A quantitative Bogomolov-type result for curves over function fields

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Motivation: The Manin–Mumford Conjecture

- K field of characteristic 0,
- X smooth projective curve of genus $g \geq 2$ defined over K,
- $D \in \mathrm{Div}^1(X)$ degree 1 divisor on X,
- $j_D \colon X \to J_X = \mathrm{Pic}^0(X),\ P \mapsto P - D$ Abel–Jacobi map.

Manin–Mumford Conjecture (Raynaud's Theorem 1983)

$$\# j_D(X(\overline{K})) \cap J_X(\overline{K})_{\mathrm{tors}} < \infty.$$

Further Questions:

a. **Positive Characteristic:** Naive analogue is false: If $\overline{K} = \overline{\mathbb{F}_p}$, every $\overline{K}$-point is torsion. Scanlon (2001) and Pink–Roessler (2004) found and proved a suitable analogue.
b. **Uniformity:** Is $\# j_D(X(\overline{K})) \cap J_X(\overline{K})_{\mathrm{tors}}$ bounded uniformly in g? Proved by Kühne (2021) in characteristic 0.
c. **Quantitative Bounds:** For example,
$\# j_D(X(\overline{K})) \cap J_X(\overline{K})_{\mathrm{tors}} \leq p^{4g} 3^g (p(2g-2) + 6g) g!$
by Buium (1996) for number fields K, where p is a prime depending on the reduction behavior of X and K.

Quantitative Bogomolov-type result

Here we want to address all three questions in the case of function fields in the more general setup of the Bogomolov Conjecture.

Theorem (Looper–Silverman–W.)

B/k smooth projective curve, $K = k(B)$ its function field, $g \geq 2$, $\epsilon \in \left[0, \frac{1}{4(g^2-1)}\right)$. For any smooth projective geometrically connected non-isotrivial curve X/K of genus g and any $D \in \mathrm{Div}^1(X)$ it holds

$$\#\left\{P \in X(\overline{K}) \mid h_{\mathrm{NT}}(j_D(P)) \leq \epsilon\omega^2\right\} \leq \left\lfloor \frac{16g^4+36g^2-26g-2}{(g-1)^2(1-4(g^2-1)\epsilon)} \right\rfloor + 1,$$

where ω^2 is the self-pairing of the admissible dualizing sheaf ω on X.

Corollary ($\epsilon = 0$)

With the same notation as in the theorem:

$$\# j_D(X(\overline{K})) \cap J_X(\overline{K})_{\mathrm{tors}} \leq 16g^2 + 32g + 124.$$

Structure and History of the proof

First, Looper–Silverman (LS) and I (W) independently proved a weaker version of the theorem. Both proofs can be structured as follows:

A. Bound of Zhang's admissible pairing for small height points

Bound of the admissible pairing of the points of small height from below in terms of the Green functions of the reduction graphs and from above in terms of $-\omega^2$. The bound in (W) improves the bound in (LS) by factor 7.

B. Bound of Green functions on metrized reduction graphs

By potential theory on metrized graphs one bounds the sup of the Green function of any metrized graph in terms of Zhang's φ-invariant for metrized graphs. The bound in (LS) improves the bound in (W) by factor 7.

C. Lower bound for ω^2 in terms of the φ-invariants

Known before for $\mathrm{char}\, K = 0$. For $\mathrm{char}\, K > 0$ only in (W) using the arithmetic Hodge index theorem for adelic line bundles on X^2.

We got the theorem by merging our work.

Preliminaries: Zhang's admissible pairing

$\mathrm{Pic}_a(X_{\overline{K}})$: Group of admissible adelic metrized line bundles on $X_{\overline{K}}$.
Zhang defined a pairing

$$(\cdot,\cdot)\colon \mathrm{Pic}_a(X_{\overline{K}}) \times \mathrm{Pic}_a(X_{\overline{K}}) \to \mathbb{R},$$

- There is a *canonical* admissible metric on the dualizing bundle ω.
- Any $D \in \mathrm{Div}(X_{\overline{K}})$ induces canonically an admissible metric on $\mathcal{O}_{X_{\overline{K}}}(D)$.

$\Rightarrow$ *Expressions like* $(n_1\omega + D_1, n_2\omega + D_2)$ *make sense.*

Some Properties:

1. $(\cdot,\cdot)$ is bilinear and symmetric.
2. Adjunction formula: $(\omega, P) = -(P, P)$ for any $P \in X(\overline{K})$.
3. If $\overline{\mathcal{L}} \in \mathrm{Pic}_a(X_{\overline{K}})$ and $\deg \mathcal{L} = 0$, then $h_{\mathrm{NT}}(\mathcal{L}) = -(\overline{\mathcal{L}}, \overline{\mathcal{L}})$.
4. For different $P, Q \in X(K)$:
   $(P, Q) = i(P, Q) + \sum_{v \in |B|} g_v(R_v(P), R_v(Q))$, where
   - $\mathcal{X}$ model of X over B.
   - $i(P, Q)$ intersection number of the closures of P and Q in $\mathcal{X}$.
   - $\Gamma_v(X)$ metrized reduction graph of $\mathcal{X}$ at $v \in |B|$.
   - $R_v\colon X(K) \to \Gamma_v(X)$ restriction map.
   - g_v (canonical) Green function on $\Gamma_v(X)$.

