Toric Degenerations

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1 Introduction

Toric varieties are popular objects in algebraic geometry due to a dictionary between their geometric properties (e.g. dimension, degree) and properties of associated combinatorial objects (e.g. fans, polytopes). For some purposes, this dictionary can be extended from toric varieties to varieties having a *toric degeneration*: a (flat) family of varieties that share many properties (e.g. dimension, degree, Hilbert-polynomial). If a variety appears as a fiber of this family, and another fiber is a toric variety, then one can hope to gain information about the original variety from the combinatorics of the toric variety.

Toric degenerations form an actively studied research area, in the intersection of algebraic geometry, representation theory, symplectic and Poisson geometry, cluster algebras, and combinatorics. In particular, currently known constructions of toric degenerations arise from techniques from commutative algebra, algebraic and tropical geometry, representation theory and combinatorics. In recent years, there have been several new applications of toric degenerations in both pure and applied mathematics, including: numerical algebraic geometry, symplectic and Poisson geometry, algebraic statistics, geometric modeling and numerical solutions to equations.

The study of toric degenerations naturally divides into two parts: constructions and applications. Moreover, for both parts, there are approaches based on different research areas. Although experts who work on these different aspects of toric degenerations do sometimes meet and exchange ideas, as far as we are aware, there have not been many events bringing together this entire community. Therefore, in this meeting, we aim to improve the communication within this community.

Guiding principles. Our goal was to invite a wide range of researchers who work on toric degenerations, equally representing the different subject areas mentioned above. The purpose of bringing together researchers who would not normally interact with one another, to share knowledge and discover new perspectives and tool kits for solving problems was achieved by a vibrant mix of mathematicians. The networking opportunities brought the potential to inspire totally new research directions and lead to new collaborations across areas.

Logistically, we planned to organize our workshop to maximize the opportunities for cross-fertilization. In particular, we invited both established and promising young researchers, maximized the number of female participants, to ensure rich diversity in perspectives and to facilitate informal mentoring and networking opportunities. In numbers, we had 78 participants, 39 of which were on site. Among the 39 in-person participants there were

- 6 PhD students (1 female, 5 male)
- 7 Postdocs (3 female, 4 male)
- 10 Tenure-Track Faculty (3 female, 7 male)

In total, we had 14 female/25 male in-person and 12 female/27 male online participants.

Scientific program. We decided to have two mini courses (3 hours each) on the symplectic respectively algebraic point of view on toric degenerations. The lectures were a great success in the sense that they provided background information enabling the participants to start discussions from a common ground. The lecture series were taught by two tenured professors

1. Susan Tolman (University of Illinois at Urbana-Champaign, U.S.)
2. Allen Knutson (Cornell University, U.S.)

Besides there were 12 complementary research talks on subjects ranging over sympelctic and complex geometry, toric geometry, the theory Newton–Okounkov bodies, the theory of cluster algebras, algebraic matroids and rigidity theory, mirror symmetry and Fano varieties, and representation theory. Although the subjects were widely spread, all talks had a similar flavour in the sense that the basic idea of toric degenerations appeared in all of them. The 12 speakers–6 female (4 tenured and 2 postdocs) and 6 male (3 tenured, 2 tenure-track and 1 postdoc)– were

1. Sandra Di Rocco (KTH Stockholm, Sweden)
2. Liana Heuberger (University of Bath, U.K.)
3. Nathan Ilten (SFU Vancouver, Canada)
4. Christopher Manon (University of Kentucky, U.S.)
5. Daniel I. Bernstein (Tulane University, U.S.)
6. Melissa Shermann-Bennett (MIT, U.S.)
7. Peter Crooks (Utah State University, U.S.)
8. Alex Küronya (University of Frankfurt, Germany)
9. Elana Kalashnikov (University of Waterloo, Canada)
10. Timothy Magee (King's College London, U.K.)
11. Yael Karshon (University of Toronto, Canada and Tel-Aviv University, Israel)
12. Laura Escobar (Washington University in St.Louis, U.S.)

In addition, we had two sessions of five-minute talks that allowed us to include almost every participant as a speaker. Especially young participants took advantage of this opportunity and presented a snapshot of their work. Although five minutes are not enough to present a deep result, it served as an icebreaker enabling participants to engage into discussions afterwards more easily. Additionally, everyone was left with the impression to know the other participants a little more. The 16 speakers–4 female (2 tenure track, 1 postdoc and 1 Phd student) and 12 male (5 PhD students, 2 postdocs and 5 tenure-track) – of the five-minute sessions were

1. Dimitra Kosta (University of Edinburgh, U.K.)
2. Takuya Murata (IPM, Iran)
3. Francesca Zaffalon (KU Leuven, Belgium)
4. Karin Schaller (FU Berlin, Germany)

5. Ahmad Mokhtar (SFU Vancouver, Canada)
6. Luca Schaffler (Roma Tre, Italy)
7. Christian Steinert (RWTH Aachen, Germany)
8. Naoki Fujita (Kumamoto University, Japan)
9. Eunjeong Lee (Chungbuk National University, South Korea)
10. Yoosik Kim (Pusan National University, South Korea)
11. Yunhyung Cho (Sungkyunkwan University, South Korea)
12. Sharon Robins (SFU Vancouver, Canada)
13. Charles Wang (Harvard, U.S.)
14. Thomas Brazelton (University of Pennsylvania, U.S.)
15. Ian Cavey (Ohio State University, U.S.)
16. Alfredo Nájera Chávez (UNAM Oaxaca, Mexico)

Lastly, we had an informal discussion lead by Frank Sottile about BIRS and how to apply for workshops. More experienced participants were sharing their insights with young participants interested in organizing events in the future. In fact, the idea of organizing this workshop arose for the first time in a similar session at a workshop in Casa Matemática Oaxaca (Mexico) also lead by Frank.

