Curvewise differentiable structure on metric measure spaces

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Recall

Every $f \in N^{1,p}(X)$ admits a (unique) p-weak differential $\mathrm{d}f : U \to (\mathbb{R}^n)^*$ wrt p-weak charts (U, φ):

p-weak differential

$$(f \circ \gamma)'_t = \mathrm{d}_{\gamma_t} f((\varphi \circ \gamma)'_t) \quad \text{a.e.} \quad t \in \gamma^{-1}(U)$$

Mod_p-a.e. γ.

Dimension of charts

If $\varphi \in \mathrm{LIP}(X, \mathbb{R}^n)$ is p-independent, then $n \leq \dim_H U$.

Corollary

If $\dim_H X < \infty$ then X admits a p-weak differentiable structure.

Existence and relationships

In general, Cheeger charts are not p-weak charts. (Not even in differentiability spaces.) But in the important PI-case they are.

PI-spaces

Let X be a p-PI space. Then (U, φ) is a Cheeger chart if and only if it is a p-weak chart.

Basic reason

PI implies: for a.e. x, $\mathrm{Lip}(\xi \cdot \varphi)(x) \simeq |D(\xi \cdot \varphi)|(x)$ for all $\xi \in (\mathbb{R}^n)^*$.

- $\implies$ Cheeger charts are p-independent. $\mathrm{Lip}(\mathrm{d}_{C,x} f \cdot \varphi - f)(x) = 0$ $\implies$ $(f \circ \gamma)'_t = \mathrm{d}_{C,x} f((\varphi \circ \gamma)'_t)$ for p-a.e. $\gamma \in \Gamma_U^+$ $\implies$ Cheeger chart is a p-weak chart.
- Conversely, p-weak differential "formally" determined by condition $|D(\mathrm{d}_x f \cdot \varphi - f)|(x) = 0$ a.e.. $\implies$ $\mathrm{Lip}(\mathrm{d}_x f \cdot \varphi - f)(x) = 0$ a.e. $\implies$ p-weak chart is a Cheeger chart.

Constructions

From now on, assume $p \in [1,\infty)$ and X has a p-weak differentiable structure (i.e. covering up to null set by p-weak charts). What consequences can we draw from this?

- Pointwise norm on $T_x^*X = (\mathbb{R}^n)^*$ (a.e. $x \in U$):

$$|\xi|_x := \left\| \frac{\xi((\varphi\circ\gamma)'_t)}{|\gamma'_t|} \right\|_{L^\infty(\boldsymbol{\pi}_x)};$$

  $\boldsymbol{\pi}_x$ a suitable probability measure concentrated on $\{(\gamma,t) : \gamma_t = x\}$;
- for every $f \in N^{1,p}(X)$ we have $|Df|(x) = |\mathrm{d}_x f|_x$ a.e. $x \in X$;
- measurable L^∞ (co-)tangent bundle $T^*X = \bigsqcup_{a.e.\ x} (T_x^*X, |\cdot|_x)$ over X.

  (a la Cheeger, GAFA '99)

Upshot

p-weak diff. str. + techniques allow us to pass from p-a.e. curvewise information to a.e. pointwise information.

Consequences

Consequences

- Concrete description of Gigli's co-cotangent module: $L^p(T^*X) = \Gamma_p(T^*X)$ (= L^p-integrable sections over T^*X) isometrically isomorphic;
- $N^{1,2}(X)$ is Hilbertian (i.e. X is inf. Hilb.) if and only if $|\cdot|_x$ is an inner product a.e. x;
- $N^{1,p}(X)$ is reflexive for $1 < p < \infty$ (renorming tangent spaces);
- $\mathrm{LIP}_b(X)$ is dense **in norm** in $N^{1,p}(X)$, also when $p = 1$ (density in energy (AGS, Rev. Mat. '13, EB, Arxiv '22)+reflexivity).

Further applications

- Tensorization of charts: if (U, φ) n-dim p-weak chart on X, (V, ψ) m-dim p-weak chart on Y, then $(U \times V, \varphi \times \psi)$ $n+m$-dim p-weak chart on $X \times Y$;
- partial progress in the tensorization problem for Sobolev spaces: isometric inclusion $N^{1,p}(X \times Y) \subset J^{1,p}(X, Y)$.

The space $J^{1,p}$ (Ambrosio–Gigli–Savare, Duke Math., '13; Ambrosio–Pinamonti–Speight, Adv.Math '15)

Consists of $f \in L^p(X \times Y)$ s.t.

- $f_x := f(x, \cdot) \in N^{1,p}(Y)$ a.e. $x \in X$;
- $f^y := f(\cdot, y) \in N^{1,p}(X)$ a.e. $y \in Y$;
- $[f]_{J^{1,p}} := \int_X \int_Y [|Df_x|^p(y) + |Df^y|^p(x)] \mathrm{d}\mu_X(x) \mathrm{d}\mu_Y(y) < \infty$.

- The differential of $f \in N^{1,p}$ “should be” $\mathrm{d}f = (\mathrm{d}f^y, \mathrm{d}f_x)$ a.e. (x, y) (only candidate, by testing along X-curves and Y-curves).
- Problem: X- and Y- differentials cannot a priori control along “diagonal” curves (γ_X, γ_Y).
- Look at $f(x, y) = h(u(x), v(y))$, $h \in C^1$, first, use approximation.
- Yields $\mathrm{d}f = (\mathrm{d}f^y, \mathrm{d}f_x)$ a.e. for Sobolev functions.
- That the norm tensorizes requires more work.
- Doesn't tell us anything about $f \in J^{1,p}$ (no known way to approximate them).

Questions and possible directions

1. $N^{1,p}(X \times Y) = J^{1,p}(X \times Y)$? Known for $p = 2$ when X, Y have p-weak diff. str.
2. $p = \infty$ case of weak diff. str. Relevant for *curve fragment* diff. str. (ongoing work).
3. Infinitesimal structure of generic spaces:
    - Density of directions $\{(\varphi \circ \gamma)'_t : \gamma_t = x\} \subset \mathbb{R}^n$ a.e. x?
    - Controlling non-injectivity of charts: $\varphi^{-1}(y)$ does not contain curves a.e. $y \in \varphi(U)$?

The end

Thank you for your attention.