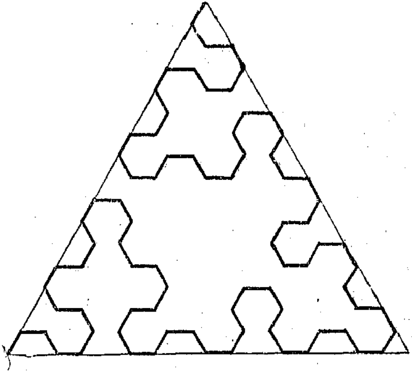
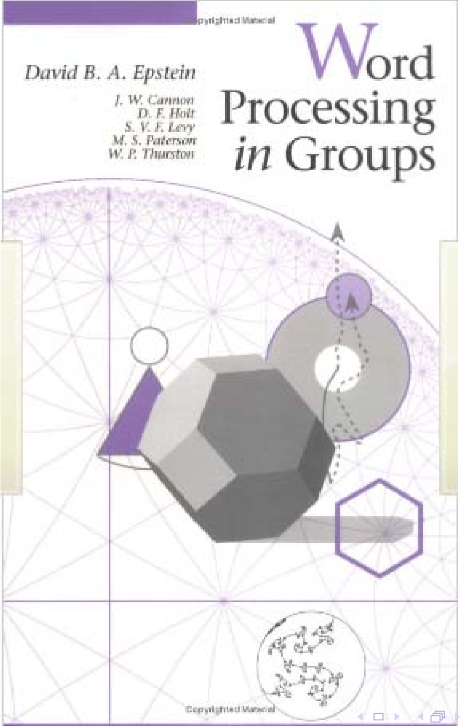


Fig. 6.





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# Initial motivation

- R. Rammal and G. Toulouse, *Random walks on fractal structures and percolation clusters.* J. Physique Letters **44** (1983)
- R. Rammal, *Spectrum of harmonic excitations on fractals.* J. Physique **45** (1984)
- E. Domany, S. Alexander, D. Bensimon and L. Kadanoff, *Solutions to the Schrödinger equation on some fractal lattices.* Phys. Rev. B (3) **28** (1984)
- Y. Gefen, A. Aharony and B. B. Mandelbrot, *Phase transitions on fractals. I. Quasilinear lattices. II. Sierpiński gaskets. III. Infinitely ramified lattices.* J. Phys. A **16** (1983)**17** (1984)

# Main early results

Sheldon Goldstein, *Random walks and diffusions on fractals.* Percolation theory and ergodic theory of infinite particle systems (Minneapolis, Minn., 1984–1985), IMA Vol. Math. Appl., 8, Springer

Summary: we investigate the asymptotic motion of a random walker, which at time $n$ is at $X(n)$, on certain 'fractal lattices'. For the 'Sierpiński lattice' in dimension $d$ we show that, as $L \to \infty$, the process $Y_L(t) \equiv X([(d+3)^L t])/2^L$ converges in distribution to a diffusion on the Sierpin'ski gasket, a Cantor set of Lebesgue measure zero. The analysis is based on a simple 'renormalization group' type argument, involving self-similarity and 'decimation invariance'. In particular,

$$|X(n)| \sim n^\gamma,$$

where $\gamma = (\ln 2)/\ln(d+3)) \leqslant 2$.

Shigeo Kusuoka, *A diffusion process on a fractal.* Probabilistic methods in mathematical physics (Katata/Kyoto, 1985), 1987.

- M.T. Barlow, E.A. Perkins, *Brownian motion on the Sierpinski gasket.* (1988)
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# Main classes of fractals considered

- $[0, 1]$

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- Sierpiński gasket

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- nested fractals

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- $[0, 1]$
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- p.c.f. self-similar sets, possibly with various symmetries
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- infinitely ramified self-similar sets, with local symmetries, and with heat kernel estimates (such as the Generalized Sierpiński carpets)

# Main classes of fractals considered

- $[0, 1]$
- Sierpiński gasket
- nested fractals
- p.c.f. self-similar sets, possibly with various symmetries
- finitely ramified self-similar sets, possibly with various symmetries
- infinitely ramified self-similar sets, with local symmetries, and with heat kernel estimates (such as the Generalized Sierpiński carpets)
- metric measure Dirichlet spaces, possibly with heat kernel estimates (MMD+HKE)

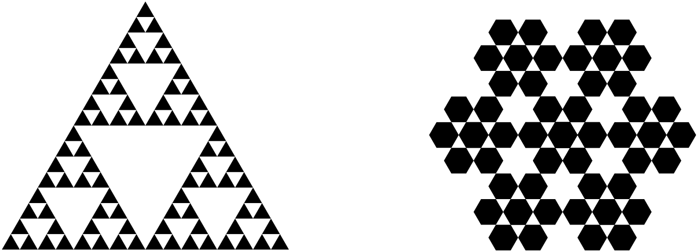

Figure: Sierpiński gasket and Lindstrøm snowflake (nested fractals), p.c.f., finitely ramified)

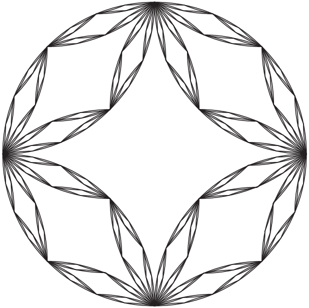

Figure: Diamond fractals, non-p.c.f., but finitely ramified

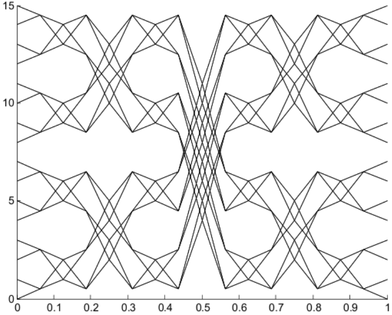


Figure: Laakso Spaces (Ben Steinhurst), infinitely ramified

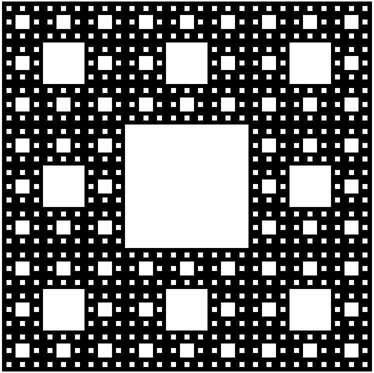

Figure: Sierpiński carpet, infinitely ramified

# Existence, uniqueness, heat kernel estimates

**Brownian motion:**
Thiele (1880), Bachelier (1900)
Einstein (1905), Smoluchowski (1906)
Wiener (1920'), Doob, Feller, Levy, Kolmogorov (1930'),
Doeblin, Dynkin, Hunt, Ito ...

Wiener process in $\mathbb{R}^n$ satisfies $\frac{1}{n}\mathbb{E}|W_t|^2 = t$ and has a Gaussian transition density:

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

$$\textit{distance} \sim \sqrt{\textit{time}}$$

"Einstein space–time relation for Brownian motion"

De Giorgi-Nash-Moser estimates for elliptic and parabolic PDEs;

Li-Yau (1986) type estimates on a geodesically complete Riemannian manifold with $\boldsymbol{Ricci} \geqslant \mathbf{0}$:

$$\boldsymbol{p_t(x,y) \sim \frac{1}{V(x,\sqrt{t})} \exp\left(-c\frac{d(x,y)^2}{t}\right)}$$

$$\boldsymbol{distance \sim \sqrt{time}}$$

Gaussian:

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

Li-Yau Gaussian-type:

$$p_t(x,y) \sim \frac{1}{V(x,\sqrt{t})} \exp\left(-c\frac{d(x,y)^2}{t}\right)$$

Sub-Gaussian:

$$p_t(x,y) \sim \frac{1}{t^{d_H/d_w}} \exp\left(-c\left(\frac{d(x,y)^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right)$$

$$distance \sim (time)^{\frac{1}{d_w}}$$

Brownian motion on $\mathbb{R}^d$: $\mathbb{E}|X_t - X_0| = ct^{1/2}$.

*Anomalous diffusion*: $\mathbb{E}|X_t - X_0| = o(t^{1/2})$, or (in regular enough situations),

$$\mathbb{E}|X_t - X_0| \approx t^{1/d_w}$$

with $d_w > 2$.

Here $d_w$ is the so-called **walk dimension** (should be called "**walk index**" perhaps).