2 Abstracts of the Talks and Lectures

2.1 Introductory Mini Courses

2.1.1 Susan Tolman

Lecture 1: Integrable systems and the n-body problem A basic introduction to symplectic geometry in general and integrable systems in particular through the perspective of the n-body problem. At the very end, I will explain how toric degenerations occupy a liminal space between the Kaehler and symplectic categories. Therefore, they can help solve a number of symplectic mysteries, which I will explain in the remaining talks.

Lecture 2: Gelfand-Cetlin systems and toric degenerations To start, I will introduce symplectic toric manifolds, classified by Delzant, and other multiplicity free actions. I will then turn to Gelfand-Cetlin systems, as introduced by Guillemin and Sternberg, and explain how they can be understood in terms of toric degenerations.

Lecture 3: Symplectic cohomological rigidity and toric degenerations I will start by describing symplectomorphisms between Hirzebruch surfaces. Next, I will discuss how results of Harada- Kaveh enable us to see that toric degenerations can be used to explain these symplectomorphisms. Finally, I will explain symplectic cohomological rigidity and my joint work with Pabiniak where we show that toric degenerations can help solve this problem.

2.1.2 Allen Knutson

Lecture 1: The Vinberg asymptotic cone vs. the Thimm trick, for enlarging group actions

Lecture 2: Bott-Samelson manifolds and their Magyar-Grossberg-Karshon-Pasquier-Parameswaran toric degeneration

Lecture 3: Branchvarieties and the Chirivi degeneration of G/P

2.2 Research Talks:

2.2.1 Sandra Di Rocco: Families of pointed toric varieties and degenerations

In this talk, we will introduce a class of polytope fibrations, which we call generalized Cayley sums. These fibrations represent a generalization of the Losev-Manin moduli space parametrizing pointed chains of projective lines.

2.2.2 Elana Kalashnikov: Mirror symmetry constructions for type A flag varieties

Mirror symmetry for Fano varieties is well understood when the variety is a toric manifold. Beyond the toric case, the next best understood example is the type A Grassmannian. In this talk, I'll survey some mirror symmetry proposals for Grassmannians and flag varieties in type A: mirrors from toric degenerations, mirrors from the Abelian/non-Abelian correspondence, and the Plucker coordinate mirror. I'll then discuss what's known and what's challenging about extending these constructions and results from Grassmannians to flag varieties and beyond.

2.2.3 Nathan Ilten-Gee: Deformation Theory for Finite Cluster Complexes

A well-known result of Stumfels says that there is a Gröbner degeneration of the homogeneous coordinate ring for the Grassmannian G(2,n) to the Stanley-Reisner ring associated to the simplicial complex dual to the n-associahedron. In this talk, I will report on recent work with A. Nájera Chávez and H. Treffinger in which we simultaneously strengthen and generalize this result. We show that for a cluster algebra A of finite cluster type with "enough" frozen coefficients, there is a canonical embedded realization of a certain torus-invariant semi-universal deformation space for the Stanley-Reisner ring of the cluster complex of A. The cluster algebra A (along with its cluster structure) may be recovered as a distinguished fiber of this deformation. In fact, the total space of this deformation may be identified with cluster algebra obtained from A by adding "universal coefficients". Among other consequences, this implies that any cluster algebra of finite cluster type is Gorenstein.

2.2.4 Christopher Manon: Toric degenerations and conformal field theory

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, and $(C, p_1, \ldots, p_n)$ be an n-marked, smooth, projective complex curve. Using some representation theory of the affine Kac-Moody algebra associated to $\mathfrak{g}$, the Wess-Zumino-Novikov-Witten model of conformal field theory associates to the data of an n-tuple of dominant weights $\lambda_1, \ldots, \lambda_n$ and a non-negative integer L a finite dimensional vector space $V_{C,\vec{p}}(\lambda_1, \ldots, \lambda_n, L)$ called a space of conformal blocks. Computing the dimension of these spaces amounts to finding a method to evaluate the so-called Verlinde formula of the WZNW theory.

A striking theorem of Pauly, and Kumar, Narasimhan, and Ramanathan realizes the conformal blocks as the spaces of global sections of line bundles on the moduli $\mathcal{M}_{C,\vec{p}}(G)$ of quasi-parabolic principal G bundles on the marked curve $(C, \vec{p})$; thus the Verlinde formula is linked to the Hilbert functions of line bundles on this moduli problem. The moduli $\mathcal{M}_{C,\vec{p}}(G)$ are themselves quite interesting. For example, if C is the projective line, their geometry is closely related to configurations of G-flags, and other spaces which carry a cluster structure.

I will give an overview of some known toric degenerations of the moduli $M_{C,\vec{p}}(G)$ when $\mathfrak{g} = sl_2, sl_3, sl_4$. These constructions have the effect of give a diagrammatic way to keep track of a basis of the spaces of conformal blocks. Time permitting, I will also describe a relationship to an integrable system studied by Hurtubise and Jeffries in the case $\mathfrak{g} = sl_2$.

2.2.5 Daniel Bernstein: Understanding algebraic matroids using tropical geometry

The algebraic matroid of an irreducible variety, embedded into affine space via a specific coordinatization, is the combinatorial structure one gets by keeping track of which coordinate projections are dominant morphisms. Certain problems in statistics and engineering require an understanding of the algebraic matroids of particular families of varieties, such as detrimental varieties. These problems are very hard to crack, partially because there aren't many general techniques. In this talk, I will discuss some successes of tropical geometry as a tool here, highlighting at least one way to think of this in terms of toric degenerations.

