On the associativity of Kontsevich's (affine) star product up to order 7

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Abstract: We show that the expansion of the Kontsevich star product $\star$ mod $\bar{o}(\hbar^6)$ found by Banks–Panzer–Pym (2018) is associative up to $\bar{o}(\hbar^6)$. We find and reduce the formula $\star_{\text{aff}}$ mod $\bar{o}(\hbar^7)$ for the expansion of the Kontsevich star-product restricted to affine Poisson brackets; it is associative up to $\bar{o}(\hbar^7)$. Moreover, we contrast the associativity mechanisms at orders $\leqslant 6$ against order 7. The results are obtained using the newly developed free software package **gcaops** (*Graph Complex Action on Poisson Structures*) for SageMath; see `https://github.com/rburing/gcaops` for the code.

Star products: deform the pointwise product

$M := \mathbb{R}^d$ with coordinates $x^1, \ldots, x^d$; $C^\infty(M) := \{f : M \to \mathbb{R} \mid f \text{ smooth}\}$ is an associative algebra w.r.t. the pointwise product $(f \cdot g)(x) = f(x) \cdot g(x)$.

Now, deform!

$A := C^\infty(M)[[\hbar]] := \{\sum_{n=0}^\infty f_n \hbar^n \mid f_n \in C^\infty(M)\}$ formal power series in $\hbar$.

Definition. A *star product* is an $\mathbb{R}[[\hbar]]$-bilinear product $\star : A \times A \to A$ given for $f, g \in C^\infty(M)$ by

$$f \star g = f \cdot g + \hbar \cdot B_1(f,g) + \frac{\hbar^2}{2!} \cdot B_2(f,g) + \frac{\hbar^3}{3!} \cdot B_3(f,g) + \ldots$$

where B_k are bi-linear bi-differential operators vanishing on constants, such that $\star$ is *associative*, i.e. $(f \star g) \star h = f \star (g \star h)$.

Example. On $\mathbb{R}^2$ with Cartesian coordinates x, y, the formula

$$f \star g = f \cdot g + \hbar \cdot x \cdot (\partial_x f \cdot \partial_y g - \partial_y f \cdot \partial_x g) + \hbar^2 \cdot x^2 \cdot (\partial_x \partial_x f \cdot \partial_y \partial_y g - 2 \cdot \partial_x \partial_y f \cdot \partial_x \partial_y g + \partial_y \partial_y f \cdot \partial_x \partial_x g)/2 + \hbar^2 \cdot x \cdot (\partial_y \partial_y f \cdot \partial_x g - \partial_x \partial_y f \cdot \partial_y g + \partial_x f \cdot \partial_y \partial_y g - \partial_y f \cdot \partial_x \partial_y g)/3 + \hbar^2 \cdot \partial_y f \cdot \partial_y g / 6 + \bar{o}(\hbar^3)$$

(where $\partial_x = \frac{\partial}{\partial x}$ and $\partial_y = \frac{\partial}{\partial y}$) defines a star product modulo $\bar{o}(\hbar^3)$.

A touch of associativity = Poisson

Proposition. If $\star$ is associative, then

$$\{f,g\}_\star := \frac{f \star g - g \star f}{2\hbar}\Big|_{\hbar=0} = \tfrac{1}{2}(B_1(f,g) - B_1(g,f))$$

is a *Poisson bracket*:

- Bi-derivation: $\{f,g\} = \sum_{i,j=1}^d P^{ij} \cdot \partial_i(f) \cdot \partial_j(g)$, $\quad P^{ij} \in C^\infty(M), \partial_i := \frac{\partial}{\partial x^i}$
- Skew-symmetry: $P^{ij} = -P^{ji}$
- Jacobi: $\sum_{s=1}^d (P^{si}\partial_s P^{jk} + P^{sj}\partial_s P^{ki} + P^{sk}\partial_s P^{ij}) = 0 \quad \forall i,j,k = 1, \ldots, d.$

Example. Continuing the Example above, we have that $\{f,g\}_\star = B_1(f,g) = x \cdot (\partial_x f \cdot \partial_y g - \partial_y f \cdot \partial_x g)$ is a Poisson bracket on $\mathbb{R}^2$.

Deformation quantization of Poisson manifolds

Problem. Given a Poisson bracket $\{-,-\}$, find a star product $\star$ such that $\{-,-\}_\star = \{-,-\}$.