This phenomena was first observed by mathematical physicists working in the transport properties of disordered media, such as (critical) percolation clusters.

$$p_t(x,y) \sim \frac{1}{t^{d_H/d_w}} \exp\left(-c\frac{d(x,y)^{\frac{d_w}{d_w-1}}}{t^{\frac{1}{d_w-1}}}\right)$$

$$distance \sim (time)^{\frac{1}{d_w}}$$

$$\begin{aligned} d_H &= \text{Hausdorff dimension} \\ \frac{1}{\gamma} = d_w &= \text{“walk dimension” } (\gamma\text{=diffusion index}) \\ \frac{2d_H}{d_w} = d_S &= \text{“spectral dimension” (diffusion dimension)} \end{aligned}$$

First example: Sierpiński gasket; Kusuoka, Fukushima, Kigami, Barlow, Bass, Perkins (mid 1980'—)

# Theorem (Barlow, Bass, Kumagai (2006)).

Under natural assumptions on the MMD (geodesic Metric Measure space with a regular symmetric conservative Dirichlet form), the sub-Gaussian ***heat kernel estimates are stable under rough isometries****, i.e. under maps that preserve distance and energy up to scalar factors.*

***Gromov-Hausdorff* + *energy***

**Theorem.** (Barlow, Bass, Kumagai, T. (1989–2010).) On any fractal in the class of generalized Sierpiński carpets there exists a unique, up to a scalar multiple, local regular Dirichlet form that is invariant under the local isometries.

Therefore there there is a unique corresponding symmetric Markov process and a unique Laplacian. Moreover, the Markov process is Feller and its transition density satisfies sub-Gaussian heat kernel estimates.

# Main difficulties:

If it is not a cube in $\mathbb{R}^n$, then

- $d_S < d_H$, $d_w > 2$
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- in fact, we can not compute any functions of finite energy;
- Fourier and complex analysis methods seem to be not applicable.

**The key result in the center of the proof: the classical elliptic Harnack inequality**. Any harmonic function (a local energy minimizer) $u \geqslant 0$ satisfies

$$\sup_{B(x,R/2)} u \ \leq \ c_1 \inf_{B(x,R/2)} u$$

where **the constant $c_1$ is determined only by the geometry of the generalized Sierpiński carpet**.

**Remark.** This lemma is a hard mix of analysis (commutativity of certain geometric projections and the Laplacian) and probability (coupling).

**Corollary.** Harmonic functions are quasi-everywhere Hölder continuous.

**Theorem.** (Grigor'yan and Telcs, also [BBK])

On a MMD space the following are equivalent

- ***(VD)***, ***(EHI)*** and ***(RES)***
- ***(VD)***, ***(EHI)*** and ***(ETE)***
- ***(PHI)***
- ***(HKE)***

and the constants in each implication are effective.

Abbreviations: Metric Measure Dirichlet spaces, Volume Doubling, Elliptic Harnack Inequality, Exit Time Estimates, Parabolic Harnack Inequality, Heat Kernel Estimates.

**Theorem 1.** Let $(\mathcal{A}, \mathcal{F})$, $(\mathcal{B}, \mathcal{F})$ be **regular local conservative** irreducible Dirichlet forms on $L^2(F, m)$ and

$$(1+\delta)\mathcal{A}(u, u) \leq \mathcal{B}(u, u) \quad \text{for all } u \in \mathcal{F}$$

where $\delta > 0$. Then $(\mathcal{B}-\mathcal{A}, \mathcal{F})$ is a regular local conservative irreducible Dirichlet form on $L^2(F, m)$.

**Technical lemma.** If $\mathcal{E}$ is a local regular Dirichlet form with domain $\mathcal{F}$, then for any $f \in \mathcal{F} \cap L^\infty(F)$ we have $\Gamma(f, f)(A) = 0$, if $A = \{x{\in}F : f(x){=}0\}$ where $\Gamma(f, f)$ is the energy measure or the "square field operator"

$$\int_F g d\Gamma(f, f) = 2\mathcal{E}(f, fg) - \mathcal{E}(f^2, g), \; g \in \mathcal{F}_b.$$

## Definition

Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(F, \mu)$. We say that $\mathcal{E}$ *is invariant with respect to all the local symmetries of* $F$ ($F$-invariant or $\mathcal{E} \in \mathfrak{E}$) if

- (1) If $S \in \mathcal{S}_n(F)$, then $U_S R_S f \in \mathcal{F}$ for any $f \in \mathcal{F}$.
- (2) Let $n \geq 0$ and $S_1, S_2$ be any two elements of $\mathcal{S}_n$, and let $\Phi$ be any isometry of $\mathbb{R}^d$ which maps $S_1$ onto $S_2$. If $f \in \mathcal{F}^{S_2}$, then $f \circ \Phi \in \mathcal{F}^{S_1}$ and $\mathcal{E}^{S_1}(f \circ \Phi, f \circ \Phi) = \mathcal{E}^{S_2}(f, f)$ where

  $$\mathcal{E}^S(g, g) = \frac{1}{m_F^n} \mathcal{E}(U_S g, U_S g)$$

  and $Dom(\mathcal{E}^S) = \{g : g \text{ maps } S \text{ to } \mathbb{R}, U_S g \in \mathcal{F}\}$.
- (3) $\mathcal{E}(f, f) = \sum_{S \in \mathcal{S}_n(F)} \mathcal{E}^S(R_S f, R_S f)$ for all $f \in \mathcal{F}$

## Lemma

*Let* $(\mathcal{A}, \mathcal{F}_1), (\mathcal{B}, \mathcal{F}_2) \in \mathfrak{E}$ *with* $\mathcal{F}_1 = \mathcal{F}_2$ *and* $\mathcal{A} \geq \mathcal{B}$. *Then* $\mathcal{C} = (1 + \delta)\mathcal{A} - \mathcal{B} \in \mathfrak{E}$ *for any* $\delta > 0$.

$$\Theta f = \frac{1}{m_F^n} \sum_{S \in \mathcal{S}_n(F)} U_S R_S f.$$

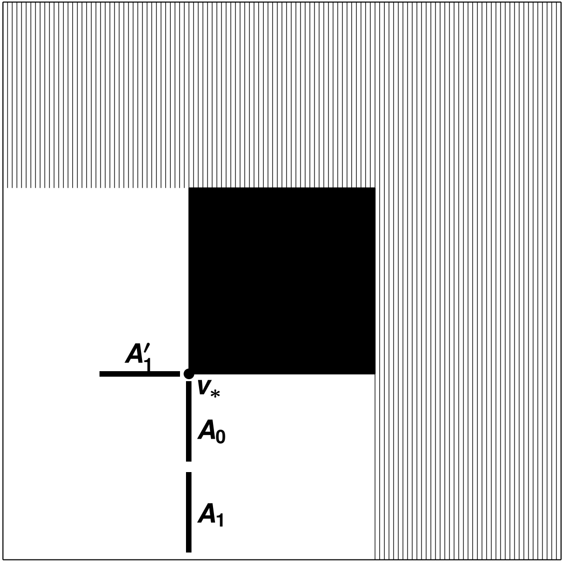
Note that $\Theta$ is a projection operator because $\Theta^2 = \Theta$. It is bounded on $C(F)$ and is an orthogonal projection on $L^2(F, \mu)$.

## Lemma

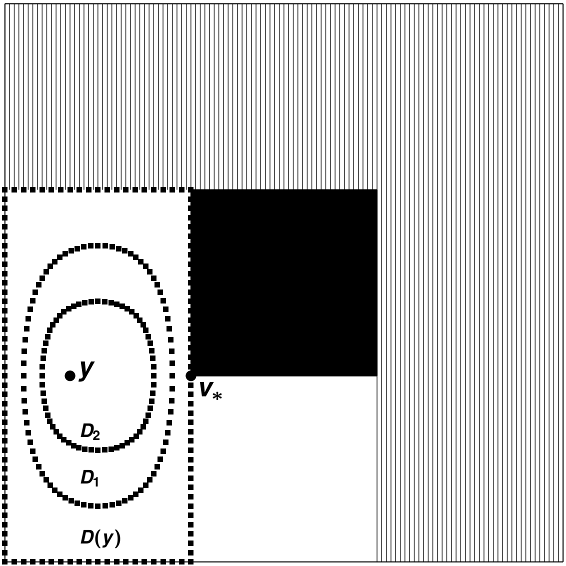
*Assume that $\mathcal{E}$ is a local regular Dirichlet form on $F$, $T_t$ is its semigroup, and $U_S R_S f \in \mathcal{F}$ whenever $S \in \mathcal{S}_n(F)$ and $f \in \mathcal{F}$. Then the following, for all $f, g \in \mathcal{F}$, are equivalent:*

$$\textbf{(a):}\ \ \mathcal{E}(f, f) = \sum_{S \in \mathcal{S}_n(F)} \mathcal{E}^S(R_S f, R_S f)$$

**(b):** $\mathcal{E}(\Theta f, g) = \mathcal{E}(f, \Theta g)$ **(c):** $T_t \Theta f = \Theta T_t f$




The half-face $A_1$ corresponds to a “slide move”,
and the half-face $A'_1$ corresponds to a “corner move”,
analogues of the “corner” and “knight’s” moves in [BB89].


y
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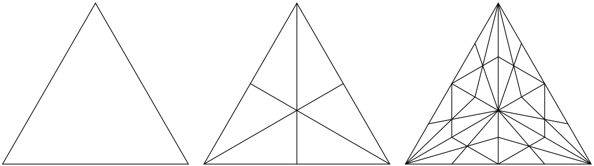


FIGURE 1. Barycentric subdivision of a 2-simplex, the graphs $G_0^T$, $G_1^T$ and $G_2^T$.