2.2.6 Melissa Sherman-Bennett: Type A braid variety cluster structures from 3D plabic graphs

Braid varieties are smooth affine varieties associated to any positive braid. Their cohomology is expected to contain information about the Khovanov-Rozansky homology of a related link. Special cases of braid varieties include Richardson varieties, double Bruhat cells, and double Bott-Samelson cells. Cluster algebras are a class of commutative rings with a rich combinatorial structure, introduced by Fomin and Zelevinsky. I'll discuss joint work with P. Galashin, T. Lam and D. Speyer in which we show the coordinate rings of braid varieties are cluster algebras, proving and generalizing a conjecture of Leclerc in the case of Richardson varieties. Seeds for these cluster algebras come from ”3D plabic graphs”, which are bicolored graphs embedded in a 3-dimensional ball and generalize Postnikov's plabic graphs for positroid varieties.

2.2.7 Peter Crooks: Gelfand-Cetlin abelianizations of symplectic quotients

Symplectic geometry is perhaps best described as a mathematical abstraction of classical mechanics. On the other hand, it is a recurring theme in mathematics that the symmetries of an object should give rise to ”quotients” of that object. Marsden-Weinstein reduction is the most fundamental incarnation of this phenomenon in symplectic geometry; it is a systematic approach to taking quotients of symplectic manifolds carrying Hamiltonian symmetries, and has important manifestations in algebraic geometry, mathematical physics, and representation theory. In this context, I will discuss some recent research on ”abelianizing” the symplectic quotients taken by any compact connected Lie group. One main ingredient will be the Gelfand-Cetlin integrable systems of Guillemin-Sternberg, as well as recent generalizations to arbitrary Lie type by Hoffman-Lane. This represents joint work with Jonathan Weitsman.

2.2.8 Alex Kuronya: Finite generation of non-toric valuation semigroups on toric surfaces

We provide a combinatorial criterion for the finite generation of a valuation semigroup associated with an ample divisor on a smooth toric surface and a non-toric valuation of maximal rank. (with Klaus Altmann, Christian Haase, Karin Schaller, and Lena Walter)

2.2.9 Liana Heuberger: Mirror Symmetry and the classification of Q-Fano threefolds

I will discuss how to use the classification of Gorenstein canonical Fano polytopes (and Laurent polynomials supported on them) in the construction of Q-Fano threefolds which admit a toric degeneration. `https://arxiv.org/abs/2210.07328`

2.2.10 Timothy Magee: Relating different LG mirrors and toric degenerations for Grassmannians

The open positroid variety in the Grassmannian has both an and cluster structure. I'll review cluster ensemble maps and discuss how such a map induces another cluster ensemble map from the mirror of to the mirror of . Using this pair of maps, we will see that we can identify two Landau-Ginzburg mirror constructions for the Grassmannian– one due to Marsh-Rietsch, and the other based on work of Gross-Hacking-Keel-Kontsevich. When one of these maps identifies the minimal model inputs of the two constructions, the other identifies the LG mirror outputs. As a corollary, we will also relate the polytopes and toric degenerations of Rietsch-Williams with those in the Gross-Hacking-Keel-Kontsevich framework. Based on ongoing joint work with Lara Bossinger, Mandy Cheung, and Alfredo Nájera Chávez.

2.2.11 Yael Karshon: Bott canonical basis?

Abstract: Together with Jihyeon Jessie Yang, we resurrected an old idea of Raoul Bott for using large torus actions to construct canonical bases for unitary representations of compact Lie groups. Our methods are complex analytic. We apply them to families of Bott-Samelson manifolds parametrized by $\mathbb{C}^n$. This application requires the vanishing of higher cohomology of sheaves of holomorphic sections of certain line bundles over the total spaces of such families; this vanishing is still conjectural. `https://arxiv.org/abs/2106.15695`

2.2.12 Laura Escobar: Geometric constructions from abstract wall-crossing

The interplay between combinatorics and algebraic geometry has immensely enriched both areas. In this context, the theory of Newton-Okounkov bodies has led to the extension of the geometry-combinatorics dictionary from toric varieties to certain varieties which admit a toric degeneration. In a recent paper with Megumi Harada, we gave a wall-crossing formula for the Newton-Okounkov bodies of a single variety. Our wall-crossing involves a collection of lattices $\{M_i\}_{i\in I}$ connected by piecewise-linear bijections $\{\mu_{ij}\}_{i,j\in I}$. In addition, in previous work Kiumars Kaveh and Christopher Manon analyze valuations into semifields of piecewise linear functions and explore their connections to families of toric degenerations, with particular attention to links to the theory of cluster varieties. Inspired by these ideas in joint work in progress with Megumi Harada and Christopher Manon we propose a generalized notion of polytopes in $\Lambda = (\{M_i\}_{i\in I}, \{\mu_{ij}\}_{i,j\in I})$, where the M_i are lattices and the $\mu_{ij} : M_i \to M_j$ are piecewise linear bijections. Roughly, these are $\{P_i \mid P_i \subseteq M_i \otimes \mathbb{R}\}_{I\in I}$ such that $\mu_{ij}(P_i) = P_j$ for all i, j. In analogy with toric varieties these generalized polytopes can encode compactifications of affine varieties as well as some of their geometric properties. In this talk, we illustrate these ideas with a concrete class of examples related to classical reflexive polytopes and Fano varieties.

3 Extended Abstracts

3.1 Sandra Di Rocco and Luca Schaffler: Families of pointed toric varieties and degenerations

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We introduce a toric generalization of the Losev–Manin moduli space building upon [AM16] and [ST21]. The Losev–Manin moduli space $\overline{\mathrm{LM}}_{m+2}$ is a projective, smooth, fine moduli space of dimension $m-1$ parametrizing chains of projective lines marked with m smooth 'light' points, which are allowed to collide, and two smooth 'heavy' points, which cannot collide with any other marked point [LM00]. Using tools from the work of Kapranov–Sturmfels–Zelevinski [KSZ91], a generalization of $\overline{\mathrm{LM}}_{0,m+2}$ to point configurations in $\mathbb{P}^2$ was used in [ST21] to describe a novel geometric and modular compactification of the moduli space of n points in $\mathbb{P}^2$. Motivated by this, we consider the general perspective of point configurations in a projective toric variety X_P associated to an arbitrary lattice polytope P.