Theorem (*Kontsevich 1997*). For every Poisson bracket P on $\mathbb{R}^d$ the following star product

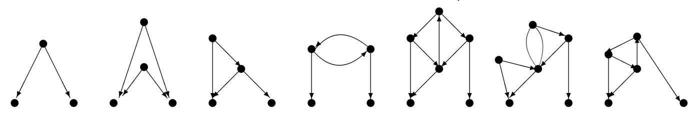
$$f \star g = f \cdot g + \sum_{k \geqslant 1} \frac{\hbar^k}{k!} \sum_{\Gamma \in G_{2,k}} w(\Gamma) \cdot \Gamma(P)(f,g)$$

is associative and $\{-,-\}_\star = P$.

Kontsevich's graphs

A *Kontsevich graph* $\Gamma \in G_{\ell,k}$ is an oriented graph made of k wedges on ℓ ordered sinks (wedges are not necessarily added one-by-one).

Examples.



Graphs ⇄ diff. operators

Let P be a Poisson structure. To $\Gamma \in G_{\ell,k}$ associate a differential operator $\Gamma(P)$ of ℓ arguments:

- Ascribe indices to edges; put P^{ij} in vertex
- Edge $m \implies \partial_m$ acts on target vertex's content
- Multiply (differentiated) contents of vertices
- Sum over all indices. $\Gamma_1 = \ \mapsto P^{ij}\ \partial_i \otimes \partial_j$

$\Gamma_2 = \ \mapsto P^{k\ell} \cdot \partial_\ell P^{ij}\ \partial_k \partial_i \otimes \partial_j$

Kontsevich graphs ↪ $\overline{\mathbb{H}}$

$V(\Gamma) \hookrightarrow \overline{\mathbb{H}}$, sinks to $\{0,1\}$, edges as geodesics.

Harmonic angle form

$$\mathrm{d}\varphi(p,q) := \mathrm{d}\,\mathrm{Arg}\left(\frac{q-p}{q-\bar{p}}\right).$$

Associate to $\Gamma \in \tilde{G}_{2,k}$ a $2k$-form

$$\omega_\Gamma := \bigwedge_{j=1}^k \mathrm{d}\varphi(p_j, p_{\mathrm{Left}(j)}) \wedge \mathrm{d}\varphi(p_j, p_{\mathrm{Right}(j)}).$$