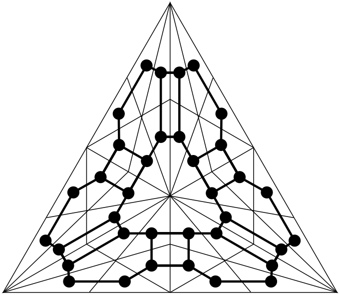


FIGURE 2. Adjacency (dual) graph $G_2$, in bold, and the barycentric subdivision graph pictured together with the thin image of $G_2^T$.

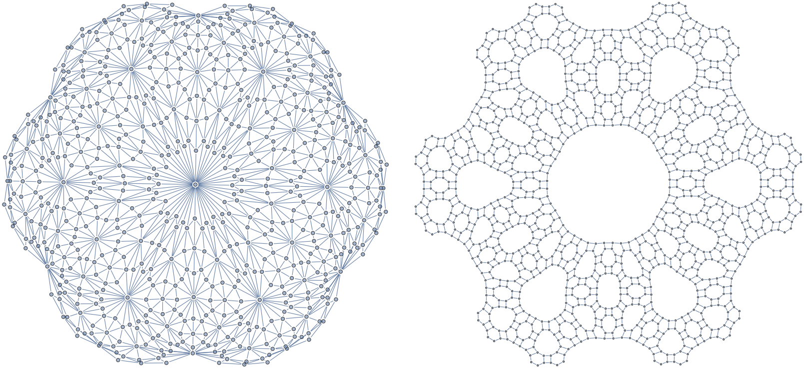


FIGURE 3. On the left: the graph $G_4^T$ for barycentric subdivision of a 2-simplex. On the right: the adjacency (dual) graph $G_4$.

**Theorem 1.1.** *The resistances across graphs $G_n^T$ and $G_n^H$ (defined in Subsection 2.2) are reciprocals, that is $R_n^T = 1/R_n$, and the asymptotic limits*

$$\log \rho^T = \lim_{n\to\infty} \frac{1}{n} \log R_n^T \qquad \textit{and} \qquad \log \rho = \lim_{n\to\infty} \frac{1}{n} \log R_n$$

*exist (and $\rho^T = 1/\rho$). Furthermore, $2/3 \leq \rho^T \leq 4/5$ and $5/4 \leq \rho \leq 3/2$.*

These estimates agree with the numerical experiments from [12], which suggest that there exists a limiting Dirichlet form on these fractals and estimates $\rho \approx 1.306$, and hence $\rho^T \approx 0.7655$.

**Conjecture 1.** *In the case $5/4 \leq \rho \leq 3/2$ ($\rho \approx 1.306$), we conjecture that the recent results of A. Grigor'yan, J. Hu, K.-S. Lau and M. Yang in [24–26, 28] can imply existence of the Dirichlet form.*

**Conjecture 2.** *Since $2/3 \leq \rho^T \leq 4/5 < 5/4 \leq \rho \leq 3/2$, we conjecture that there is essentially no uniqueness of the Dirichlet forms, spectral dimensions, resistance scaling factors etc for repeated barycentric subdivisions.*

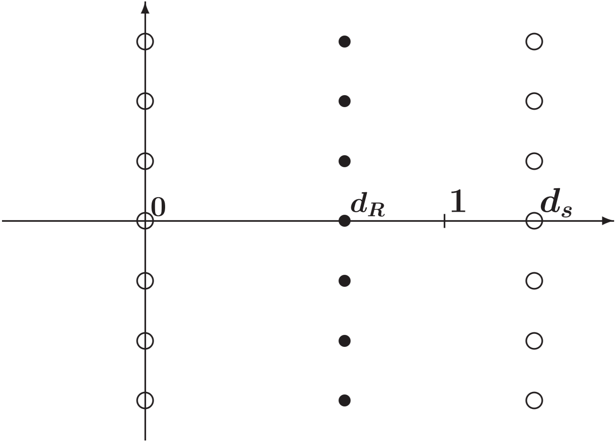
# Selected results: spectral analysis

**Theorem.** (Derfel, Grabner, Vogl; T.; Kajino (2007–2011)) For a large class of **finitely ramified symmetric fractals**, which includes the Sierpiński gaskets, and may include the Sierpiński carpets, the spectral zeta function

$$\zeta(s) = \sum \lambda_j^{s/2}$$

has a meromorphic continuation from the half-pain $Re(s) > d_S$ to $\mathbb{C}$. Moreover, all the poles and residues are computable from the geometric data of the fractal. Here $\lambda_j$ are the eigenvalues if the unique symmetric Laplacian.

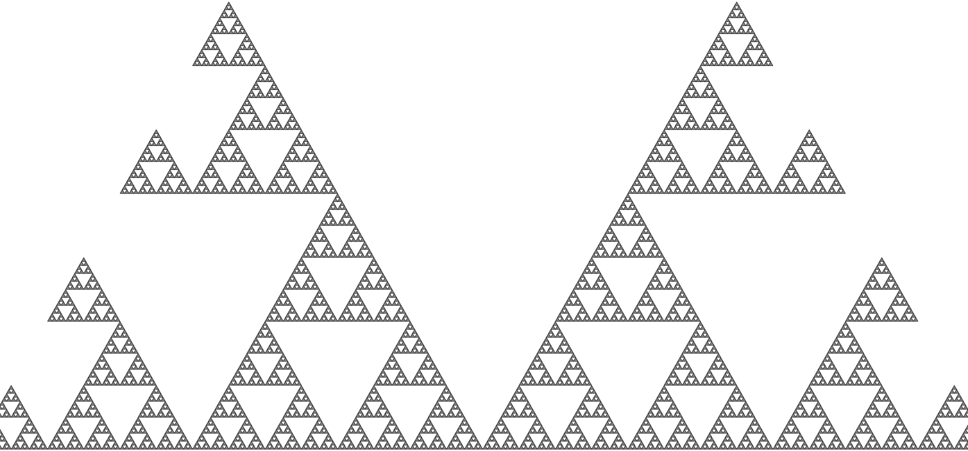
- Example: $\zeta(s)$ is the Riemann zeta function up to a trivial factor in the case when our fractal is $[0, 1]$.
- In more complicated situations, such as the Sierpiński gasket, there are infinitely many non-real poles, which can be called complex spectral dimensions, and are related to oscillations in the spectrum.




$$d_s = \frac{\log 9}{\log 5}$$

$$d_R = \frac{\log 4}{\log 5}$$

Poles (white circles) of the spectral zeta function of the Sierpiński gasket.



A part of an infinite Sierpiński gasket.

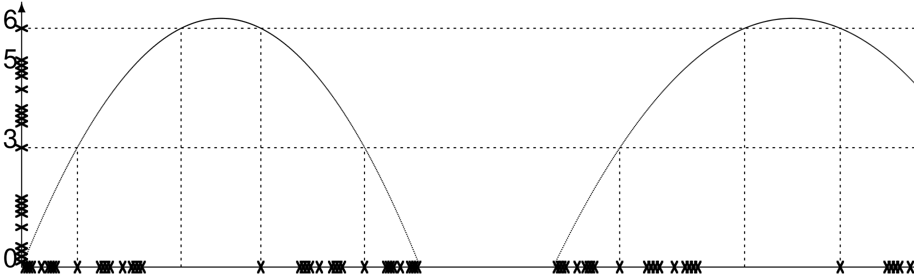
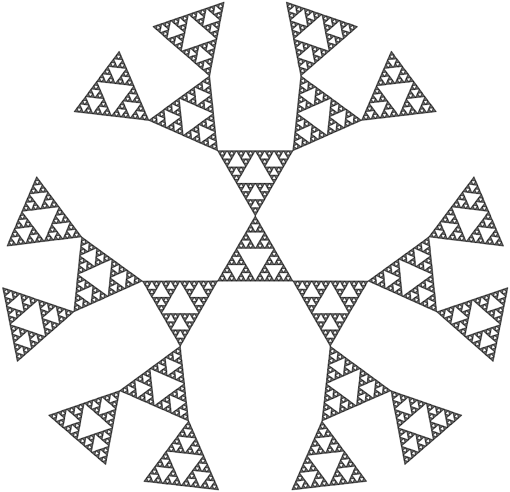
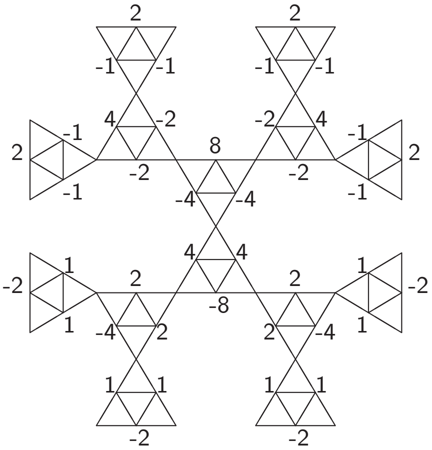



Figure: An illustration to the computation of the spectrum on the infinite Sierpiński gasket. The curved lines show the graph of the function $\mathfrak{R}(\cdot)$.