A1: Bound of admissible pairings by ω^2

For simplicity we assume $\epsilon = 0$. The general case works similar. Let

$P_1, \dots, P_s \in X(\overline{K})$ with $j_D(P_i) \in J_X(\overline{K})_{\mathrm{tors}}$ and $F = P_1 + \dots + P_s \in \mathrm{Div}(X)$.

We have to bound s and may assume $s \geq 2$. We compute the Néron–Tate height of $s\omega - (2g-2)F \in \mathrm{Div}^0(X)$ by Zhang's admissible pairing

$$0 \leq h_{\mathrm{NT}}(s\omega - (2g-2)F) = -(s\omega - (2g-2)F, s\omega - (2g-2)F).$$

By bilinearity and the adjunction formula $(\omega, P_j) = -(P_j, P_j)$ this implies

$$\omega^2 \leq -\frac{4(g-1)}{s} \sum (P_j, P_j) - \frac{4(g-1)^2}{s^2} \sum (P_j, P_j) - \frac{4(g-1)^2}{s^2} \sum_{j \neq k} (P_j, P_k).$$

Since $P_j - P_k = j_D(P_j) - j_D(P_k)$ is also torsion,

$$0 = h_{\mathrm{NT}}(P_j - P_k) = -(P_j - P_k, P_j - P_k) = 2(P_j, P_k) - (P_j, P_j) - (P_k, P_k).$$

Summing over all $j \neq k$ we get $\sum (P_j, P_j) = \frac{1}{s-1} \sum_{j \neq k} (P_j, P_k)$ and hence,

$$\omega^2 \leq -\frac{4g(g-1)}{s(s-1)} \sum_{j \neq k} (P_j, P_k).$$

A2: Bound of admissible pairings by Green function

From the previous slide:

$$\omega^2 \leq -\tfrac{4g(g-1)}{s(s-1)} \sum_{j \neq k} (P_j, P_k). \tag{1}$$

We may assume $P_1, \ldots, P_s \in X(K)$ after a finite field extension. For $P_j \neq P_k$ the pairing (P_j, P_k) can be computed by

$$(P_j, P_k) = i(P_j, P_k) + \sum_{v \in |B|} g_v(R_v(P_j), R_v(P_k)) \geq \sum_{v \in |B|} g_v(R_v(P_j), R_v(P_k)).$$

By Elkies' lower bound for metrized graphs by Baker–Rumely

$$\sum_{j \neq k} g_v(R_v(P_j), R_v(P_k)) \geq -s \cdot \sup_{x \in \Gamma_v(X)} g_v(x, x).$$

$$\Rightarrow \quad \sum_{j \neq k} (P_j, P_k) \geq -s \cdot \sum_{v \in |B|} \sup_{x \in \Gamma_v(X)} g_v(x, x).$$

Combining with (1),

$$\omega^2 \leq \tfrac{4g(g-1)}{s-1} \sum_{v \in |B|} \sup_{x \in \Gamma_v(X)} g_v(x, x). \tag{2}$$

B & C: Bounding Green functions and ω^2 by φ

By potential theory on metrized graphs one bounds the Green function

$$\sup_{x,y\in\Gamma_v(X)} g_v(x,y) \leq \frac{8g^4+18g^2-13g-1}{2g(2g+1)(g-1)^2}\varphi(\Gamma_v(X)), \tag{3}$$

where $\varphi(\Gamma_v(X))$ is Zhang's φ-invariant of the metrized graph $\Gamma_v(X)$.
By the Hodge index theorem for adelic line bundles on X^2 one proves

$$\omega^2 \geq \frac{g-1}{2g+1}\sum_{v\in|B|}\varphi(\Gamma_v(X)). \tag{4}$$

Combining (2), (3) and (4)

$$\frac{g-1}{2g+1}\sum_{v\in|B|}\varphi(\Gamma_v(X)) \leq \omega^2 \leq \frac{4g(g-1)}{s-1}\cdot\frac{8g^4+18g^2-13g-1}{2g(2g+1)(g-1)^2}\sum_{v\in|B|}\varphi(\Gamma_v(X)).$$

As $\omega^2 > 0$, solving for s yields

$$s \leq \left\lfloor \frac{16g^4+37g^2-28g-1}{(g-1)^2}\right\rfloor \leq 16g^2+32g+124,$$

or $s \leq 1$ if $\sum_{v\in|B|}\varphi(\Gamma_v(X)) = 0$
($\Leftrightarrow$ if X has everywhere good reduction).

Thank you for your attention!