The *graph weight* of Γ is

$$w(\Gamma) := \frac{1}{(2\pi)^{2k}} \int_{C_k(\mathbb{H})} \omega_\Gamma,$$

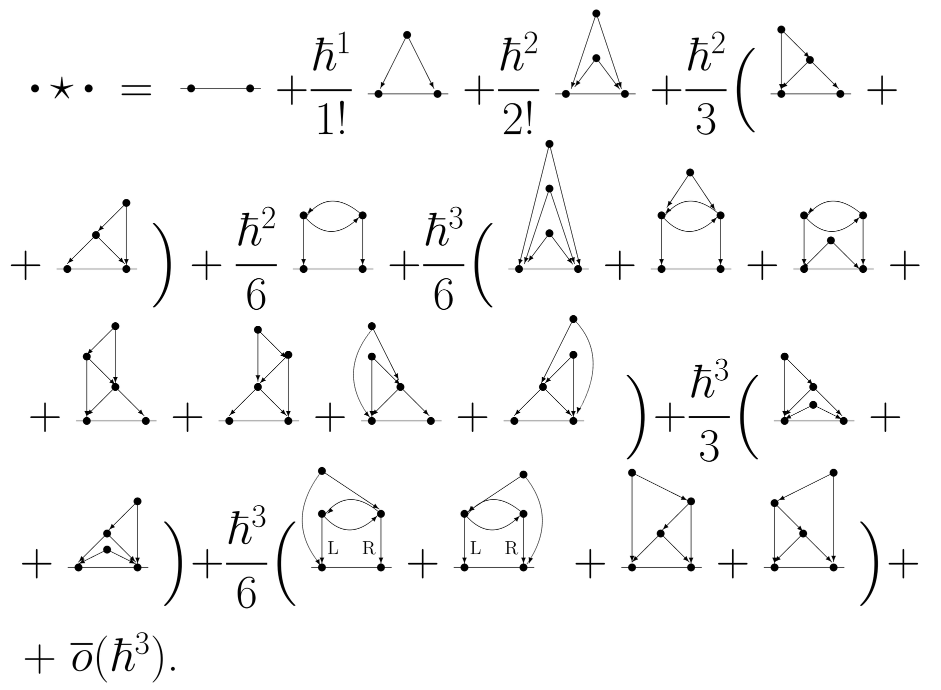
where the integral is taken over

$$C_k(\mathbb{H}) := \{(p_1, \ldots, p_k) \in \mathbb{H}^k : p_i \neq p_j\}.$$

Examples. $\quad \omega_{\Gamma_1} = \mathrm{d}\varphi(p_1,0) \wedge \mathrm{d}\varphi(p_1,1)$,

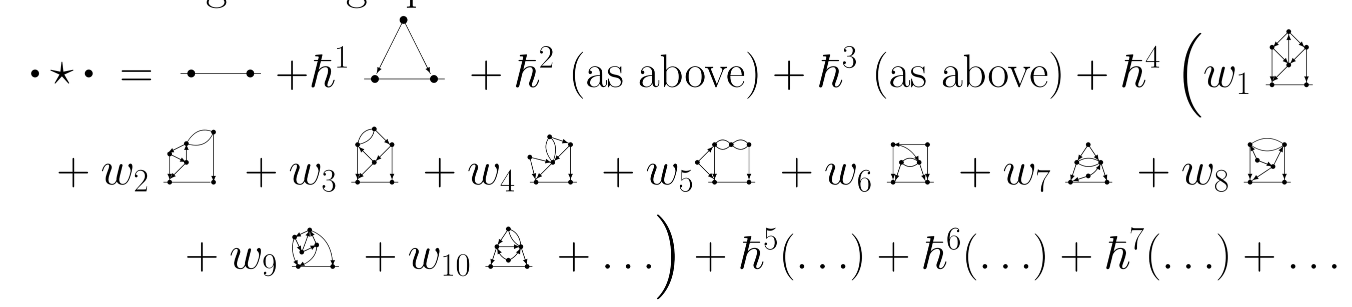
$\omega_{\Gamma_2} = \mathrm{d}\varphi(p_1,0) \wedge \mathrm{d}\varphi(p_1,1) \wedge \mathrm{d}\varphi(p_2,0) \wedge \mathrm{d}\varphi(p_2,p_1)$

Star product: graphically



The hunting of the star product

Let the weights of graphs at $\hbar^k$ be undetermined variables:



Strategy. Find relations between $w(\Gamma)$'s and solve the system of equations. This is made possible by the new **gcaops** software.

Known relations between weights

- Multiplicativity: $w(\ ) = w(\ )^2 \rightsquigarrow$ prime and composite graphs.
- Skew-symmetry: $w(\ ) = -w(\ ) \rightsquigarrow$ also "zero graphs" with $\omega_\Gamma = 0$.
- Mirror-reflection: $w(\ ) = w(\ )$, in general with sign $(-1)^k$.
- Any sink receives no edges $\implies$ weight is zero (by dimension count).
- Cyclic weight relations (Shoikhet–Felder–Willwacher 2008):

$$w(\Gamma) = (-1)^n \sum_{\substack{E \subset \mathrm{Edge}(\Gamma) \\ \forall e \in E, \mathrm{target}(e) \neq 0}} (-1)^{N_0(\Gamma_E)} \cdot w(\Gamma_E).$$

Γ_E: Γ but edges in E directed to 0, $N_0(\Gamma_E) = \#\{e \in \Gamma_E \mid \mathrm{target}(e) = 0\}$.

- Graphs containing an "eye on ground" have zero weight.
- Families of graphs with known weights (e.g. Bernoulli).
- Relations between weights from associativity:

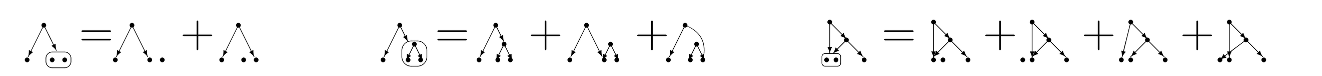
M1. $\mathrm{Assoc}_\star(P)(f,g,h)(x) = 0$ as a number $\in \mathbb{R}$, for fixed P, f, g, h, x.
M2. $\mathrm{Assoc}_\star(P)(f,g,h) = 0$ as polynomial for $P^{ij}, f, g, h \in \mathbb{R}[x_1, \ldots, x_d]$.
M3. $\mathrm{Assoc}_\star(P[f_1, \ldots, f_r]) \equiv 0$ as a differential operator on $f_1, \ldots, f_r$ if P^{ij} is differential polynomial in $f_1, \ldots, f_r$, e.g. $P^{ij} = \varepsilon_{ijk} u \partial_k \varphi$ on $\mathbb{R}^3$.