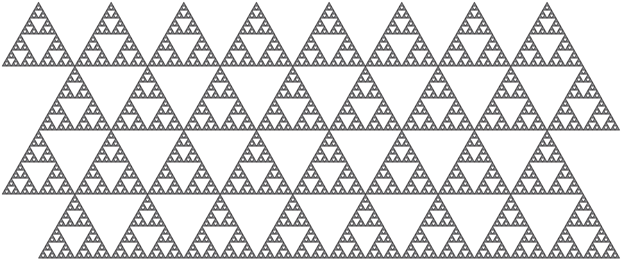
**Theorem.** (T. 1998, Quint 2009) On the Barlow-Perkins infinite Sierpiński fractafold the spectrum of the Laplacian consists of a **dense set of eigenvalues** $\mathfrak{R}^{-1}(\Sigma_0)$ **of infinite multiplicity** and a **singularly continuous component of spectral multiplicity one supported on** $\mathfrak{R}^{-1}(\mathcal{J}_R)$.

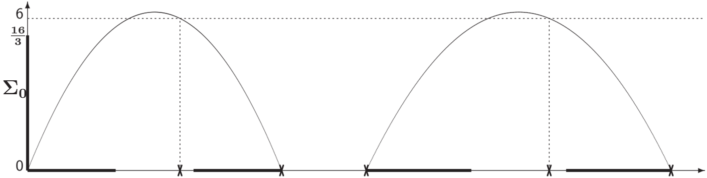


The Tree Fractafold.




An eigenfunction on the Tree Fractafold.






**Theorem.** (Strichartz, T. 2010) The Laplacian on the periodic triangular lattice finitely ramified Sierpiński fractal field consists of absolutely continuous spectrum and pure point spectrum. The **absolutely continuous spectrum is $\mathfrak{R}^{-1}[0, \frac{16}{3}]$.** The **pure point spectrum** consists of two infinite series of eigenvalues of infinite multiplicity. The spectral resolution is given in the main theorem.

# Open problems

- Existence of self-similar diffusions on finitely ramified fractals? on any self-similar fractals? on limit sets of self-similar groups? is there a natural diffusion on any connected set with a finite Hausdorff measure?

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- Differential geometry of fractals?
- PDEs involving derivatives, such as the Navier-Stokes equation.

# More on motivations and connections to other areas: Cheeger, Heinonen, Koskela, Shanmugalingam, Tyson


J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces,* Geom. Funct. Anal. **9** (1999) J. Heinonen, *Lectures on analysis on metric spaces.* Universitext. Springer-Verlag, New York, 2001. J. Heinonen, *Nonsmooth calculus,* Bull. Amer. Math. Soc. (N.S.) **44** (2007)

J. Heinonen, P. Koskela, N. Shanmugalingam, J. Tyson, *Sobolev classes of Banach space-valued functions and quasiconformal mappings*. J. Anal. Math. 85 (2001)

# Further directions (global)

- Fractal behavior of processes in algebra and geometry and probabilistic approach to stability under Hölder continuous transformations (Gromov, Perelman).

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- Mathematical physics, in particular, more general diffusion processes than in Einstein theory, behavior of fractals in magnetic field, Feynman integrals and field theories in general spaces.
- Computational tools for natural sciences, such as geophysics, chemistry, biology etc.

# Recent/current exciting new developments (local)

- Mario Bonk/Dimitrios Ntalampekos:
  **Potential theory on Sierpiński carpets with applications to uniformization**
  (compare to Koskela/Zhou **Geometry and analysis of Dirichlet forms**: Sierpiński gasket in harmonic coordinates)

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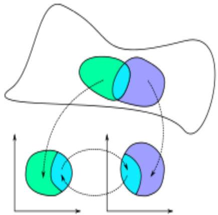
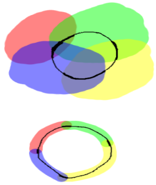
- Jun Kigami:
  **Weighted partition of a compact metrizable space, its hyperbolicity, and Ahlfors regular conformal dimension**

- **Long term goals:** Geometric analysis on **fractal** *Dirichlet metric measure spaces* and 'elements of intrinsic differential geometry', in particular **vector analysis and differential forms**.
- **Motivation:**
  - 'Items of Riemannian flavor already studied' (e.g. by Ambrosio, Bakry, Cheeger, Emery, Gigli, Hino, Kajino, Kigami, Koskela, Ledoux, Sturm, Zhou and others)
  - 'Items of deRham or Hodge type flavor hardly looked at', but in principle accessible using first order derivations (Cipriani, Sauvageot, Weaver and others)
  - Potential applications in physics (magnetic fields, fluid dynamics, optical waveguides) and data science.
  - A number of papers on sub-Riemannian and hypoelliptic setting, at UConn: Baudoin, Chousionis, Gordina.

# Smooth manifold case

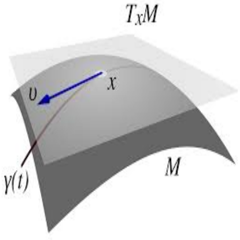
- dimension $\mathbf{dim}\, M$ of $M$ defined as the dimension of the Euclidean space containing its charat images as open sets
- $\mathbf{dim}\, M$ equals its topological dimension $\mathbf{dim_{topo}}\, M$





(Separable and metrizable $X$ has $\mathbf{dim_{topo}}\, X = n$ if $n$ is minimal value s.t. any finite open cover of $X$ has refinement s.t. each $x \in X$ is contained in at most $n + 1$ sets of the refinement.)

- tangent space $T_x M$ at every $x \in M$ is $n$-dim vector space, similarly for cotangent space $T_x^* M$
- in particular, $\Lambda^k T_x^* M = \{0\}$ for $k > n$ (there are no nontrivial $k$-forms)
- $\mathbf{dim}\, T_x M = \mathbf{dim}_{topo} M$ for all $x \in M$



Can talk about 'dimension of (co-)tangent spaces' using concepts of *AF-martingale dimension* $\mathbf{dim_{mart}}$ (Motoo, Watanabe, ...) resp. *index* of Dirichlet form (Hino):

- There is an equiv class of (mutually abs. cont.) minimal energy dominant measures $m$
- The index $p$ of $(\mathcal{E}, \mathcal{F})$ is the smallest integer such that for any $N \in \mathbb{N}$ and any $f_1, \ldots, f_N \in \mathcal{F}$,

$$\mathrm{rank}\left(\frac{d\Gamma(f_i, f_j)}{dm}(x)\right)_{i,j=1}^{N} \leq p \quad \text{for } m\text{-a.e. } x \in X.$$

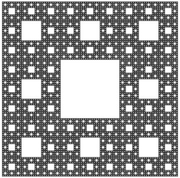
- Hino '08, '10: $p$ well def (indep of choice of $m$) and $\mathbf{dim_{mart}} = p$

'Energy dominant measure': For any $f \in \mathcal{F} \cap C_c(X)$, energy measure $\Gamma(f)$ satisfies $\Gamma(f) << m$; recall

$$\int_X \varphi \, d\Gamma(f) = \mathcal{E}(f\varphi, f) - \frac{1}{2}\mathcal{E}(f^2, \varphi), \quad \varphi \in \mathcal{F} \cap C_c(X).$$

'Minimal': If $m'$ has same property, then $m << m'$.

- Kusuoka '89: $\mathbf{dim}_{\mathrm{mart}} = 1$ for $d$-dim standard Sierpinski gasket (although $\mathbf{dim}_H = \frac{\log(d+1)}{\log 2}$ may be very large)
- Hino '08, '10, '13:
  - P.c.f. self-similar fractals: $\mathbf{dim}_{\mathrm{mart}} = 1$
  - Self-similar generalized Sierpinski carpets: $1 \leq \mathbf{dim}_{\mathrm{mart}} \leq d_s$



- $\mathbf{dim}_{\mathbf{mart}}$ may be interpreted as $m$-essential supremum of dimensions of tangent spaces in a measurable bundle sense (papers of Hino, also Eberle '99, H./Röckner/Teplyaev '13)

## Examples

For $X = \mathbb{R}^n$ with $\mathcal{E}(f) = \int_{\mathbb{R}^n} (\nabla f)^2 dx$, $f \in H^1(\mathbb{R}^n)$, have $\mathbf{dim}_{\mathbf{mart}} = n$.