Let H be the dense open subtorus of X_P, diagonally embedded in X_P^m. The Chow quotient $X_P^m/\!/H$ can be interpreted as a compactification of the moduli space of m light and k heavy points in X_P up to H-action, where k is the number of torus fixed points in X_P. The normalization of $X_P^m/\!/H$ is isomorphic to the toric variety $X_{\mathcal{Q}(P,m)}$ associated to the *quotient fan* $\mathcal{Q}(P,m)$. By work of Billera–Sturmfels [BS92], $X_{\mathcal{Q}(P,m)}$ is a projective toric variety associated to the normal fan $\mathcal{Q}(P,m)$ to a polytope $Q(P,m)$ called the *fiber polytope*. For example, if P is the 1-dimensional simplex Δ_1, then $Q(\Delta_1, m)$ is the $(m-1)$-dimensional permutohedron and $X_{\mathcal{Q}(\Delta_1,m)} \cong \overline{\mathrm{LM}}_{0,m+2}$. We introduce toric families of pointed, degenerate toric varieties over $X_{\mathcal{Q}(P,m)}$ for an arbitrary lattice polytope P. These varieties are governed by projective toric families $f\colon X_{\mathcal{R}(P,m)} \to X_{\mathcal{Q}(P,m)}$ which resolve the indeterminacies of the rational map $X_P^m \dashrightarrow X_{\mathcal{Q}(P,m)}$. The fiber of f over a point in the dense torus of $X_{\mathcal{Q}(P,m)}$ is an appropriate translate by H-action of the diagonal $X_P \subseteq X_P^m$.

The definition of $\mathcal{R}(P,m)$ implies the existence of a toric embedding $X_{\mathcal{R}(P,m)} \hookrightarrow X_P^m \times X_{\mathcal{Q}(P,m)}$ which induces a polarization on $X_{\mathcal{R}(P,m)}$. Let $R(P,m)$ be the lattice polytope with normal fan $\mathcal{R}(P,m)$ corresponding to this polarization. We find that the polytopes $R(P,m)$ can be described as a generalization of the class of π-*twisted Cayley sums*.

Definition 1. [[CDR08, Definition 3.5]] Let $\pi\colon M \to \Lambda$ be a surjective map of lattices and denote by $\pi_{\mathbb{R}}$ its extension $M_{\mathbb{R}} = M \otimes \mathbb{R} \to \Lambda \otimes \mathbb{R} = \Lambda_{\mathbb{R}}$. Let $R_1, \dots, R_\ell \subseteq M_{\mathbb{R}}$ be lattice polytopes which are *normally equivalent*. Assume that $\pi_{\mathbb{R}}(R_i) = v_i \in \Lambda$ are distinct and are the vertices of the lattice polytope $F = \mathrm{Conv}(v_1, \dots, v_\ell)$. Then $R = \mathrm{Conv}(R_1, \dots, R_\ell)$ is called π-*twisted Cayley sum* of $R_1, \dots, R_\ell$.

If Y is the toric variety associated to the normally equivalent polytopes $R_1, \dots, R_\ell$ in the definition, then we have a toric morphism $X_R \to Y$ with fibers isomorphic to X_F [CDR08, Lemma 3.6]. This toric morphisms $X_R \to Y$ do not describe $X_{\mathcal{R}(P,m)} \to X_{\mathcal{Q}(P,m)}$ as the latter may have reducible fibers. By not requiring that the polytopes R_i are only mapped to vertices we obtain an enlarged class of polytopes which we call *generalized* π-twisted Cayley sums. The following results give a geometric characterization of these polytopes.

Theorem 2. *Let $R = R(R_1, \dots, R_k, \pi)$ be a generalized π-twisted Cayley sum and let $F = \pi_{\mathbb{R}}(R)$. Let Y be the toric variety associated to the normally equivalent lattice polytopes R_i. Then there exists a toric morphism $X_R \to Y$ with generic fiber X_F.*

Theorem 3. *Let X_P be the polarized toric variety associated to a lattice polytope P and let $H \subseteq X_P$ be the dense open torus. Let $\pi\colon M \to \Lambda$ be the map of character lattices induced by the diagonal embedding $H \hookrightarrow H^m$. The polytope $R(P,m)$ has the structure of generalized π-twisted Cayley sum with $\pi_{\mathbb{R}}(R(P,m)) = mP$ and $R(P,m) = \mathrm{Conv}(R_1, \dots, R_k)$ with R_i normally equivalent to $Q(P,m)$.*

The family $f\colon X_{\mathcal{R}(P,m)} \to X_{\mathcal{Q}(P,m)}$ has interesting geometric features. f is equidimensional and it can be endowed with light and heavy sections analogous to the ones of the family of the Losev–Manin moduli space. Moreover, we have forgetful morphisms $X_{\mathcal{Q}(P,m+1)} \to X_{\mathcal{Q}(P,m)}$. Note that these last two propositions are immediate generalizations of [ST21, Lemma 8.6 and Proposition 8.9]. Flatness and reducedness of the fibers of f are explored via a combinatorial characterization by translating in our context Molcho's weak semistability in [Mol21]. As an application of this criterion, we show that f is flat with reduced fibers if P is the d-dimensional simplex Δ_d.

As $X_{\mathcal{R}(P,m)} \to X_{\mathcal{Q}(P,m)}$ together with the light and heavy sections is a generalization of the Losev–Manin moduli space $\overline{\mathrm{LM}}_{m+2}$, it is natural to ask whether $X_{\mathcal{R}(\Delta_1,m)}$ coincides with the universal family $\overline{\mathrm{LM}}_{m+3}$ over $\overline{\mathrm{LM}}_{m+2}$. This is indeed the case, implying that $R(\Delta_1, m)$ is the m-dimensional permutohedron. Note that the recursion $\mathcal{R}(P,m) \cong \mathcal{Q}(P,m+1)$ is not true in general: a counterexample is provided by $P = \Delta_2$ and $m = 2$.

