Schubert calculus and toric degenerations of flag varieties

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Let

- X : an irreducible non-singular projective variety,
- Δ : a rational convex polytope,
- $Z(\Delta)$: the normal toric variety corresponding to Δ,

and assume that there exists a flat degeneration of X to $Z(\Delta)$.

Definition

If a subvariety $Y \subseteq X$ degenerates into a union of irreducible toric closed subvarieties of $Z(\Delta)$ under the degeneration of X to $Z(\Delta)$, then it is called a **semi-toric degeneration** of Y.

Such semi-toric limit of Y corresponds to a union of faces of Δ. We denote by $\mathcal{F}(Y)$ the set of faces of Δ included in this union.

Aim

to study geometric or combinatorial properties of the cohomology class $[Y] \in H^*(X; \mathbb{Z})$ using combinatorics of $\mathcal{F}(Y)$.

A motivating example is given by Kogan–Miller (2005).

- They constructed semi-toric degenerations of (opposite) Schubert varieties X^w in type A for $w \in \mathfrak{S}_n$ from Knutson–Miller's semi-toric degenerations (2005) of (opposite) matrix Schubert varieties.
- These are subfamilies of a toric degeneration of the flag variety $SL_n(\mathbb{C})/B$ to the normal toric variety corresponding to the Gelfand–Tsetlin polytope $GT(\lambda)$.
- The maximal faces in $\mathcal{F}(X^w)$ are naturally parametrized by the set $RP(w)$ of reduced pipe dreams.

The set $RP(w)$ inherits combinatorial information about the Schubert class $[X^w] \in H^*(SL_n(\mathbb{C})/B; \mathbb{Z})$ in several ways:

- by Kogan (2000) using the Gelfand–Tsetlin integrable system;
- by Kiritchenko–Smirnov–Timorin (2012) using the polytope ring.

The set $RP(w)$ has two kinds of remarkable combinatorial properties, following Bergeron–Billey (1993), Knutson–Miller (2005), ...

- (**subword complex**) There exists a natural bijection between $RP(w)$ and the set of reduced expressions of w appearing as subexpressions of

$$w_0 = s_n s_{n-1} s_n s_{n-2} s_{n-1} s_n \cdots s_1 s_2 \cdots s_n.$$

- (**mitosis recursion**) The set $RP(w)$ is obtained from $RP(w_0) = \{*\}$ by a sequence of mitosis operators.

Question

Can we generalize these combinatorics of reduced pipe dreams to toric degenerations of more general projective varieties?

- Caldero (2002) constructed toric degenerations of the flag variety G/B in general Lie type using string polytopes in representation theory.
- Morier-Genoud (2008) proved that Caldero's toric degeneration induces semi-toric degenerations of Schubert and opposite Schubert varieties.

Theorem (F. 2022)

The semi-toric limit of the opposite Schubert variety X^w gives a subword complex for an arbitrary string polytope in general Lie type.

Theorem (Kiritchenko 2016, F. 2022, F.–Nishiyama in preparation)

Combinatorics of mitosis recursion is naturally extended to the semi-toric limit of the Schubert variety X_w for the Gelfand–Tsetlin polytope in type C that is a specific example of a string polytope.