## Examples

For $X = M$ compact RMf with $\mathcal{E}(f) = \int_M (\nabla f)^2 \, \mathrm{dvol}$, $f \in H^1(M)$, have $\mathbf{dim}_{\mathbf{mart}} = n$.

- Generally $\mathbf{dim}_{\mathbf{topo}}$ and $\mathbf{dim}_{\mathbf{mart}}$ may differ
- In particular: Topo one-dim spaces might carry nontrivial **2**-forms
  ... Somehow counterintuitive (would expect $\mathbf{dim}_{\mathbf{mart}} \leq \mathbf{dim}_{\mathbf{topo}}$)
  ... What happens ?
- Connected to behaviour of (analogs of) the exterior derivation

$$\boldsymbol{d} : \boldsymbol{L}^2(\boldsymbol{M}, \boldsymbol{T}^*\boldsymbol{M}, \mathbf{dvol}) \to \boldsymbol{L}^2(\boldsymbol{M}, \boldsymbol{\Lambda}^2 \boldsymbol{T}^*\boldsymbol{M}, \mathbf{dvol})$$

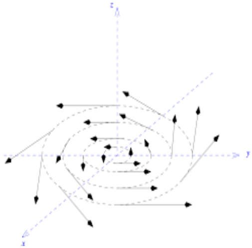
  taking **1**-forms into **2**-forms, $\boldsymbol{a_i}\, \boldsymbol{dx^i} \mapsto \frac{\partial \boldsymbol{a_i}}{\partial \boldsymbol{x_j}}\, \boldsymbol{dx^j} \wedge \boldsymbol{dx^i}$
- Phenomenon does not occur in classical theory
- For simplicity, illustrate issue for **curl**-operator

# Curl of vector fields

- $U \subset \mathbb{R}^3$ open, connected, $\boldsymbol{v} = (v_1, v_2, v_3) : U \to \mathbb{R}^3$ vector field
- $\textbf{curl}\, \boldsymbol{v} = \nabla \times \boldsymbol{v} = \left( \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}, \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right)$

If $\boldsymbol{v}$ velocity field of a fluid flow

- Small ball is made rotate by flow
- axis points in direction of vector field **curl** $\boldsymbol{v}$ (right hand rule)
- Angular speed is $\frac{1}{2}$ of length of **curl** $\boldsymbol{v}$



- Connection to differential forms by duality argument: Given $v = (v_1, v_2, v_3)$, consider $\omega := v_i dx^i$. Then

$$\begin{aligned} d\omega &= \frac{\partial v_i}{\partial x^j}\, dx^j \wedge dx^i \\ &= \left(\frac{\partial v_3}{\partial x^2} - \frac{\partial v_2}{\partial x^3}\right) dx^2 \wedge dx^3 + \ldots \end{aligned}$$

- Two-dim curl: $U \subset \mathbb{R}^2$ open, connected and vector field $u = (u_1, u_2) : U \to \mathbb{R}^2$
- consider $v := (u_1, u_2, 0)$ then function $\mathbf{curl}\, u : U \to \mathbb{R}$,

$$\mathbf{curl}\, u = \frac{\partial u_2}{\partial x}(x, y) - \frac{\partial u_1}{\partial y}(x, y),$$

is third component of $\mathbf{curl}\, v = (0, 0, \mathbf{curl}\, u)$

- In terms of differential forms,

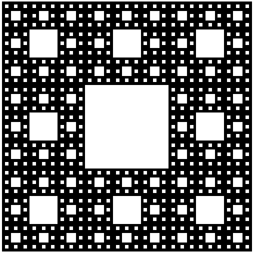
$$d(u_1(x,y)dx + u_2(x,y)dy) = \operatorname{curl} u(x,y)\; dx \wedge dy$$

- Can consider $\operatorname{curl} : L^2(U, \mathbb{R}^2) \to L^2(U)$ as closed unbounded operator
- Next idea: Replace $U$ by a generalized Sierpinski carpet with $2 = \dim_{\mathrm{mart}} > \dim_{\mathrm{topo}} = 1$

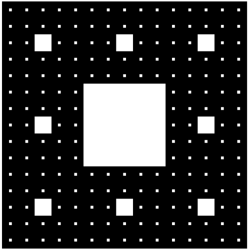
# Sierpinski carpets

Consider non-self-similar generalized Sierpinski carpets studied by Mackay/Tyson/Wildrick '13.

- $\mathbf{a} = (a_1, a_2, \ldots)$ sequence of reals $a_i > 0$ s.t. $\frac{1}{a_i} > 1$ odd integer
- Rewrite $S_{\mathbf{a},0} := [0,1]^2$ as union of congruent closed subsquares of side lengths $a_1$, touching only at boundaries, remove middle one to get a set $S_{\mathbf{a},1}$
- Rewrite $S_{\mathbf{a},1}$ as union of congruent closed subsquares of side lengths $a_1 a_2$, touching only at boundaries, remove middle ones (w.r.t. the subsquares) to get a set $S_{\mathbf{a},2}$
- $S_{\mathbf{a}} := \bigcap_{m \geq 0} S_{\mathbf{a},m}$ generalized Sierpinski carpet associated with sequence $\mathbf{a}$



Standard self-similar carpet $\boldsymbol{S}_{\mathbf{a}}$ with $\boldsymbol{a} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \dots)$



Non self-similar carpet $\boldsymbol{S}_{\mathbf{a}}$ with $\boldsymbol{a} = (\frac{1}{3}, \frac{1}{5}, \frac{1}{7} \dots)$

## Proposition

*(Mackay/Tyson/Wildrick'13)*
*If* $\mathbf{a} \in l^2$ *then* $S_{\mathbf{a}}$ *has positive two dim Lebesgue measure and "all classical-type Sobolev inequalities".*

## Examples

$a_n := \frac{1}{2n+1}$.

Let $\mathbf{a} \in l^2$ be fixed, write $S := S_{\mathbf{a}}$ and $L^2(S)$ for $L^2$-space on $S$ w.r.t. two dim Lebesgue.

# Energy form

Consider

$$\mathcal{E}_S(f) := \int_S (\nabla f(x,y))^2 \, d(x,y), \quad f \in C^1(\mathbb{R}^2).$$

Polarization yields bilinear form.

The form $(\mathcal{E}_S, C^1(\mathbb{R}^2))$ is closable, and its closure $(\mathcal{E}_S, \mathcal{D}_S)$ is a strongly local regular Dirichlet form on $L^2(S)$.

(Follows as in Koskela/Shanmugalingam/Tyson '04, Shanmugalingam '00; Newtonian Sobolev spaces; for $f \in C^1(\mathbb{R}^2)$ the function $|\nabla f|$ is minimal upper gradient of $f$.)

For a vector field $u = (u_1, u_2)$ with $u_1, u_2 \in C^1(\mathbb{R}^2)$, $\mathbf{curl}\, u$ is a continuous function and can be restricted to $S$, and

$$(\mathbf{curl}\, u)|_S \in L^2(S)$$

Therefore: May view $\mathbf{curl}$ as densely defined unbounded operator

$$\mathbf{curl} : L^2(S, \mathbb{R}^2) \to L^2(S)$$

with domain $C^1(\mathbb{R}^2, \mathbb{R}^2)$

Slightly reformulated: Endow $\mathbf{curl}$ with abstract domain $\mathbf{dom}\,\mathbf{curl}$ and let $\mathbf{curl}^*$ be its adjoint with domain $\mathbf{dom}(\mathbf{curl}^*)$

## Theorem (Hinz/T. '15)

*1-dim Hodge-Helmholtz composition holds (despite that* $\dim_H = 2$*).*

## Theorem (Hinz/T. '17)

*Let* $\mathbf{a} \in l^2$ *be such that*

$$\lim_{n\to\infty} \frac{a_1 \cdots a_{n-1}}{a_n} = 0$$

*If* $\mathrm{dom\,curl}$ *contains all smooth vector fields, then* $\mathrm{dom}(\mathrm{curl}^*) \subset L^2(S)$ *is* $\{0\}$ *and, in particular, the operator* $(\mathrm{curl}, \mathrm{dom\,curl})$ *is not closable.*

('$\mathbf{a}$ decays fast enough but not too fast'.)

## Examples

$a_n := \frac{1}{2n+1}$.