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3.2 Christopher Manon: Toric degenerations and conformal field theory

Conformal blocks are vector spaces which serve as the correlation functions of the Wess-Zumino-Novikov-Witten model of conformal field theory. These vector spaces depend on two types of data. From geometry, we require an n-pointed, smooth, projective algebraic curve $(C, \vec{p})$. From representation theory, we choose a simple complex Lie algebra $\mathfrak{g}$, and n dominant weights $\lambda_1, \ldots, \lambda_n \in \Lambda_+$. We think of the weight λ_i as giving the internal structure of the point p_i. Finally, we choose a non-negative integer L; these data determine the space of conformal blocks $V_{C,\vec{p}}(\vec{\lambda}, L)$.

The space $V_{C,\vec{p}}(\vec{\lambda}, L)$ is constructed as a certain space of invariant vectors as follows. First one associates to λ_i, L the integral highest-weight module $H(\lambda_i, L)$ of the Kac-Moody algebra $\hat{\mathfrak{g}}$ associated to $\mathfrak{g}$. An action of the Lie algebra $\mathbb{C}[C \setminus \vec{p}] \otimes \mathfrak{g}$ can then be constructed on the tensor product $H(\lambda_1, L) \otimes \mathbb{C}dots \otimes H(\lambda_n, L)$, and $V_{C,\vec{p}}(\vec{\lambda}, L)$ is the space of invariant vectors in the dual of this representation. Notably, the conformal blocks for fixed $\vec{\lambda}, L$ are all subspaces of $[H(\lambda_1, L) \otimes \mathbb{C}dots \otimes H(\lambda_n, L)]^*$, and the information that is changing in their construction is Lie algebra action by $\mathbb{C}[C \setminus \vec{p}] \otimes \mathfrak{g}$. The following summary of results of Tsuchiya, Ueno, and Yamada says that things don't change too much.

Theorem 1. [[TUY89]] *The following hold for any* $\mathfrak{g}$.

1. *The spaces $V_{C,\vec{p}}(\vec{\lambda}, L)$ are the fibers of a finite rank vector bundle $V(\vec{\lambda}, L)$ over $\mathcal{M}_{g,n}$.*
2. *This vector bundle extends to $\overline{\mathcal{M}}_{g,n}$.*
3. *The space $V_{\mathbb{P}^1,\vec{p}}(\vec{\lambda}, L)$ can be realized as a subspace of $[V(\lambda_1) \otimes \mathbb{C}dots \otimes V(\lambda_n)]^{\mathfrak{g}}$.*

As a consequence of this result, there is a space of conformal blocks $V_\Gamma(\vec{\lambda}, L)$ for every semistable curve determined by a trivalent graph Γ with $\beta_1\Gamma = g$ and n leaves. This leads to a strategy for studying conformal blocks: if one wants to understand conformal blocks over $C, \vec{p}$, one can try to degenerate the curve $C, \vec{p}$ to the curve determined by some Γ, where the issues can become more combinatorial. The following is an example of how this strategy can work out for $\mathfrak{g} = sl_2$.

Example 1. [The sl_2 case] *Let $\mathfrak{g} = sl_2$, let Γ be a trivalent graph with n leaves and $\beta_1\Gamma = g$, and let $r_1, \ldots, r_n, L \in \mathbb{Z}_{\geq 0}$, then the dimension of $V_\Gamma(\vec{r}, L)$ is equal to the number of functions $w : E\Gamma \to \mathbb{Z}_{\geq 0}$ which satisfy the following conditions:*

1. *for any leaf edge ℓ_i we have $w(\ell_i) = r_i$*
2. *for any trinode with incident edges i, j, k we must have $w(i) + w(j) + w(k) \leq 2L$*
3. *with i, j, k as above, $w(i) + w(j) + w(k) \in 2\mathbb{Z}$*
4. *with i, j, k as above, $w(i), w(j), w(k)$ form the sides of a triangle.*

We let $P_\Gamma(\vec{r}, L)$ be the convex polytyope defined by conditions $1, 2,$ and 4. Condition 3 defines a lattice $\mathcal{L} \subset \mathbb{Z}^{E\Gamma}$. The dimension of $V_\Gamma(\vec{r}, L)$ is then equal to the order of the finite set $P_\Gamma(\vec{r}, L) \cap \mathcal{L}$.

The polyhedra $P_\Gamma(\vec{r}, L)$ in Example 3.2 have appeared in many places, some of which indicate a connection to a second moduli problem. In the case $g = 0$ the $P_\Gamma(\vec{r}, L)$ appear in work of Sturmfels and Xu [SX10] in the theory of toric degenerations of the Cox ring of a certain blow-up of projective space. Work of Bauer [Bau91] then shows that the Cox ring in question is also the Cox ring of the moduli of semistable rank 2 vector bundles. Moreover, the same polyhedra in the general genus case appear in work of Hurtubise and Jeffrey [HJ00] as the momentum image of a certain integrable system on the moduli of flat $SU(2)$ bundles on the Riemann surface defined by C. The following is the connection between conformal blocks and moduli of bundles suggested by these appearances.

Theorem 2. [[KNR94], [Pau96], [BLS98], ...] *For every $(\vec{\lambda}, L) \in \Lambda^n \times \mathbb{Z}$ there is a line bundle $\mathcal{L}_{\vec{\lambda},L}$ on $\mathcal{M}_{C,\vec{p}}(G)$, moreover:*

$$H^0(\mathcal{M}_{C,\vec{p}}(G), \mathcal{L}_{\vec{\lambda},L}) = V_{C,\vec{p}}(\vec{\lambda}, L).$$

It can be shown that the sheaf of algebras $V(G)$ extends to a locally free sheaf on $\overline{\mathcal{M}}_{g,n}$. The fiber $V_\Gamma(G)$ has been shown to be the Cox ring of the moduli $M_\Gamma(G)$ of singular principal G-bundles on the nodal curve associated to Γ. If $n = 0$ it can be shown that $V_\Gamma(G)$ is a Rees algebra of the coordinate ring of the so-called free group character variety $\mathcal{X}(F_g, G)$, the following is a geometric restatement of this fact.

