Coarea Inequality for Sobolev Functions on Metric Spaces

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Smooth Functions on Rough Spaces and Fractals, with Connections to Curvature Functional Inequalities

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JYVÄSKYLÄN YLIOPISTO

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- **Question**: Does there exist $C = C(n)$ such that for a reasonable class of "n-dimensional" metric spaces we have

$$\int_{\mathbb{R}}^{*} \int_{u^{-1}(t)} g \, d\mathcal{H}^{n-1} dt \leq C \int_{X} g \rho \, d\mathcal{H}^n$$

  for all $u\colon X \to \mathbb{R}$ and any upper gradient ρ of u?

Eilenberg's Inequality

- Federer: for any X, any $s \geq 1$, any Lipschitz function $u\colon X \to \mathbb{R}$, and any $A \subset X$,

$$\int_{\mathbb{R}}^{*} \mathcal{H}^{s-1}(u^{-1}(t) \cap A)dt \leq \frac{2\omega_{s-1}}{\omega_s} \mathrm{Lip}\,(u)\mathcal{H}^s(A)\,.$$

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where

$$\mathrm{lip}(u)(x) := \limsup_{x \neq y \to x} \frac{|u(y) - u(x)|}{d(y, x)}.$$

Coarea Inequality vs. Eilenberg's Inequality

- By work of **Cheeger**, $\mathcal{H}^n$ is doubling + a $(1,1)$-Poincaré inequality, then local lip is a minimal upper gradient, so, Eilenberg's Inequality implies the Coarea Inequality.

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- But in general, we could have that the minimal upper gradient ρ_u is strictly smaller than $\mathrm{lip}(u)$ on a set of positive measure.
- Motivated by **uniformization theory**, we wish to avoid stringent geometric assumptions.

Our Main Theorems

X homeomorphic to $\mathbb{R}^2$, and $\mathcal{H}^2$ locally finite (*metric surface* for short)

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Theorem (Esmayli-Ikonen-Rajala, 2022)

If $u\colon X \to \mathbb{R}$ is MONOTONE and has a p-integrable upper gradient ρ, for some $p \geq 1$, then with $\kappa = (4/\pi) \cdot 200$,

$$\int_{\mathbb{R}} \int_{u^{-1}(t)} g \, d\mathcal{H}^1 \, dt \leq \kappa \int_X g \rho \, d\mathcal{H}^2.$$

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Theorem (same paper)

There exists a Lipschitz function u on a metric surface X such that

$$\int_{\mathbb{R}} \int_{u^{-1}(t)} g \, d\mathcal{H}^1 \, dt > 0, \quad \text{while} \quad \int_X g \rho \, d\mathcal{H}^2 = 0.$$

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(g is the characteristic function of a closed subset $A \subset X$.)

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Theorem (Esmayli-Ikonen-Rajala, 2022)

Suppose X is a metric surface and $u\colon X \to \mathbb{R}$ is WEAKLY monotone. If u has a locally p-integrable upper gradient for some $p \geq 2$, then u is continuous.

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- Take $u(x, y) = x$. Use Fubini to finish.

Main References

- ESMAYLI, B., HAJŁASZ, P.: The coarea inequality. Annales Fennici Mathematici, 46(2), 965–991. (2021)

- ESMAYLI, B., IKONEN, T., RAJALA K.: Coarea inequality for monotone functions on metric surfaces (Preprint)

- MEIER, D., WENGER, S.: Quasiconformal almost parametrizations of metric surfaces (Preprint)

- NTALAMPEKOS, D., ROMNEY, M.: Polyhedral approximation of metric surfaces and applications to uniformization, Duke Math. J..

- RAJALA, K. Uniformization of two-dimensional metric surfaces. Invent. Math., 207(3):1301–1375, 2017.