# Proof (by contradiction)

Suppose $\mathbf{0} \neq \boldsymbol{u} \in \mathbf{dom}(\mathbf{curl}^*) \subset \boldsymbol{L}^2(\boldsymbol{S})$ and $\mathbf{curl}^* \, \boldsymbol{u} = \boldsymbol{w} \in \boldsymbol{L}^2(\boldsymbol{S}, \mathbb{R}^2)$. Then ex. smooth function $\boldsymbol{f}$ such that $\langle \boldsymbol{u}, \boldsymbol{f} \rangle_{L^2(S)} > \mathbf{0}$.

**Claim**: Can construct sequence $(\boldsymbol{v}_n)_n$ of smooth vector fields $\boldsymbol{v}_n$ s.t.

(a) $\lim_n \mathbf{curl}\, \boldsymbol{v}_n = \boldsymbol{f}$ in $\boldsymbol{L}^2(\boldsymbol{S})$

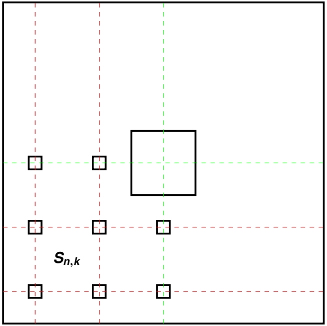
(b) $\lim_n \boldsymbol{v}_n = \mathbf{0}$ in $\boldsymbol{L}^2(\boldsymbol{S}, \mathbb{R}^2)$.

If so, then

$$\begin{aligned} 0 = \lim_n \langle w, v_n \rangle_{L^2(S,\mathbb{R}^2)} = \lim_n \langle \mathrm{curl}^* \, u, v_n \rangle_{L^2(S,\mathbb{R}^2)} &= \lim_n \langle u, \mathrm{curl}\, v_n \rangle_{L^2(S)} \\ &= \langle u, f \rangle_{L^2(S)} \\ &> 0, \end{aligned}$$

what cannot be true. Suffices to show claim.

Cover $S$ by compact subsets $S_{n,k}$ obtained by taking parallels to the axes through the midpoints of all holes of size $\delta_n = a_1 \cdots a_n$. Intersections are Cantor sets and $\mathbf{diam}\, S_{n,k} \leq \sqrt{2}\delta_{n-1}$.

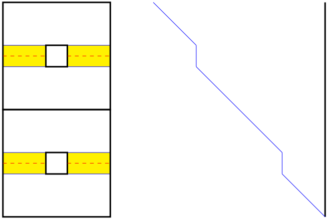


**Step 1:** We show how to choose small nbhs $U_{n,k}$ of the boundaries of the sets $S_{n,k}$ and construct sequence of energy finite functions $g_n$ s.t.

(i) $\nabla g_n$ arbitrarily close to vector field $(0, 1)$ in $L^2(S, \mathbb{R}^2)$

(ii) each $g_n$ is locally constant on each $U_{n,k}$.

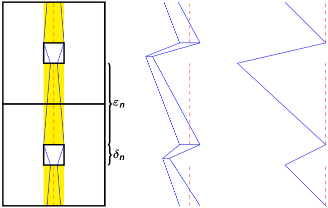
- For fixed $n$, consider Cantor-set parts of boundaries of the $S_{n,k}$ parallel to $x$-axis, let $F_n$ be union of their vertical parallel sets
- Let $\varphi_n$ be a continuous function that is constant in $x$ on $S$, constant in $y$ on $F_n$, and on each connected component of $S \setminus F_n$ differs from $g(x, y) := y$ by an additive constant.



Each $\varphi_n$ is restriction to $S$ of a Lipschitz function, hence of finite energy. Moreover

$$\lim_n \mathcal{E}(g - \varphi_n) = \lim_n \int_{F_n} (\nabla(g - \varphi_n))^2 \, d\lambda^2$$
$$= \lim_n \lambda^2(F_n) \leq \lim_n \frac{\delta_n}{a_1 \cdots a_{n-1}} = \lim_n a_n = 0.$$

Now consider vertical Cantor set parts of the boundaries of the sets $S_{n,k}$. Connect two vertically adjacent holes by rectangles with horizontal side length $\delta_n$ and vertical side length $\varepsilon_n := (1 - a_n)(a_1 \cdots a_{n-1})$. Inscribe trapezoids with lower edge length $\delta_n$ and upper edge length $\delta_{n/2}$ ...

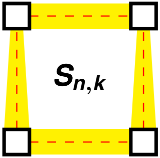



Let $\psi_n$ be the function on $[0,1]^2$ created by putting little tents over each rectangle such that $\psi_n$ is zero on left, right and lower edge of each rectangle, has value $\varepsilon_n$ on the upper (short) edge of the trapezoid and is linear in between. For boundary pieces proceed similarly by 'mirroring to the outside'. Number of such tents is $\leq \frac{2}{a_1 \cdots a_{n-1}}$.

The functions $g_n := \varphi_n - \psi_n$ now satisfy

$$\lim_n \mathcal{E}_S(g - g_n)^{1/2} \leq \lim_n \mathcal{E}_S(g - \varphi_n)^{1/2} + \lim_n \mathcal{E}_S(\psi_n)^{1/2} = 0,$$

what is (i). Each $g_n$ is locally constant on the neighborhood $U_{n,k}$ of $S_{n,k}$ consisting of two rectangles and two trapezoids (with modifications at the boundary of $S$), what shows (ii).



**Step 2:** Let $f_{n,k}$ be one of the values of the function $f$ on $S_{n,k}$, and let $x_{n,k}$ be one of the values of the $x$ coordinate on $S_{n,k}$. There exists a sequence of smooth functions $h_n$ such that

$$\|h_n\|_{\sup} \leqslant a_1 \cdots a_{n-1} \|f\|_{\sup}$$

and on each set $S_{n,k} \setminus U_{n,k}$ we have

$$h_n(x, y) = f_{n,k}(x - x_{n,k}).$$

Then we define

$$v_n = h_n \nabla g_n.$$

Obviously (b) is satisfied, $\lim_n v_n = 0$ in $L^2(S, \mathbb{R}^2)$.

# Strongly local forms on compact spaces

This is a part of the broader program to develop **probabilistic, spectral and vector analysis on singular spaces** by **carefully building approximations by graphs or manifolds**.

$X$ compact metric space, $\mu$ finite Radon measure, full support, $(\mathcal{E}, \mathcal{F})$ strongly local regular Dirichlet form. We consider *differential forms with respect to a 'coordinate sequence' and an energy dominant measure*.

## Theorem

*(Hinz/T. '15)*
*Suppose that $X$ is topologically one-dimensional. Then, under some natural conditions, either the martingale dimension of $(\mathcal{E}, \mathcal{F})$ is one or $(\partial_1, \mathcal{F} \otimes \mathcal{A}_{\mathrm{Lip}})$ is not closable.*

# Canonical diffusions on the pattern spaces of aperiodic Delone sets (Patricia Alonso-Ruiz, Michael Hinz, Rodrigo Trevino, T.)

A subset $\boldsymbol{\Lambda} \subset \mathbb{R}^d$ is a **Delone set** if it is **uniformly discrete**:

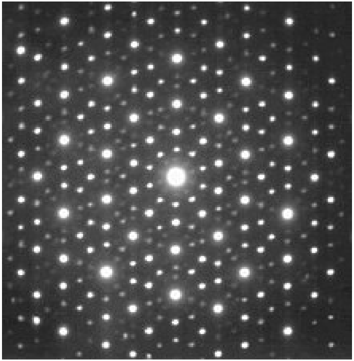
$$\exists \varepsilon > 0 : \; |\vec{x} - \vec{y}| > \varepsilon \quad \forall \vec{x}, \vec{y} \in \Lambda$$

and relatively dense:

$$\exists R > 0 : \; \Lambda \cap B_R(\vec{x}) \neq \varnothing \quad \forall \vec{x} \in \mathbb{R}^d.$$

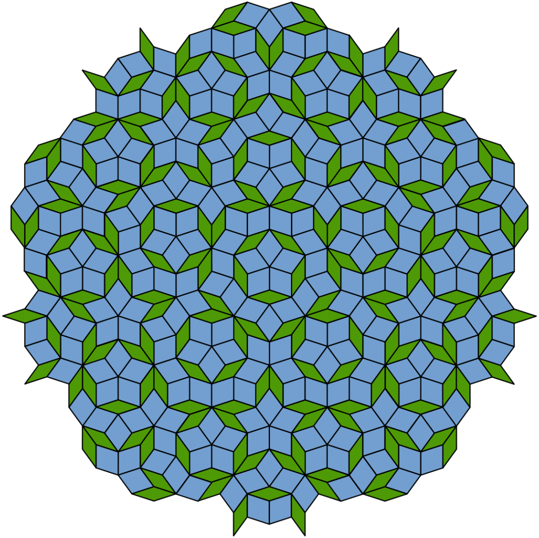
A Delone set has **finite local complexity** if $\forall R > 0 \exists$ finitely many clusters $P_1, \ldots, P_{n_R}$ such that for any $\vec{x} \in \mathbb{R}^d$ there is an $i$ such that the set $B_R(\vec{x}) \cap \Lambda$ is translation-equivalent to $P_i$. A Delone set $\Lambda$ is **aperiodic** if $\Lambda - \vec{t} = \Lambda$ implies $\vec{t} = \vec{0}$. It is **repetitive** if for any cluster $P \subset \Lambda$ there exists $R_P > 0$ such that for any $\vec{x} \in \mathbb{R}^d$ the cluster $B_{R_P}(\vec{x}) \cap \Lambda$ contains a cluster which is translation-equivalent to $P$. These sets have applications in crystallography ($\approx$ **1920**), coding theory, approximation algorithms, and the theory of quasicrystals.