Theorem 3. [[Man15]] *The coarse moduli of singular bundles $M_\Gamma(G)$ contains $\mathcal{X}(F_g, G)$ as a dense, open subset.*

There is then a further degeneration of V_Γ to an algebra constructed from $V_{0,3}(G)$. By imagining a copy of $V_{0,3}(G)$ at each trinode of G, we can ask that a copy of a maximal torus $T \subset G$ acts on the two copies of $V_{0,3}(G)$ corresponding to two vertices of Γ which meet at an edge.

Theorem 4. [[Man18]] *There is a flat degeneration from $V_\Gamma(G)$ to the algebra of $T^{E\Gamma}$-invariants in $\bigotimes_{v \in V\Gamma} V_{0,3}(G)$.*

These techniques have been used to great effect to show that $V_{C,\vec{p}}(G)$ is finitely generated in types A and C.

Theorem 5. [[BG19], [MY20], [Wil]] *The algebra $V_{C,\vec{p}}(G)$ is finitely generated for G of types A or C.*

We can also find toric degenerations of the algebras of conformal blocks which realize known counting rules for these objects, provided one knowns toric degenerations of the $0, 3$ case. The following are some results along these lines.

Proposition 1. [[Man18]] *The algebra $V_{0,3}(sl_2)$ is a polynomial ring on four generators. As a consequence, for generic $C, \vec{p}$ there is a toric degeneration of $M_{C,\vec{p}}(SL_2)$ for each graph Γ.*

Proposition 2. [[Man13]] *The algebra $V_{0,3}(sl_3)$ is isomorphic to:*

$$\mathbb{C}[P_{12}, P_{23}, P_{31}, P_{21}, P_{32}, P_{12}, X, S, T]/\langle P_{12}P_{23}P_{31} + P_{21}P_{32}P_{12} - XST\rangle$$

As a consequence, for generic $C, \vec{p}$ there is a toric degeneration of $M_{C,\vec{p}}(SL_3)$ for each graph Γ and one of the three toric degenerations of $V_{0,3}(sl_3)$.

Proposition 3. [[HM]] *The algebra $V_{0,3}(sl_4)$ is presented by 20 generators with 40 relations. The corresponding tropical variety has 544 maximal cones, and 528 of these define normal T^3 equivariant toric degenerations of $V_{0,3}(sl_4)$. As a consequence, for generic $C, \vec{p}$ there is a toric degeneration of $M_{C,\vec{p}}(SL_4)$ for each graph Γ and one of the 528 toric degenerations of $V_{0,3}(sl_4)$.*

prime$'$

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3.3 Ahmad Mokhtar: Fano schemes of singular symmetric matrices

Linear subspaces of singular symmetric matrices. Let CK be an algebraically closed field of characteristic not equal to 2. For an integer $n \geq 3$, consider the CK-vector space S_n of symmetric $n \times n$ matrices with entries in CK. We are interested in studying those subspaces $Q \subset S_n$ such that every matrix in Q is singular.
Example 1. Fix $n \geq 3$ and $0 \leq s \leq \lfloor \frac{n-1}{2} \rfloor$. Any subspace of S_n that is $\mathrm{GL}(n)$-equivalent to a subspace of the form

$$\begin{array}{c c} & \begin{array}{ccc} s & n-2s-1 & s+1 \end{array} \\ \begin{array}{c} s \\ n-2s-1 \\ s+1 \end{array} & \begin{pmatrix} * & * & * \\ * & * & 0 \\ * & 0 & 0 \end{pmatrix} \end{array},$$

consists only of singular matrices. Here each star block is filled with linear forms. Such a subspace is called a s-compression space.

There is a moduli space parametrizing subspaces of singular symmetric matrices: Denote by SD_n the closed subscheme of $CP_{CK}^{\binom{n+1}{2}-1}$ cut out by the determinant of the generic symmetric matrix

$$\begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{1,2} & x_{2,2} & \cdots & x_{2,n} \\ \vdots & & \ddots & \vdots \\ x_{1,n} & x_{2,n} & \cdots & x_{n,n} \end{pmatrix},$$

filled with $\binom{n+1}{2}$ independent forms $x_{i,j}$. For a fixed integer k, consider the subset of the Grassmannian $\mathrm{Gr}(k+1, S_n)$ consisting of those points $[Q]$ such that the corresponding k-plane lies on SD_n (i.e. the subspace $Q \subset S_n$ contains only singular matrices). There is a natural scheme structure (see [EH16, Chapter 6]) on this subset of the Grassmannian and the resulting scheme $\mathbf{F}_k(SD_n)$ is called the *Fano scheme* of k-planes of SD_n.

In this study, we investigate the irreducibility, connectedness and smoothness of the schemes $\mathbf{F}_k(SD_n)$. These schemes provide a rich family of examples of *general Hilbert schemes*. Moreover, by understanding the schemes $\mathbf{F}_k(SD_n)$ we are able to reprove a number of results on linear subspaces of singular symmetric matrices using geometric arguments rather than cumbersome computations in linear algebra. These proofs have the potential to expand the boundaries of what we know about subspaces of singular matrices in S_n.

Torus action on $\mathbf{F}_k(SD_n)$. The group of invertible diagonal matrices $T \cong (CK\setminus\{0\})^n$ acts on S_n via $B \cdot M \mapsto BMB^t$ for $B \in T$ and $M \in S_n$. This induces a natural torus action on the schemes $\mathbf{F}_k(SD_n)$. By using the fixed points of this action, we can characterize the integers k for which $\mathbf{F}_k(SD_n)$ is non-empty, a result first proved by Meshulam [M89, Theorem 1].