$\star$ mod $\bar{o}(\hbar^k)$ for $k = 4, 5, 6, 7$: progress

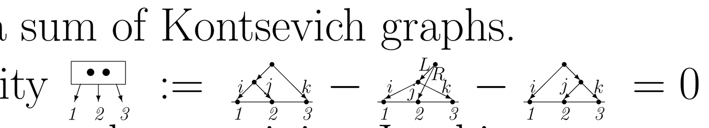
$\star$ mod $\bar{o}(\hbar^4)$: Buring–Kiselev (2017) up to 10 parameters, Banks–Panzer–Pym (2017) full. We verify ✓.
$\star$ mod $\bar{o}(\hbar^5)$: Banks–Panzer–Pym (2017–18). We verify relations (2018).
$\star$ mod $\bar{o}(\hbar^6)$: Banks–Panzer–Pym (2018). We verify associativity (2022).
Restriction to Poisson brackets with affine coefficients:
$\star_{\text{aff}}$ mod $\bar{o}(\hbar^7)$: We verify associativity and rationality (2022); cf. Ben-Amar (2003) for rationality.

Associator and Jacobi: graphically

$\mathrm{Assoc}(\star) = (\bullet \star \bullet) \star \bullet - \bullet \star (\bullet \star \bullet)$, calculated by bilinearity and e.g.:

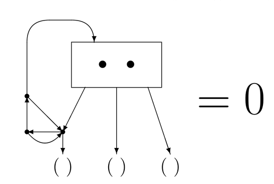


Assoc($\star$) is not identically zero as a sum of Kontsevich graphs.
Need to factor through Jacobi identity



and its differential consequences, by graphs containing Jacobiator.

$\rightsquigarrow$ *Leibniz* graphs, e.g.

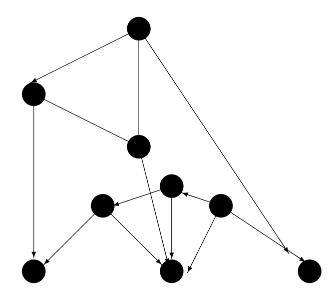


Assoc($\star$) = $\sum$ Leibniz

Theorem (Buring–Kiselev 2017-2022).
- Assoc($\star$) mod $\bar{o}(\hbar^4)$ is a sum of 0th layer Leibniz graphs
- Assoc($\star$) mod $\bar{o}(\hbar^5)$ is a sum of 0th layer Leibniz graphs
- Assoc($\star$) mod $\bar{o}(\hbar^6)$ is a sum of 0th layer Leibniz graphs
- Assoc($\star$) mod $\bar{o}(\hbar^7)$ needs 1st layer of Leibniz graphs.

Layers of Leibniz graphs

Necessary Leibniz graphs are not all obtained by contraction of an edge between aerial vertices in Kontsevich graphs in the associator (0th layer).
Example. At $\hbar^7$, the following Leibniz graph



has weight $-3/128 \cdot \zeta(3)^2/\pi^6 + 31/725760$; yet its expansion does not appear in Assoc($\star$) itself. Indeed, contracting Kontsevich subgraphs over $\{0,1\}$ or $\{1,2\}$ in its expansion $\implies$ the outer graph is composite, with one of the factors (the one containing the 3-cycle) having zero weight.

Reduce $\star_{\text{aff}}$ by Jacobi

$\star_{\text{aff}}$ mod $o(\hbar^7)$: # 1423 K. graphs. Coefficients $\in \mathbb{Q} + \mathbb{Q} \cdot \zeta(3)^2/\pi^6$. Assimilating as much as possible into Leibniz graphs $\rightsquigarrow \star_{\text{aff}}^{\text{red}}$ mod $\bar{o}(\hbar^7)$ with 326 Kontsevich graphs and $\mathbb{Q}$-coefficients. NB: All such $\sum$ K. graphs encode same formula $f \star_{\text{aff}} g$! *You can use it.*

References

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