# Electron diffraction picture of a Zn-Mg-Ho quasicrystal



Aperiodic tilings were discovered by mathematicians in the early 1960s, and, some twenty years later, they were found to apply to the study of natural quasicrystals (1982 Dan Shechtman, 2011 Nobel Prize in Chemistry).

# Penrose tiling



# pattern space of a Delone set

Let $\Lambda_0 \subset \mathbb{R}^d$ be a **Delone set**. The **pattern space (hull)** of $\Lambda_0$ is the closure of the set of translates of $\Lambda_0$ with respect to the metric $\varrho$, i.e.

$$\Omega_{\Lambda_0} = \overline{\left\{\varphi_{\vec{t}}\,(\Lambda_0) : \vec{t} \in \mathbb{R}^d\right\}}.$$

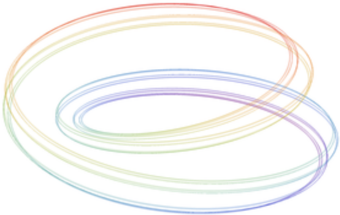
## Definition

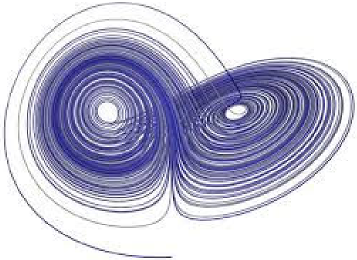
Let $\Lambda_0 \subset \mathbb{R}^d$ be a Delone set and denote by $\varphi_{\vec{t}}\,(\Lambda_0) = \Lambda_0 - \vec{t}$ its translation by the vector $\vec{t} \in \mathbb{R}^d$. For any two translates $\Lambda_1$ and $\Lambda_2$ of $\Lambda_0$ define $\varrho(\Lambda_1, \Lambda_2) = \inf\{\varepsilon > 0 : \ \exists\, \vec{s}, \vec{t} \in B_\varepsilon(\vec{0}) : B_{\frac{1}{\varepsilon}}(\vec{0}) \cap \varphi_{\vec{s}}(\Lambda_1) = B_{\frac{1}{\varepsilon}}(\vec{0}) \cap \varphi_{\vec{t}}(\Lambda_2)\} \wedge 2^{-1/2}$

## Assumption

*The action of $\mathbb{R}^d$ on $\Omega$ is uniquely ergodic:*
*$\Omega$ is a compact metric space with the unique $\mathbb{R}^d$-invariant probability measure $\mu$.*

# Topological solenoids
# (similar topological features as the pattern space $\Omega$):





## Theorem

(i) *If* $\vec{W} = (\vec{W}_t)_{t\geq 0}$ *is the standard Gaussian Brownian motion on* $\mathbb{R}^d$*, then for any* $\Lambda \in \Omega$ *the process* $X_t^\Lambda := \varphi_{\vec{W}_t}(\Lambda) = \Lambda - \vec{W}_t$ *is a conservative Feller diffusion on* $(\Omega, \varrho)$*.*

(ii) *The semigroup* $P_t f(\Lambda) = \mathbb{E}[f(X_t^\Lambda)]$ *is*

$$\boxed{\textbf{self-adjoint on } L^2_\mu\textbf{, Feller but not strong Feller.}}$$

*Its associated Dirichlet form is regular, strongly local, irreducible, recurrent, and has Kusuoka-Hino dimension* $d$*.*

(iii) *The semigroup* $(P_t)_{t>0}$ **does not admit heat kernels with respect to** $\mu$*. It does have Gaussian heat kernel with respect to the not-*$\sigma$*-finite (no Radon-Nykodim theorem) pushforward measure* $\lambda_\Omega^d$

$$p_\Omega(t, \Lambda_1, \Lambda_2) = \begin{cases} p_{\mathbb{R}^d}(t, h_{\Lambda_1}^{-1}(\Lambda_2)) & \textit{if } \Lambda_2 \in \mathrm{orb}(\Lambda_1), \\ 0 & \textit{otherwise.} \end{cases} \tag{1}$$

(iv) **There are no semi-bounded or** $L^1$ **harmonic functions (Liouville-type)**.

# no classical inequalities

**Useful versions of the Poincare, Nash, Sobolev, Harnack inequalities DO NOT HOLD**, except in orbit-wise sense.

# spectral properties

## Theorem

*The unitary **Koopman operators** $U_{\vec{t}}$ on $L^2(\Omega, \mu)$ defined by $U_{\vec{t}}f = f \circ \varphi_{\vec{t}}$ commute with the heat semigroup*

$$\boxed{U_{\vec{t}}P_t = P_tU_{\vec{t}}}$$

*hence commute with the Laplacian $\boldsymbol{\Delta}$, and all spectral operators, such as the unitary Schrödinger semigroup.*

**... hence we may have continuous spectrum (no eigenvalues) under some assumptions even though $\mu$ is a probability measure on the compact set $\Omega$.**

Under special conditions $P_t$ is connected to the evolution of a **Phason**:

"Phason is a quasiparticle existing in quasicrystals due to their specific, quasiperiodic lattice structure. Similar to phonon, phason is associated with atomic motion. However, whereas phonons are related to translation of atoms, phasons are associated with atomic rearrangements. As a result of these rearrangements, waves, describing the position of atoms in crystal, change phase, thus the term "phason" (from the wikipedia)".

# Phason evolution

## Corollary

*The unitary* ***Koopman operators*** $U_{\vec{t}}$ *on* $L^2(\Omega, \mu)$ *defined by* $U_{\vec{t}} f = f \circ \varphi_{\vec{t}}$ *commute with the heat semigroup*

$$\boxed{U_{\vec{t}} P_t = P_t U_{\vec{t}}}$$

*hence commute with the Laplacian* $\Delta$, *and all spectral operators, including the unitary* **Schrödinger semigroup** $e^{i\Delta t}$

$$\boxed{U_{\vec{t}} e^{i\Delta t} = e^{i\Delta t} U_{\vec{t}}}$$

Recent physics work on phason ("accounts for the freedom to choose the origin"): Topological Properties of Quasiperiodic Tilings (Yaroslav Don, Dor Gitelman, Eli Levy and Eric Akkermans Technion Department of Physics)
`https://phsites.technion.ac.il/eric/talks/`
J. Bellissard, A. Bovier, and J.-M. Chez, Rev. Math. Phys. 04, 1 (1992).

# Helmholtz, Hodge and de Rham

## Theorem

*Assume* $d = 1$*. Then the space* $L^2(\Omega, \mu, \mathbb{R}^1)$ *admits the orthogonal decomposition*

$$L^2(\Omega, \mu, \mathbb{R}^1) = \operatorname{Im} \nabla \oplus \mathbb{R}(dx). \tag{2}$$

In other words, the $L^2$-cohomology is 1-dimensional, which is surprising because the **de Rham cohomology is not one dimensional**.


M. Hinz, M. Röckner, T., Vector analysis for Dirichlet forms and quasilinear PDE and SPDE on fractals, Stoch. Proc. Appl. (2013). M. Hinz, T., Local Dirichlet forms, Hodge theory, and the Navier-Stokes equation on topologically one-dimensional fractals, Trans. Amer. Math. Soc. (2015,2017).

Lorenzo Sadun. Topology of tiling spaces, volume 46 of University Lecture Series. American Mathematical Society, Providence, RI, 2008. Johannes Kellendonk, Daniel Lenz, and Jean Savinien. Mathematics of aperiodic order, volume 309. Springer, 2015.

# BV and Besov spaces on fractals with Dirichlet forms (Patricia Alonso-Ruiz, Fabrice Baudoin, Li Chen, Luke Rogers, Nages Shanmugalingam, T.)