Theorem 2. *The Fano scheme $\boldsymbol{F}_k(SD_n)$ is non-empty if and only if $k \leq k_{\max} = \frac{(n+1)(n-2)}{2}$.*

Meshulam's proof uses linear algebra and graph theory and has the advantage that it works for all fields with more than n elements. The geometric proof on the other hand uses torus fixed points and is much shorter and insightful. This systematic use of torus fixed points was employed by Ilten and Chan [CI15] in their study of Fano schemes of rectangular matrices of bounded rank.

A glimpse of the Geometry of $\mathbf{F}_k(SD_n)$. For a fixed n, we understand the geometry of $\mathbf{F}_k(SD_n)$ better when $k = 1$ or k is near $k_{\max}$. Here we mention what holds in the two extremal cases. The first result concerns the Fano scheme of lines of SD_n.

Theorem 3. *The Fano scheme $\boldsymbol{F}_1(SD_n)$ has $\lfloor \frac{n-1}{2} \rfloor + 1$ irreducible components, each of the expected dimension $n^2 - 5$. For each $0 \leq s \leq \lfloor \frac{n-1}{2} \rfloor$, the s-compression spaces form an irreducible component.*

The next result gives the picture for $k = k_{\max}$.

Theorem 4. *Let $n \geq 4$. Then the Fano scheme $\boldsymbol{F}_{k_{\max}}(SD_n)$ is irreducible of dimension $n - 1$ and is generically non-reduced. The scheme is the $\mathrm{GL}(n)$ orbit of a 0-compression space. For $n = 3$, the scheme $\boldsymbol{F}_k(SD_n)$ has two disjoint irreducible components of dimension 2. One component is the $\mathrm{GL}(n)$ orbit of a 1-compression space and is smooth. The other is the $\mathrm{GL}(n)$ orbit of a 0-compression space and is generically non-reduced.*

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3.4 Takuya Murata: The local structure of a toric degeneration

Constructing toric degenerations

Example. [relative elliptic curve] Let $\pi : X \to S$ be a relative elliptic curve over a variety S. Assume $X \subset CP^2 \times S$ and π the projection (e.g., a cubic pencil).

If S has a toric generation, then so does $CP^2 \times S$. Then we lift the toric degeneration under the finite morphism $i : X \hookrightarrow CP^2 \times S$ to get a toric degeneration of X. Namely, we use $i^{-1}(H) = X \cap H$ that is irreducible, H a hypersurface.

The approach in my Ph.D. thesis is to reduce the general case to the relative curve case like the above, using a variant of the degeneration to a normal cone. It uses a part of the following idea.

Suppose we are given a degeneration $\pi : \mathfrak{X} \to CA^1$ of a variety X such that $\mathfrak{X}$ is a CG_m-equivariant finite reduced covering of the degeneration to a normal cone, or a *degenerations of Rees type*. (This degeneration is not proper so, in practice, we use a proper analog of such a degeneration.) Then $\mathfrak{X}$ is the relative Spec of $\mathcal{R}$ with $\mathcal{O}_X[t] \subset \mathcal{R} \subset \mathcal{O}_X[t, t^{-1}]$ (here, the inclusions are assumed to preserve the CZ-grading) and such an $\mathcal{R}$ is equivalent to giving a filtration $\mathcal{O}_X = \mathcal{I}_0 \supset \mathcal{I}_1 \supset \cdots$ that is multiplicative (or equivalent to a quasi-valuation if X is an affine variety), called an *ideal filtration* in my Ph.D. thesis.

Given a scheme S, Alexeev and Knutson [AK10] consider the stack of all finite morphisms from geometrically reduced schemes to S (which they call *branchvarieties*). Thus, the ideal filtrations of finite type or equivalently the degenerations of Rees type of a fixed variety X form an algebraic stack that is an analog of a Hilbert scheme, which I call the *intrinsic toric degeneration* of X (to be precise, a proper analog of such).

A map to a toric variety

A (normal) toric variety W can be thought of as a moduli space; i.e., it is determined by maps to W, called classifying maps (cf. [Cox95] (I am indebted to Nathan Ilten for letting me know about Cox's paper) and [Kol06].) For exampe, when W is a projective space and classifying maps are algebraic maps, W is the classifying space for the linear systems.

We ask (without assuming that W is normal) whether a toric degeneration $X \rightsquigarrow W$ can be viewed as a classifying map and, conversely, whether a toric degeneration can be obtained from some classifying map, the first question investigated by Lara Bossinger and me ([BM20]).

Since W contains a dense torus, each map $X \to W$ is a compactification of a monomial map. It follows that there is always a morphism (algebraic map) from any variety $X - B$ to W up to some base locus B, at least when W is quasi-projective. Over the complex numbers (or over characteristic zero), when the degeneration $X \rightsquigarrow W$ is embedded into a smooth variety,[1] following Goresky and MacPherson ([GM83], [Gor81]), we explicitly construct a surjective continuous map $\varphi : X \to W$ using the tubular neighborhoods to the orbits on W and has the following two consequences. First, φ can be thought of as giving a local structure to the degeneration $X \rightsquigarrow W$ indexed by the orbits on W. Second, not only we recover (modulo some verification) the construction of integrable systems of [HK15], we also construct integrable systems on singular strata. (In fact, we only need W is semi-toric.)

The second question seems interesting and is possibly related to the construction in the first part.