References:


**Besov class via heat semigroup on Dirichlet spaces**

**I: Sobolev type inequalities**
arXiv:1811.04267 *J. Funct. Analysis (2020)*

**II: BV functions and Gaussian heat kernel estimates**
arXiv:1811.11010 *Calc. Var. PDE (2020)*

**III: BV functions and sub-Gaussian heat kernel estimates**
arXiv:1903.10078 *Calc. Var. PDE (2021)*

**BV functions and fractional Laplacians on Dirichlet spaces**
arXiv:1910.13330 *revised December 2022*

**+ recent papers by Alonso-Ruiz, Fabrice Baudoin, Li Chen**

# sub-Gaussian Heat Kernel Estimates (sGHKE)

$$p_t(x,y) \sim \frac{1}{t^{d_H/d_w}} \exp\left(-c\frac{d(x,y)^{\frac{d_w}{d_w-1}}}{t^{\frac{1}{d_w-1}}}\right)$$

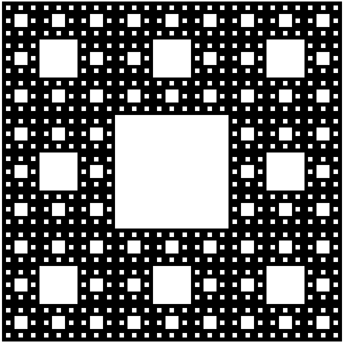
$$distance \sim (time)^{\frac{1}{d_w}}$$

$$d_H = \text{Hausdorff dimension}$$

$$\frac{1}{\gamma} = d_w = \text{“walk dimension” } (\gamma\text{=diffusion index})$$

$$\frac{2d_H}{d_w} = d_S = \text{“spectral dimension” (diffusion dimension)}$$

First example: Sierpiński gasket; Kusuoka, Fukushima, Kigami, Barlow, Bass, Perkins (mid 1980’—)



$$1 = d_t = d_{mart} < d_{tH} = \frac{\ln 2}{\ln 3} + 1 < d_S < d_H = \frac{\ln 8}{\ln 3} < 2 < d_w$$

For Sierpinski carpets there exists a unique Dirichlet form and diffusion process due to [Barlow and Bass 1998, 1999] (see also [B-B-Kumagai-T 2010])

# Open questions:

On the Sierpinski carpet,

$$\kappa = d_W - d_H + d_{tH} - 1 = d_W - d_H + \frac{\log 2}{\log 3}$$

would give the best Hölder exponent for harmonic functions?
[*Strongly supported by numerical results: L.Rogers et al*]

Note that $(d_W - d_H)$ –Hölder continuity follows from the classical results:

**Martin Barlow. Diffusions on fractals. In Lectures on probability theory and statistics (Saint-Flour, 1995)**, volume 1690 of Lecture Notes in Math., pages 1–121. Springer, Berlin, 1998.

**Martin Barlow. Heat kernels and sets with fractal structure. In Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002)**, volume 338 of Contemp. Math., pages 11–40. Amer. Math. Soc., Providence, RI, 2003

Here $d_{tH} = \dfrac{\ln 2}{\ln 3} + 1$ is the topological-Hausdorff dimension of the Sierpinski carpet defined in [Theorem 5.4, **R.Balka, Z.Buczolich, M.Elekes.** ***A new fractal dimension: the topological Hausdorff dimension*****. Adv. Math., 274:881–927, 2015**.]

Roughly speaking,

$$d_{tH} :=$$

$$1 + \inf\{\textit{Hausdorff dim. of boundaries of a base of open sets}\}$$

# BV and weak Bakry-Émery non-negative curvature

## Definition

$BV(X) := KS^{\lambda_1^\#,1}(X) = \mathbf{B}^{1,\alpha_1^\#}(X)$ with $\alpha_1^\# = \frac{\lambda_1^\#}{d_W}$ the $L^1$–Besov critical exponent, and for $f \in BV(X)$

$$\mathrm{Var}(f) := \liminf_{r \to 0^+} \iint_{\Delta_r} \frac{|f(y) - f(x)|}{r^{\lambda_1^\#} \mu(B(x,r))} \, d\mu(y) \, d\mu(x).$$

## Definition

We say that $(X, \mu, \mathcal{E}, \mathcal{F})$ satisfies the weak Bakry-Émery non-negative curvature condition $wBE(\kappa)$ if there exist a constant $C > 0$ and a parameter $0 < \kappa < d_W$ such that for every $t > 0$, $g \in L^\infty(X, \mu)$ and $x, y \in X$,

$$|P_t g(x) - P_t g(y)| \leq C \frac{d(x,y)^\kappa}{t^{\kappa/d_W}} \|g\|_{L^\infty(X,\mu)}. \tag{3}$$

For **nested fractals** we do have $\kappa = d_W - d_H > 0$.
A set has finite perimeter if and only if it has finite boundary, $P(E) \sim \#(\partial E)$.

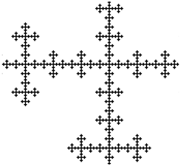
## Theorem (research in progress)

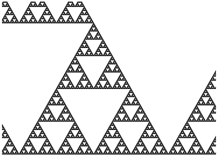
*$f \in BV$ iff $\nabla f$ is a "vector valued Radon measure".*

This is understood in the distributional sense (Hinz, Rogers, Strichartz et al)

## Corollary (research in progress)

1. *on the Vicsek set, any BV function is $\mathbb{R}^1$-$BV$ along each geodesic path.*
2. *on the Sierpiński gasket, any* **BV function is discontinuous**.





For Sierpinski carpets, the critical exponent

$$\alpha_1^* \geq (d_H - d_{tH} + 1)/d_W, \tag{4}$$

where $d_{tH}$ is the topological-Hausdorff dimension. For the classical Sierpinski carpet $d_{tH} = \dfrac{\log 2}{\log 3} + 1$. However the Barlow-Bass theory only yields $wBE(\kappa)$ for $\kappa = d_W - d_H$, not for

$$\kappa = d_W - d_H + d_{tH} - 1.$$

We believe equality holds in (4) for $\alpha_1^*$ and post an open question about the weak Bakry-Émery estimate at criticality. Note that, if $1 < d_S = 2\frac{d_H}{d_W} < 2$, proving $wBE(\kappa)$ for $\kappa > d_W - d_H$ would involve improving the Hölder continuity estimates for harmonic functions in [BB89,BB99,Ba98].
Improved Hölder continuity estimates for harmonic functions are strongly supported by numerical calculations in [L.Rogers et al].
**Conjecture**: for generalized Sierpinski carpets

$$\alpha_1^* = (d_H - d_{tH} + 1)/d_W$$

and the condition $wBE(\kappa)$ is valid for some $\kappa > (d_W - d_H)_+$.

## Open question (Martin Barlow): Are there two fractals with the same values of $d_H$, $d_w$ but different critical exponents $\alpha_1^*$?

Preliminary answer is ***Yes***! ***Compare the SC with*** :

**Martin Barlow. *Which values of the volume growth and escape time exponent are possible for a graph*?**
Rev. Mat. Iberoamericana, 20(1):1–31, 2004.

**Ben Hambly.**
On the asymptotics of the eigenvalue counting function for **random recursive Sierpinski gaskets**.
Probab. Theory Related Fields 117 (2000), no. 2, 221–247.
Brownian motion on a **random recursive Sierpinski gasket**.
Ann. Probab. 25 (1997), no. 3, 1059–1102.
**Brownian motion on a homogeneous random fractal**.
Probab. Theory Related Fields 94 (1992), no. 1, 1–38.

Further examples of spaces to which our theory applies can be constructed by taking products of nested fractals where the condition $\boldsymbol{wBE(d_W - d_H)}$ is valid. The $\boldsymbol{n}$-fold product $\boldsymbol{X^n}$ supports a heat kernel obtained by tensoring, the walk dimension remains $\boldsymbol{d_W}$ on the product and $\boldsymbol{wBE(d_W - d_H)}$ is still true. All that has changed is that the Hausdorff dimension is now $\boldsymbol{nd_H}$.

**Theorem**. If $\boldsymbol{X}$ is a nested fractal, then for every $\boldsymbol{n} \in \mathbb{N}$, the space $\boldsymbol{BV(X^n)} = \mathbf{B}^{1,d_H/d_W}(\boldsymbol{X^n})$ is dense in $\boldsymbol{L^1(X^n, \mu^{\otimes n})}$ and our $\boldsymbol{wBE}$ Assumption is satisfied.

# Why do we care?

Many reasons, including

- **Martin Barlow, Thierry Coulhon, Alexander Grigor'yan. Manifolds and graphs with slow heat kernel decay**. Invent. Math. 144 (2001), no. 3, 609–649.
- Joint Spectra and related Topics in Complex Dynamics and Representation Theory: BIRS Banff 23w5033 May 21–26, 2023
- Quantum gravity and other topics in physics
- Applied mathematics

end of the talk :-)

# Thank you!

reminder: 8th Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals: June 2025