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3.5 Francesca Zaffalon: Toric degenerations of partial flag varieties and combinatorial mutations of matching field polytopes

Motivation. In [CMZ22] we study toric degenerations of Grassmannian and partial flag varieties. The *partial flag variety* $\mathcal{F}\ell_n(J)$, for $J \subset [n] = \{1, \dots, n\}$, is the set of flags $V_{j_1} \subsetneq \cdots \subsetneq V_{j_s}$ where $J = \{j_1 < \cdots < j_s\}$ and V_{j_i} is a j_i-dimensional vector subspace of CC^n. The *Grassmannian* $\mathrm{Gr}(k, n)$ is the partial flag variety $\mathcal{F}\ell_n(\{k\})$. We are interested in studying toric degenerations of partial flag varieties from a computational perspective.

A toric degeneration [And13] of an algebraic variety X is a flat family $\mathcal{F} \to CA^1$ whose fibers $\mathcal{F}_t$ over all points $t \in CA^1 \setminus \{0\}$ are isomorphic to X and whose fiber over 0 is a toric variety $Y := \mathcal{F}_0$. Toric degenerations are particularly useful because many important algebraic invariants of X, such as the Hilbert polynomial and the degree, coincide with those of Y.

[1] In particular, the degeneration need not be quasi-projective.

The classes of toric degenerations that we study arise from matching fields [SZ93]. These have been introduced and studied for various kinds of homogeneous spaces, such as Grassmannians [MS19, CHM22a, CHM22b], Schubert varieties [CM21a] and full flag varieties [CM21b].

Matching fields. Matching fields contain the combinatorial information necessary to construct a toric degeneration of Grassmannians and partial flag varieties. A partial flag variety $\mathcal{F}\ell_n(J)$ can be in a product of projective spaces, where at the level of rings the embedding is given by

$$\psi : CC[p_I \mid I \subset [n], |I| \in J] \to CC[x_{i,j} \mid i \in [n-1], j \in [n]], \quad p_I \mapsto \det(X_I)$$

where $X = (x_{i,j})$ is an $(n-1) \times n$ matrix of variables and X_I is the submatrix given by the first $|I|$ rows and columns indexed by I. A good candidate for a Gröbner degeneration of $\mathcal{F}\ell_n(J)$ is given by deforming ψ to a monomial map. This is done by sending each Plücker variable p_I to one of the summands of $\det(X_I)$.

A matching field is a map Λ that sends each Plücker variable p_I to a permutation $\sigma \in S_{|I|}$. In particular this gives a natural monomial map $\phi_\Lambda : CC[p_I] \to CC[x_{i,j}]$, the kernel of which is a toric ideal, i.e., binomial and prime. In general this is not a toric degeneration of the partial flag variety $\mathcal{F}\ell_n(J)$. To each matching field Λ we associate the polytope P_Λ, which is the toric polytope of $V(\ker(\phi_\Lambda))$.

In order for a matching field Λ to give rise to a toric degeneration of $\mathcal{F}\ell_n(J)$, it is necessary for Λ to be *coherent*, i.e., induced by a *weight matrix* M_Λ. To obtain a toric degeneration, one takes the initial ideal of the Plücker ideal with respect to the image of M_Λ under the *tropical Stiefel map* [FR15]. If this is a prime ideal we obtain a toric degeneration, with corresponding toric variety given by $V(\ker(\phi_\Lambda))$.

We consider the family of coherent matching fields Λ^σ labeled by permutations $\sigma \in S_n$, where if $I = \{p < q < i_3 < \cdots < i_k\}$, then

$$\Lambda^\sigma(I) = \begin{cases} id & \text{if } \sigma(p) > \sigma(q) \text{ or } |I| = 1 \\ (12) & \text{otherwise.} \end{cases}$$

Combinatorial mutations. The main tool we use to prove that a matching field Λ^σ give rise to a toric degeneration of partial flag varieties are combinatorial mutations. These are a special kind of piecewise linear maps between polytopes. We say that two polytopes are combinatorial mutation equivalent if there exists a sequence of combinatorial mutation between them. Combinatorial mutation were originally introduced, [ACGK12], in the context of classifying Fano varieties. They have been shown to connect families of Newton-Okounkov bodies for flag varieties [CM21b, CHM22b] and adjacent tropical cones [EH21].

We prove that a coherent matching field Λ inherits the property of giving rise to a toric degeneration from a matching field Λ' whenever the associated polytopes $\mathcal{P}_\Lambda$ and $\mathcal{P}_{\Lambda'}$ are combinatorial mutation equivalent.

Theorem 1. *Fix a natural number n and a subset $J \subseteq [n]$. Let Λ be a matching field for the partial flag variety $\mathcal{F}\ell(J;n)$. If the matching field polytope $\mathcal{P}_\Lambda$ is combinatorial mutation equivalent to the Gelfand-Tsetlin polytope, then Λ gives rise to a toric degeneration of $\mathcal{F}\ell(J;n)$.*

Note that the matching field Λ^{w_0} for $w_0 = (n, n-1, \dots, 1)$ is the diagonal matching field which classically gives rise to a toric degeneration of Gr(k,n) and $\mathcal{F}\ell_n$ [MS04]. The corresponding matching field polytopes are the well-studied Gelfand-Tsetlin polytopes.

In the case of the Grassmannians, we show that all matching fields Λ^σ give rise to toric degenerations.

Theorem 2. *Fix $\sigma \in S_n$. The matching field polytope $\mathcal{P}^k_{\Lambda^\sigma}$ is combinatorial mutation equivalent to Gelfand-Tsetlin polytope. In particular, each matching field Λ^σ gives rise to a toric degeneration of Gr(k,n).*

For more general partial flag varieties, we have to restrict to a smaller class of matching fields, labeled by permutations avoiding specific patterns.

Theorem 3. *If $\sigma \in S_n$ is a permutation that avoids the patterns $4123, 3124, 1423$ and 1324, then the polytope $\mathcal{P}^J_{\Lambda^\sigma}$ is combinatorial mutation equivalent to the Gelfand-Tsetlin polytope. In particular, Λ^σ gives rise to a toric degeneration of $\mathcal{F}\ell_n(J)$.*

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